Central European Journal of Mathematics
DOI: 10.2478/s11533-006-0037-2
Research article
CEJM 5(1) 2007 134–153

Scattering properties for a pair of Schrödinger type operators on cylindrical domains

Michael Melgaard*

Department of Mathematics,
Uppsala University,
S-751 06 Uppsala, Sweden

Received 17 May 2006; accepted 29 September 2006

Abstract: Strong asymptotic completeness is shown for a pair of Schrödinger type operators on a cylindrical Lipschitz domain. A key ingredient is a limiting absorption principle valid in a scale of weighted (local) Sobolev spaces with respect to the uniform topology. The results are based on a refined version of Mourre's method within the context of pseudo-selfadjoint operators.
© Versita Warsaw and Springer-Verlag Berlin Heidelberg. All rights reserved.

Keywords: Lipschitz domains, scattering, limiting absorption principle, weighted Sobolev spaces
MSC (2000): 35P25, 47A40, 47F05, 35P05

1 Introduction and main theorem

We investigate scattering properties of two Schrödinger type operators H and H_0 defined on a cylindrical Lipschitz domain $\mathcal{M} = \mathbb{R} \times Q$, where $Q \subset \mathbb{R}^{n-1}$, $n \geq 2$, is an open and bounded Lipschitz domain [12]. Specifically, the operators act as

$$H = -\partial_i m^{ij} \partial_j + V, \quad H_0 = -\partial_i \delta^{ij} \partial_j \tag{1}$$

on $L^2(\mathcal{M})$ with Dirichlet boundary conditions; sums over indices are suppressed. The matrix-valued function $M \equiv (m^{ij})$ is real-valued and symmetric on $\mathcal{M}$ and V is a multiplication operator induced by a real-valued function on $\mathcal{M}$. Henceforth $x = (x_1, \tilde{x})$ is a vector of $\mathbb{R} \times Q$ and $\langle x \rangle := (1 + |x|^2)^{\frac{1}{2}}$. We impose the following conditions on M and V; here the symbol δ_{ij} refers to the components of the Euclidean metric matrix 1.

* E-mail: melgaard@math.uu.se

Assumption 1.1. The following inequalities are understood in the sense of matrices.
(i) There exist positive constants c and C such that

$$c \leq M(x) \leq C \quad \text{for a.e. } x \in \mathcal{M}.$$

(ii) There exists $\mu_1 > 1$ and a positive constant C_1 such that $v^{ij}(x) := m^{ij} - \delta^{ij}(x)$ satisfies

$$|v^{ij}(x)| \leq C_1 \langle x_1 \rangle^{-\mu_1} \ \textit{for a.e. } x \in \mathcal{M}.$$

(iii) There exists $\mu_2 > 1$ and a positive constant C_2 such that

$$|\partial_1 m^{ij}(x)| \leq C_2 \langle x_1 \rangle^{-\mu_2} \ \text{for a.e. } x \in \mathcal{M}.$$

In particular, Assumption 1.1(ii) implies that
(ii)'

$$\lim_{l \to \infty} \|\chi(\pm x_1 \geq l)(m^{ij}(x) - \delta^{ij})\|_{L^\infty(\mathcal{M})} = 0 \quad \forall i, j = 1, \ldots, n.$$

Assumption 1.2.
(i) Let $V \in L^\infty(\mathcal{M})$.
(ii) There exists $\nu_1 > 1$ and a positive constant C_3 such that

$$|V(x)| \leq C_3 \langle x_1 \rangle^{-\nu_1} \ \textit{for a.e. } x \in \mathcal{M}.$$

(iii) There exists $\nu_2 > 1$ and a positive constant C_4 such that

$$|\partial_1 V(x)| \leq C_4 \langle x_1 \rangle^{-\nu_2} \ \text{for a.e. } x \in \mathcal{M}.$$

In particular, Assumption 1.2(ii) implies that
(ii)'

$$\lim_{l \to \infty} \|\chi(\pm x_1 \geq l)\, V(x)\|_{L^\infty(\mathcal{M})} = 0.$$

We then define the sesquilinear form $\widetilde{\mathfrak{h}}$ with domain $\mathbf{H}^1_0(\mathcal{M}) \times \mathbf{H}^1_0(\mathcal{M})$ by

$$\widetilde{\mathfrak{h}}[\varphi, \psi] = \langle \partial_i \varphi, m^{ij} \partial_j \psi \rangle, \quad \varphi, \psi \in \mathbf{H}^1_0(\mathcal{M})$$

It is clearly densely defined and symmetric. In view of Assumption 1.1 the matrix M is bounded and uniformly positive and, consequently, the form $\widetilde{\mathfrak{h}}$ is non-negative and closed. Invoking Kato's representation theorem [12, Theorem VI.2.4], we get an unique self-adjoint operator $\widetilde{H}$. Since, moreover, Assumption 1.2 ensures that V is bounded, the KLMN theorem [28, Theorem X.17] asserts that the sesquilinear form sum

$$\mathfrak{h}[\varphi, \psi] := \widetilde{\mathfrak{h}}[\varphi, \psi] + \langle \varphi, V\psi \rangle, \quad \varphi, \psi \in \mathfrak{Q}(\mathfrak{h}) = \mathfrak{Q}(\widetilde{\mathfrak{h}}) = \mathbf{H}^1_0(\mathcal{M}),$$

is closed and semi-bounded from below and hence it generates a self-adjoint operator H.

Furthermore, we introduce the unperturbed Hamiltonian H_0 as follows. On $L^2(\mathcal{M})$ we consider the sesquilinear form $\mathfrak{h}_0$ with domain $\mathfrak{Q}(\mathfrak{h}_0) = \mathbf{H}^1_0(\mathcal{M})$ defined by

$$\mathfrak{h}_0[\varphi, \psi] := \langle \partial_i \varphi, \delta^{ij} \partial_j \psi \rangle, \quad \varphi, \psi \in \mathbf{H}^1_0(\mathcal{M}). \tag{2}$$

It is a densely defined, symmetric, non-negative closed form. A form core of $\mathfrak{h}_0$ is $C_0^\infty(\mathcal{M})$. Kato's representation theorem gives a unique self-adjoint operator H_0 in $L^2(\mathcal{M})$ with domain

$$\begin{aligned} \mathfrak{D}(H_0) := \big\{ \psi \in \mathbf{H}^1_0(\mathcal{M}) \,:\, & \exists \varphi \in L^2(\mathcal{M}) \text{ such that} \\ & \mathfrak{h}_0[\psi, u] = \langle \varphi, u \rangle_{L^2(\mathcal{M})} \quad \forall u \in \mathbf{H}^1_0(\mathcal{M}) \big\}. \end{aligned} \tag{3}$$

Our aim is to establish scattering theory for the pair (H, H_0) governed by the Schrödinger equation

$$i \frac{d}{dt} \psi(t) = H \psi(t), \quad \psi(0) = \psi_0. \tag{4}$$

The solution to (4) is given by $\psi(t) = U(t)\psi_0 = e^{-itH}\psi_0$. If one replaces H by H_0, the "free" Hamiltonian, the corresponding solution to (4) can be expressed as $U_0(t) = \exp(-itH_0)$; the free evolution. The study of scattering consists in comparing the two evolutions $U_0(t)$ and $U(t)$ for large positive and negative times t, using the Møller wave operators

$$W^\pm = s - \lim_{t \to \pm\infty} U(-t) U_0(t) P_{\mathrm{ac}}(H_0). \tag{5}$$

Here $P_{\mathrm{ac}}(H_0)$ denotes the projection onto the subspace of absolute continuity of H_0. Assuming that the wave operators exist, one says that they are *asymptotically complete* if the ranges of W^+ and W^-, denoted $\operatorname{Ran} W^\pm$, coincide with the subspace of continuity of H, denoted $\mathcal{H}_{\mathrm{c}}(H)$. It the ranges of $W^\pm$ equal the subspace of absolute continuity of H, then the wave operators are said to be *strongly complete*; in other words, the singular continuous spectrum of H is empty. In that case the absolutely continuous parts of H_0 and H are unitarily equivalent via the wave operators. Our main result is:

Theorem 1.3. *Let Assumption 1.1 and Assumption 1.2 be satisfied. Then*
1. The wave operators

$$W^\pm = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}$$

exist and are strongly asymptotically complete.
2. If φ is an admissible function, then

$$W^\pm = s - \lim_{t \to \pm\infty} e^{it\varphi(H)} e^{-it\varphi(H_0)}.$$

We recall that a real-valued function φ, defined on $\mathbb{R}_+$, is *admissible* provided

$$\lim_{t \to \infty} \int_0^\infty \left| \int_I e^{-it\varphi(\lambda) - is\lambda} \, d\lambda \right|^2 ds = 0$$

for any bounded interval $I \subset \mathbb{R}_+$.

The key ingredient in the proof of Theorem 1.3 is a limiting absorption principle (abbrev. LAP) in a framework of weighted Sobolev spaces $\mathbf{H}^s_{(\beta)}(\mathcal{M}) := \{\psi \in \mathcal{D}'(\mathcal{M}) : \langle x\rangle^\beta \psi \in \mathbf{H}^s(\mathcal{M})\}$ equipped with its natural norm; here $\mathcal{D}'(\mathcal{M})$ denotes the space of distributions. This version of the LAP, suitable for the study of scattering theory, is the content of Theorem 6.1, assertion 4. Since Eidus' classic paper [13], the LAP has been extensively considered in spectral and scattering theory (see, e.g., [29]). To prove Theorem 27 we apply an abstract version of Mourre's method. This method was first developed to prove the LAP for three-body Schrödinger operators [25] (see also [26] and [31]). Froese and Herbst [14] showed that N-body Schrödinger operators have no positive eigenvalues. Later use of the Mourre method by Sigal, Soffer, and Derezinski lead to breakthroughs in N-body quantum scattering as described in [10]. Iwashita [17] and Weder [33] proved the LAP for first order symmetric systems. Ben-Artzi *et al* [3] and Tamura [32] used Mourre's method to derive the LAP for the acoustic wave operators (for further developments, see the survey [9]). To establish Theorem 6.1 we apply a refined version of Mourre's method within the context of pseudo-selfadjoint operators (see, e.g., [16]). In fact, we shall establish another version of Theorem 6.1(4), see Theorem 6.2, wherein the set $\mathcal{M}$ is replaced by $\mathbb{R}^n$. This version of the LAP, valid on the whole Euclidean space, has independent interest. Another version of the LAP for H, valid in a scale of Besov spaces, can be found in [24]. Therein a simpler version of the abstract Mourre method is applied, avoiding the notion of pseudo-selfadjoint operators. Applications of the main theorems to mesoscopic physics will be published elsewhere.

There seems to be few results on this kind of problem in the literature, but related results for other kinds of operators are found in [2, 8, 11, 15, 18, 20, 22, 23].

The paper is organized as follows. In Section 2 we introduce weighted Sobolev spaces and a class of Besov spaces. An abstract version of Mourre's method is summarized in Section 3 within the context of pseudo-selfadjoint operators. In Section 4 we define a dilation operator A and a (strict) Mourre estimate for the operator H_0 is established. Auxiliary properties of the Hamiltonians H and H_0 are shown in Section 5. In Section 6 we give the proof of Theorem 6.2, which enables us to give a fairly short proof of the LAP in Theorem 6.1 within the framework of weighted, local Sobolev spaces. The main result on scattering theory, Theorem 1.3, is proven in Section 7.

Finally, let us fix some basic notation. Let $\mathcal{H}$ be a separable complex Hilbert space. We denote its scalar product and norm by $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}}$, resp. If $\mathcal{K}$ is another Hilbert space, then we write $\mathcal{K} \subset \mathcal{H}$ if $\mathcal{K}$ is embedded in $\mathcal{H}$, and we write $\mathcal{K} \hookrightarrow \mathcal{H}$ provided the embedding is continuous and dense. Let T be a self-adjoint operator on a Hilbert space $\mathcal{H}$ with domain $\mathfrak{D}(T)$. The spectrum and resolvent set are denoted by $\sigma(T)$ and $\rho(T)$, resp. We use standard terminology for the various parts of the spectrum, see, e.g., [27]. The resolvent is $R(\zeta) = (T-\zeta)^{-1}$. The spectral family associated to T is denoted by $E_T(\xi)$, $\xi \in \mathbb{R}$. The spaces of bounded operators from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X},\mathcal{Y})$ and the upper, resp. lower, half-plane is denoted by $\mathbb{C}_\pm := \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}$.

2 Weighted Sobolev spaces and Besov spaces

We adopt the usual notation for function spaces: C_0^∞, L^2, etc. The Schwartz space of rapidly decreasing functions and its adjoint space of tempered distributions are denoted by $\mathcal{S}$ and $\mathcal{S}'$, resp. The Fourier transformation is denoted by $\mathcal{F}$.

The weighted Sobolev spaces on $\mathbb{R}^n$ are defined by

$$\mathbf{H}^s_{(t)}(\mathbb{R}^n) = \{\, \psi \in \mathcal{S}'(\mathbb{R}^n) \,:\, \langle p\rangle^s \langle x\rangle^t \psi \in L^2(\mathbb{R}^n)\,\}.$$

Here $\langle x\rangle$ denotes the operator of multiplication by the function $(1+|x|^2)^{\frac{1}{2}}$ and $\langle p\rangle = \mathrm{F}^*\langle x\rangle\mathcal{F}$.

In order to state an optimal form of the limiting absorption principle we introduce a class of Besov spaces. For this aim we let $\theta_1, \theta_2 \in C_0^\infty(\mathbb{R}^n)$ be two functions such that $\theta_1(x) > 0$ for $|x| < 2$, $\theta_1(x) = 0$ otherwise and $\theta_2(x) > 0$ if $1/2 < |x| < 2$, $\theta_2(x) = 0$ otherwise. Then, for $s, t \in \mathbb{R}$ and $1 \le q \le \infty$, we define

$$\mathbf{H}^s_{t,q}(\mathbb{R}^n) = \Big\{ \psi \in \mathcal{S}'(\mathbb{R}^n) \,:\, \|\theta_1(x)\psi\|_{\mathbf{H}^s(\mathbb{R}^n)} + \left[\int_1^\infty \left\| r^t \theta_2\left(\frac{x}{r}\right)\psi\right\|^q_{\mathbf{H}^s(\mathbb{R}^n)} \frac{dr}{r}\right]^{\frac{1}{q}} < \infty \Big\}.$$

For $q = \infty$ the term containing the integral must be understood as being $\sup_{r>1} \|r^t\theta_2(x/r)\psi\|_{\mathbf{H}^s(\mathbb{R}^n)}$. The spaces $\mathbf{H}^s_{t,q}(\mathbb{R}^n)$ have Banach structure and we note that

$$(\mathbf{H}^s_{t,q}(\mathbb{R}^n))^* = \mathbf{H}^{-s}_{-t,q'}(\mathbb{R}^n) \text{ for } 1 \le q < \infty \text{ and } \frac{1}{q} + \frac{1}{q'} = 1.$$

The weighted Sobolev spaces $\mathbf{H}^s_{(t)}(\mathbb{R}^n)$ coincide algebraically and topologically with $\mathbf{H}^s_{t,2}(\mathbb{R}^n)$. Observe also that we have for all $\beta > 1/2$,

$$L^2_{(\beta)}(\mathbb{R}^n) \subsetneq \mathbf{H}^{-1}_{(\beta)}(\mathbb{R}^n) \subsetneq \mathbf{H}^{-1}_{\frac{1}{2},1}(\mathbb{R}^n) \subsetneq \mathbf{H}^{-1}_{(\frac{1}{2})}(\mathbb{R}^n) \subsetneq \mathbf{H}^{-1}(\mathbb{R}^n) \quad \text{(continuously and densely)} \tag{6}$$

$$\mathbf{H}^1(\mathbb{R}^n) \subsetneq \mathbf{H}^1_{(-\frac{1}{2})}(\mathbb{R}^n) \subsetneq \mathbf{H}^1_{-\frac{1}{2},\infty}(\mathbb{R}^n) \subsetneq \mathbf{H}^1_{(-\beta)}(\mathbb{R}^n) \subsetneq L^2_{(-\beta)}(\mathbb{R}^n) \quad \text{(continuously)} \tag{7}$$

For further information, we refer to [1, Section 4.1].

3 Pseudo-selfadjoint operators and Mourre's method

We begin by recalling some definitions from the theory of pseudo-selfadjoint operators [16]. Then we summarize a refined abstract version of Mourre's method [25] for pseudo-selfadjoint operators found in [5] and [7].

Definition 3.1. A pseudo-selfadjoint operator in $\mathcal{H}$ is a linear operator $T : \mathfrak{D}(T) \subset \mathcal{H} \to \mathcal{H}$ such that

(i) $T\mathfrak{D}(T) \subset \overline{\mathfrak{D}(T)}$.
(ii) The operator T is self-adjoint as an operator in the Hilbert space $\overline{\mathfrak{D}(T)}$.

Definition 3.2. If $\varphi \in C_\infty(\mathbb{R})$ (that is, $\varphi : \mathbb{R} \to \mathbb{C}$ is continuous and tends to zero at infinity) and T is a pseudo-selfadjoint operator in $\mathcal{H}$, then $\varphi(T)$ denotes the operator in $\mathcal{B}(\mathcal{H})$ defined by
(i) $\varphi(T)|_{\overline{\mathfrak{D}(T)}}$ is the operator given by the functional calculus applied to T, where T is reduced to $\overline{\mathfrak{D}(T)}$.
(ii) $\varphi(T)|_{\mathcal{H}\ominus\overline{\mathfrak{D}(T)}} = 0$.

One defines $R(\zeta) := (T-\zeta)^{-1}$ for every $\zeta \in \mathbb{C}\setminus\mathbb{R}$ with $R(\zeta) = 0$ on $\mathcal{H} \ominus \overline{\mathfrak{D}(T)}$. The family $\{R(\zeta); \zeta \in \mathbb{C}\setminus\mathbb{R}\}$ is a "self-adjoint" pseudo-resolvent in the sense that
(i) $R(\zeta) \in \mathcal{B}(\mathcal{H})$ for every $\zeta \in \mathbb{C}\setminus\mathbb{R}$.
(ii) $R(\zeta)^* = R(\bar{\zeta})$ for every $\zeta \in \mathbb{C}\setminus\mathbb{R}$.
iii) $R(\zeta_1) - R(\zeta_2) = (\zeta_1 - \zeta_2)R(\zeta_1)R(\zeta_2)$ for every $\zeta_1, \zeta_2 \in \mathbb{C}\setminus\mathbb{R}$.

The spectrum of T reduced to $\overline{\mathfrak{D}(T)}$ will be denoted by $\sigma(T)$. Then, for any $\zeta \in \mathbb{C}\setminus\sigma(T)$, one has $R(\zeta)\mathcal{H} = \mathfrak{D}(T)$, $R(\zeta)(T-\zeta)\psi = \psi$ for every $\psi \in \mathfrak{D}(T)$, $(T-\zeta)R(\zeta)$ is the orthogonal projection of $\mathcal{H}$ onto $\overline{\mathfrak{D}(T)}$.

Definition 3.3. Let $\{T_\gamma\}_{\gamma>0}$ be a family of pseudo-selfadjoint operators in $\mathcal{H}$. It is said that $\lim_{\gamma\to\infty} T_\gamma = T$ (pseudo-selfadjoint operators in $\mathcal{H}$) holds in the strong resolvent sense provided $s-\lim_{\gamma\to\infty}(T_\gamma-\zeta)^{-1} = (T-\zeta)^{-1}$ for some $\zeta \in \mathbb{C}$ obeying $\inf_{\gamma>0} \operatorname{dist}(\zeta, \sigma(T_\gamma)) > 0$.

The following two results allow us to construct pseudo-selfadjoint operators. The first one is found in [7, Lemma 3.7]

Proposition 3.4. *Let $\widetilde{T}$ be a self-adjoint, bounded from below, densely defined operator in $\mathcal{H}$. Denote by $\mathcal{H}_1$ the form domain of $\widetilde{T}$ and $\mathcal{H}_{-1} = \mathcal{H}_1^*$ (and thus $\mathcal{H}_1 \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{H}_{-1}$). Let $\chi \in \mathcal{B}(\mathcal{H})$ and $T_\gamma := \widetilde{T} + \gamma\chi^*\chi$ for every $\gamma > 0$. Then the following statements are true:*
1. There exists a pseudo-selfadjoint operator T in $\mathcal{H}$ such that $T = \lim_{\gamma\to\infty} T_\gamma$ in the sense of Definition 3.3.
2. For any $\lambda > -\inf \widetilde{T}$, $(T+\lambda)^{-1} = s - \lim_{\gamma\to\infty}(T_\gamma+\lambda)^{-1}$ in the norm topology of $\mathcal{B}(\mathcal{H}_{-1}, \mathcal{H}_1)$. In particular, $(T+\lambda)^{-1} \in \mathcal{B}(\mathcal{H}_{-1}, \mathcal{H}_1)$.

The second one is due to Simon [30].

Proposition 3.5. *Let $\{\mathfrak{t}_k\}_{k\geq 1}$ be a sequence of (not necessarily densely defined) non-negative closed forms in $\mathcal{H}$ such that $\mathfrak{Q}(\mathfrak{t}_k) \supset \mathfrak{Q}(\mathfrak{t}_{k+1})$ and $\mathfrak{t}_k[\varphi,\varphi] \leq \mathfrak{t}_{k+1}[\varphi,\varphi]$ for every*

$\varphi \in \mathfrak{Q}(\mathfrak{t}_{k+1})$ and $k \geq 1$. Define the form $\mathfrak{t}_\infty$ by

$$\mathfrak{t}_\infty[\varphi, \psi] = \lim_{k\to\infty} \mathfrak{t}_k[\varphi, \psi], \qquad \textit{with} \tag{8}$$

$$\varphi, \psi \in \mathfrak{Q}(\mathfrak{t}_\infty) := \left\{ \varphi \in \bigcap_{k\geq 1} \mathfrak{Q}(\mathfrak{t}_k) \ : \ \sup_k \mathfrak{t}_k[\varphi, \varphi] < \infty \right\} \tag{9}$$

Then:
1. The form $\mathfrak{t}_\infty$ is non-negative and closed.
2. If T_k denotes the pseudo-selfadjoint operator corresponding to $\mathfrak{t}_k$ by the representation theorem on $\overline{\mathfrak{Q}(\mathfrak{t}_k)}$ ($1 \leq k \leq \infty$), then $T_\infty = \lim_{\kappa\to\infty} T_k$ in the sense of Definition 3.3.

To analyze the spectral properties of T, one introduces a self-adjoint (densely defined) operator A in $\mathcal{H}$ and one considers the operator $e^{-i\tau A} T e^{i\tau A} =: W(\tau)T$, where $e^{i\tau A}$ is the unitary group generated by A. We introduce several notions of regularity of T with respect to $e^{i\tau A}$.

Definition 3.6. Let $B \in \mathcal{B}(\mathcal{H})$.
(i) The operator B is said to be of class $C^1(A)$ (resp., $C^1_u(A)$) if the map

$$\tau \mapsto W(\tau)B \in \mathcal{B}(\mathcal{H})$$

is of class C^1 in the strong (resp., norm) topology of $\mathcal{B}(\mathcal{H})$ or, equivalently, if the following limit exists strongly, resp., in norm, in $\mathcal{B}(\mathcal{H})$:

$$[B, A] := \lim_{\tau\to 0} \frac{e^{-i\tau A} B e^{i\tau A} - B}{i\tau}.$$

(ii) The operator B is said to be of class $\mathcal{C}^1(A)$ (or A-regular) if

$$\int_0^1 \| e^{-i\tau A} B e^{i\tau A} + e^{i\tau A} B e^{-i\tau A} - 2B \| \, \frac{d\tau}{\tau^2} < \infty.$$

(iii) The operator B is said to be of class $C^{1+\beta}(A)$ for $\beta \in (0, 1]$ if B is of class $C^1(A)$ and the map

$$\tau \mapsto W(\tau)[B, A] \in \mathcal{B}(\mathcal{H})$$

is Hölder continuous of order β, i.e., there exists a constant C such that

$$\| W(\tau)[B, A] - [B, A] \|_{\mathcal{B}(\mathcal{H})} \leq C\tau^\beta.$$

If $B \in \mathcal{B}(\mathcal{H})$ we may define the following quadratic form on $\mathfrak{D}(A)$:

$$\langle \psi, [B, A]\psi \rangle = \langle B\psi, A\psi \rangle - \langle A\psi, B\psi \rangle, \quad \psi \in \mathfrak{D}(A). \tag{10}$$

Then $B \in C^1(A)$ if and only if this form is bounded for the $\mathcal{H}$-topology on $\mathfrak{D}(A)$. In this case, the bounded operator on $\mathcal{H}$ defined in part (i) of Definition 3.6 is precisely the operator associated to the form (10). It is denoted by $[B, A]$.

It is well-known [5] that

$$C^{1+\beta}(A) \subset \mathcal{C}^1(A) \subset C^1_u(A) \subset C^1(A). \tag{11}$$

Definition 3.7. A pseudo-selfadjoint operator T in $\mathcal{H}$ is said to be of class $C^1(A)$, $C^1_u(A)$, $C^{1+\beta}(A)$, or $\mathcal{C}^1(A)$, if the resolvent $R(\zeta)$ is of class $C^1(A)$, $C^1_u(A)$, $C^{1+\beta}(A)$, or $\mathcal{C}^1(A)$ for some, and thus all, $\zeta \in \mathbb{C} \setminus \sigma(T)$. In the affirmative case, we write $T \in C^1(A)$ (or $T \in C^1_u(A)$, $T \in C^{1+\beta}(A)$, $T \in \mathcal{C}^1(A)$, resp.).

To verify that $T \in C^1(A)$, one may use the quadratic form $[A,T]$ defined on $\mathfrak{D}(A) \cap \mathfrak{D}(T)$ by

$$\langle \varphi, [T,A]\psi \rangle = \langle T\varphi, A\psi \rangle - \langle A\varphi, T\psi \rangle, \quad \varphi, \psi \in \mathfrak{D}(A) \cap \mathfrak{D}(T). \tag{12}$$

The following criterion is found in [7, Lemma 5.5]; it generalizes the well-known one [1, Theorem 6.2.10].

Proposition 3.8. *Let T be a pseudo-selfadjoint operator in $\mathcal{H}$ such that*

$$A[\mathfrak{D}(A) \cap \mathfrak{D}(T)] \subset \overline{\mathfrak{D}(T)}. \tag{13}$$

Then $T \in C^1(A)$ if and only if the following two requirements are fulfilled:
(i) $R(\zeta)\mathfrak{D}(A) \subset \mathfrak{D}(A)$ for some $\zeta \in \mathbb{C} \setminus \sigma(T)$.
(ii) There exists a positive constant c such that

$$|\langle [A,T]\varphi, \varphi \rangle| \leq c \left(\|T\varphi\|^2_{\mathcal{H}} + \|\varphi\|^2 \right), \quad \varphi \in \mathfrak{D}(A) \cap \mathfrak{D}(T). \tag{14}$$

Moreover, we may substitute (i) by
(i') There exists $\zeta \in \mathbb{C} \setminus \sigma(T)$ such that $\{\varphi \in \mathfrak{D}(A) : R(\zeta)\varphi, R(\overline{\zeta})\varphi \in \mathfrak{D}(A)\}$ is a core for A.

Under these conditions the space $R(\zeta)\mathfrak{D}(A)$ does not depend on $\zeta \in \mathbb{C} \setminus \sigma(T)$. It is a core for T and a dense subspace of $\mathfrak{D}(A) \cap \mathfrak{D}(T)$ for the intersection topology (associated with the norm $\|\cdot\|_{\mathcal{H}} + \|A \cdot\|_{\mathcal{H}} + \|T \cdot\|_{\mathcal{H}}$). In addition, $\mathfrak{D}(A) \cap \mathfrak{D}(T)$ is a core for T, and in the sense of forms on $\mathcal{H}$ one has:

$$[A, R(\zeta)] = R(\zeta)[T,A]R(\zeta), \quad \zeta \in \mathbb{C} \setminus \sigma(T), \tag{15}$$

where the same symbols denote the forms $[A, R(\zeta)]$, $[T,A]$ and their continuous extensions to the spaces $\mathcal{H}$ or $\mathfrak{D}(T)$ (endowed with the graph topology).

One has the Virial theorem (see, e.g., [5]):

Theorem 3.9. *If $T \in C^1(A)$ is a pseudo-selfadjoint operator, then*

$$\langle \varphi, [T,A]\psi \rangle = 0 \tag{16}$$

for each $\lambda \in \mathbb{R}$ and $\varphi, \psi \in \mathfrak{D}(T)$ satisfying $T\varphi = \lambda\varphi$, $T\psi = \lambda\psi$.

It is convenient to introduce two functions $\varrho_T \equiv \varrho_T^A$ and $\tilde{\varrho}_T \equiv \tilde{\varrho}_T^A$ defined on $\mathbb{R}$ with values in $(-\infty, +\infty]$. Let T be any pseudo-selfadjoint operator of class $C^1(A)$. For any $\xi \in \mathbb{R}$ we denote

$$\begin{aligned}\varrho_T^A(\xi) = \sup\{\,&\alpha \in \mathbb{R} \,:\, \exists f \in C_0^\infty(\mathbb{R}) \text{ real }, f(\xi) \neq 0, \text{ such that} \\ & f(T)i[T,A]f(T) \geq \alpha f(T)^2\}\,, \\ \tilde{\varrho}_T^A(\xi) = \sup\{\,&\alpha \in \mathbb{R} \,:\, \exists f \in C_0^\infty(\mathbb{R}) \text{ real }, f(\xi) \neq 0, \\ & \text{and a compact operator } K \text{ such that} \\ & f(T)i[T,A]f(T) \geq \alpha f(T)^2 + K\}\,.\end{aligned}$$

The significance of these two functions are reflected by the following simple fact [1, Proposition 7.2.6].

Proposition 3.10. *The functions ϱ_T and $\tilde{\varrho}_T$ are lower semicontinuous, $-\infty < \varrho_T(\xi) \leq \tilde{\varrho}_T(\xi) \leq +\infty$, and*

$$\varrho_T(\xi) < \infty \Longleftrightarrow \xi \in \sigma(T), \quad \tilde{\varrho}_T(\xi) < \infty \Longleftrightarrow \xi \in \sigma_{\mathrm{ess}}(T).$$

We may now introduce the main notion of Mourre's method.

Definition 3.11 ("Mourre estimate")**.** Let A be a self-adjoint operator in $\mathcal{H}$ and let T be a pseudo-selfadjoint operator in $\mathcal{H}$ of class $C^1(A)$. If $\xi \in \mathbb{R}$ and $\tilde{\varrho}_T^A(\xi) > 0$, then the operator A is said to be conjugate to T at ξ and A is said to be strictly conjugate to T at ξ if $\varrho_T^A(\xi) > 0$.

The latter motivates the following definition.

Definition 3.12 (Mourre set)**.** If T is a pseudo-selfadjoint operator of class $C^1(A)$, then $\mu^A(T) := \{\xi \in \mathbb{R} : A \text{ is conjugate to } T \text{ at } \xi\}$ is called the Mourre set.

As we shall see now, if $T \in \mathcal{C}^1(A)$, then T has nice spectral properties on the open set $\mu^A(T)$ [1, Theorem 7.4.2]:

Theorem 3.13. *Let T be a pseudo-selfadjoint operator in $\mathcal{H}$.*
1. If $T \in C^1(A)$, then $\mu^A(T) \cap \sigma_{\mathrm{p}}(T)$ is discrete in $\mu^A(T)$, and each of the included eigenvalues of T has finite multiplicity.
2. If $T \in \mathcal{C}^1(A)$ such that $\sigma(T) \neq \mathbb{R}$, then $\sigma_{\mathrm{sc}}(T) \cap \mu^A(T) = \emptyset$.

The limiting absorption principle takes the following form [1, Proposition 7.4.4]:

Theorem 3.14. *Let T be a pseudo-selfadjoint operator in $\mathcal{H}$ of class $\mathcal{C}^1(A)$ such that $\sigma(T) \neq \mathbb{R}$. Assume that there exist two Hilbert spaces $\mathcal{K}$ and $\mathcal{K}_1$ such that*

(i) $\mathcal{K}_1 \hookrightarrow \mathcal{K}$ *and* $\mathcal{H} \hookrightarrow \mathcal{K}$, *then* $\mathcal{K}^* \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{K}$,
(ii) For some $\lambda_0 \in \mathbb{R} \setminus \sigma(T)$, $R(\lambda_0)$ *may be extended to an operator of* $\mathcal{B}(\mathcal{K}, \mathcal{K}^*)$.
(iii) $R(\lambda_0)\mathcal{K}_1 \subset \mathfrak{D}(A)$.
Let $\mathcal{K}_{\frac{1}{2},1}$ *denote the interpolation space* $(\mathcal{K}, \mathcal{K}_1)_{\frac{1}{2},1}$ *obtained by a real interpolation (see, e.g. [4]); in particular* $\mathcal{K}_{\frac{1}{2},1} \hookrightarrow \mathcal{K}$, $\mathcal{K}^* \subset \mathcal{K}^*_{\frac{1}{2},1}$ *and* $\mathcal{B}(\mathcal{K}, \mathcal{K}^*) \subset \mathcal{B}(\mathcal{K}_{\frac{1}{2},1}, \mathcal{K}^*_{\frac{1}{2},1})$. *Then*
1. For every $\zeta \in \mathbb{C}^{\pm}$, $R(\zeta) \in \mathcal{B}(\mathcal{K}, \mathcal{K}^*)$ *and the function*

$$\mathbb{C}^{\pm} \ni \zeta \mapsto R(\zeta) \in \mathcal{B}(\mathcal{K}, \mathcal{K}^*) \tag{17}$$

is holomorphic.
2. The above function, considered in $\mathcal{B}(\mathcal{K}_{\frac{1}{2},1}, \mathcal{K}^*_{\frac{1}{2},1})$*-valued, may be extended to a weak***-continuous function defined on* $\mathbb{C}^{\pm} \cup [\mu^A(T) \setminus \sigma_{\rm p}(T)]$.

Finally, we mention that the following perturbative result is often useful for establishing Mourre estimates in applications [1, Theorem 7.2.9].

Theorem 3.15. *Let* A, T *and* T_0 *be self-adjoint operators in a Hilbert space* $\mathcal{H}$ *such that both* T *and* T_0 *are of class* $C^1_u(A)$. *If* $(T+i)^{-1} - (T_0+i)^{-1}$ *is compact, then* $\tilde{\rho}^A_T = \tilde{\rho}^A_{T_0}$. *In particular,* A *is conjugate to* H *at* λ *if and only if it is conjugate to* T_0 *at* λ.

4 Mourre estimate for H_0

The Dirichlet Laplacian $-\Delta_{D,Q}$ on $L^2(Q)$ generated by the sesquilinear form

$$\mathfrak{q}[\varphi, \psi] := \langle \partial_i \varphi, \delta^{ij} \partial_j \psi \rangle, \quad \varphi, \psi \in \mathbf{H}^1_0(Q), \tag{18}$$

has purely discrete spectrum consisting of eigenvalues $(0 <)\upsilon_1 < \upsilon_2 \leq \upsilon_3 \leq \cdots$. The latter constitute the threshold set $\Upsilon := \{\upsilon_n : n \in \mathbb{N}\}$ of H_0. The unperturbed Hamiltonian H_0 clearly has the tensor decomposition

$$H_0 = p_1^2 \otimes I + I \otimes (-\Delta_{D,Q}), \tag{19}$$

where $p_1 = -i\partial_1$ is the momentum operator in $L^2(\mathbb{R})$. Moreover, one has

$$\sigma(H_0) = \sigma_{\rm ac}(H_0) = \sigma_{\rm ess}(H_0) = [\upsilon_1, \infty). \tag{20}$$

Let A be the self-adjoint extension of $(1/2)(x_1 p_1 + p_1 x_1)$ initially defined on $C_0^\infty(\mathcal{M})$. The operator A is the infinitesimal generator of the dilation (with respect to x_1) group $\exp(-itA)$ defined by

$$e^{itA}\psi(x_1, \widetilde{x}) = e^{-\frac{t}{2}}\psi(e^{-t}x_1, \widetilde{x})$$

for all $t \in \mathbb{R}$ and all $\psi \in L^2(\mathcal{M})$. Using the isomorphism $\mathbf{H}^1_0(\mathcal{M}) \simeq \mathbf{H}^1(\mathbb{R}) \otimes \mathbf{H}^1_0(Q)$ we can write

$$e^{itA} = e^{itA} \otimes 1 \tag{21}$$

where, on the right-hand side, we regard A as an operator in $L^2(\mathbb{R})$. Since e^{itA} leaves invariant $\mathbf{H}^1(\mathbb{R})$ [1, Proposition 4.2.4], we infer from (21) that e^{itA} maps $\mathbf{H}^1_0(\mathcal{M})$, the form domain of both H_0 and H, into itself.

We shall show that the dilation operator A is conjugate to H_0 away from the set of thresholds Υ. Below we shall, with a slight abuse of notation, occasionally regard A as an operator in $L^2(\mathbb{R})$.

The classic (strict) Mourre estimate for the one-dimensional Laplacian [25] immediately gives us the following result.

Proposition 4.1. *The operator A is conjugate to $T_1 := p_1^2$ at each point $\mathbb{R} \setminus \{0\}$. In particular,*

$$\varrho^A_{T_1}(\xi_1) = \begin{cases} 2\xi_1 & \text{for } \xi_1 \geq 0, \\ +\infty & \text{for } \xi_1 < 0. \end{cases} \tag{22}$$

Next we state [6, Theorem 3.4].

Theorem 4.2. *Let T_1, T_2 be two self-adjoint, bounded from below operators in the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. Assume that A_j, $j = 1, 2$, is a self-adjoint operator in $\mathcal{H}_j$ such that T_j is of class $C^k(A_j)$, $k \in \mathbb{N}\setminus\{0\}$. Then the operators $T := T_1 \otimes I + I \otimes T_2$ and $A := A_1 \otimes I + I \otimes A_2$ are self-adjoint operators in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Moreover, the operator T is of class $C^k(A)$ and $\forall \xi \in \mathbb{R}$:*

$$\varrho^A_T(\xi) = \inf_{\xi=\xi_1+\xi_2} \left[\varrho^{A_1}_{T_1}(\xi_1) + \varrho^{A_2}_{T_2}(\xi_2)\right].$$

With these auxiliary results in place, we can prove the Mourre estimate for H_0.

Proposition 4.3. *The operator A is conjugate to the unperturbed operator H_0 at each point $\mathbb{R} \setminus \Upsilon$.*

Proof. We apply Theorem 4.2. Bearing in mind the decomposition of H_0 in (19), we set $A_1 := A$, $A_2 := 0$, which are self-adjoint operators in $L^2(\mathbb{R})$, resp. $L^2(Q)$. The operator $T_1 := p_1^2$, resp. $T_2 = -\Delta_{D,Q}$, is a non-negative self-adjoint operator in $L^2(\mathbb{R})$, resp. $L^2(Q)$. Proposition 4.1 gives us an explicit expression for $\varrho^A_{T_1}(\xi_1)$. Moreover, one easily sees that

$$\varrho^0_{T_2}(\xi_2) = \begin{cases} 0 & \text{for } \xi_2 \in \Upsilon, \\ +\infty & \text{for } \xi_2 \notin \mathbb{R} \setminus \Upsilon. \end{cases}$$

Then, by setting $\gamma(\xi) := \xi - \sup\{\zeta \in \Upsilon : \zeta \leq \xi\}$, Theorem 4.2 gives for all $\xi \in \mathbb{R}$:

$$\varrho^A_{H_0}(\xi) = \begin{cases} 2\gamma(\xi) & \text{for } \xi \geq \upsilon_1, \\ +\infty & \text{for } \xi < \upsilon_1. \end{cases}$$

Since $\gamma(\xi)$ is strictly positive on $\mathbb{R} \setminus \Upsilon$, the assertion follows. □

A Mourre estimate for a matrix-valued Hamiltonian with Schrödinger operators as component Hamiltonians was established in [23]. Since the latter Hamiltonian has the same structure as H_0, the approach above applies and gives a less cumbersome proof.

5 Auxiliary properties of the Hamiltonians

Let T be a self-adjoint operator in $L^2(\Omega)$ with form domain $\mathbf{H}^1_0(\Omega)$. A priori, its resolvent $R(\zeta)$ defined for Im $\zeta \neq 0$ is an operator in $\mathcal{B}(L^2(\Omega))$. It can be regarded as an operator in $\mathcal{B}(\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega))$. Indeed, we have embeddings

$$\mathbf{H}^1_0(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow [\mathbf{H}^1_0(\Omega)]^* = \mathbf{H}^{-1}(\Omega)$$

because $[L^2(\Omega)]^* = L^2(\Omega)$ and $\mathbf{H}^1_0(\Omega)$ is a dense subspace of $L^2(\Omega)$. By hypothesis, $\mathbf{H}^1_0(\Omega)$ is the form domain of the operator T; thus, for Im $\zeta \neq 0$ the map $T - \zeta : \mathfrak{D}(T) \to L^2(\Omega)$ extends to a continuous bijective operator from $\mathbf{H}^1_0(\Omega)$ onto $\mathbf{H}^{-1}(\Omega)$, whose inverse is an extension of $R(\zeta)$ to an element of $\mathcal{B}(\mathbf{H}^{-1}(\Omega), \mathbf{H}^1_0(\Omega))$, which we shall also denote by $R(\zeta)$.

Under Assumption 1.1 and Assumption 1.2 we now consider H and $\widetilde{H} := H - V$. It follows from above that H (resp., $\widetilde{H}$) may be regarded as a bounded operator from $\mathbf{H}^1_0(\mathcal{M})$ into $\mathbf{H}^{-1}(\mathcal{M})$ and, for $\lambda > -\inf H$ (resp., $\nu > 0$), the operator $H + \lambda$ (resp., $\widetilde{H} + \nu$)) is a bijection (resp. isomorphism) from $\mathbf{H}^1_0(\mathcal{M})$ to $\mathbf{H}^{-1}(\mathcal{M})$

The operator H (resp. $\widetilde{H}$) may be uniquely extended to an bounded operator, also designated by H, from the local Sobolev space $\mathbf{H}^1_{\mathrm{loc}}(\mathcal{M})$ into $\mathbf{H}^{-1}_{\mathrm{loc}}(\mathcal{M})$, defined by

$$H\psi := -\partial_i m^{ij} \partial_j \psi + V\psi \text{ in } \mathcal{D}'(\mathcal{M}), \quad \psi \in \mathbf{H}^1_{\mathrm{loc}}(\mathcal{M}). \tag{23}$$

Lemma 5.1. *If $\psi \in \mathbf{H}^1_{\mathrm{loc}}(\mathcal{M})$ and $\varphi \in C^1(\mathcal{M})$, then*

$$\widetilde{H}(\varphi\psi) = \varphi\widetilde{H}\psi - m^{ij}(\partial_i\varphi)(\partial_j\psi) - \partial_i(m^{ij}\psi(\partial_j\varphi)) \text{ in } \mathcal{D}'(\mathcal{M}). \tag{24}$$

and

$$H(\varphi\psi) = \varphi H\psi - m^{ij}(\partial_i\varphi)(\partial_j\psi) - \partial_i(m^{ij}\psi(\partial_j\varphi)) \text{ in } \mathcal{D}'(\mathcal{M}). \tag{25}$$

Proof. Follows from a computation in $\mathcal{D}'(\mathcal{M})$, using (23). □

Next we consider the pseudo-selfadjoint extensions (also denoted by $\widetilde{H}$ and H) of $\widetilde{H}$ and H in $L^2(\mathbb{R}^n)$ by identifying $L^2(\mathcal{M})$ with the closed subspace of $L^2(\mathbb{R}^n)$ for which its elements vanish on $\mathbb{R}^n \setminus \mathcal{M}$. Then $\mathbf{H}^1_0(\mathcal{M})$ will be a closed subspace of $\mathbf{H}^1(\mathbb{R}^n)$.

Lemma 5.2. *Let Assumption 1.1 and Assumption 1.2 hold. Then:*
1. $(\widetilde{H} + \nu)^{-1} \in \mathcal{B}(\mathbf{H}^{-1}(\mathbb{R}^n), \mathbf{H}^1(\mathbb{R}^n))$ *for any* $\nu > 0$.
2. $(H + \lambda)^{-1} \in \mathcal{B}(\mathbf{H}^{-1}(\mathbb{R}^n), \mathbf{H}^1(\mathbb{R}^n))$ *for any* $\lambda > -\inf H$.

Proof. We prove the assertions separately.
1. For any $\alpha > 0$ we set $m_\alpha^{ij} : \mathbb{R}^n \to (0,\infty]$ by $m_\alpha^{ij} = m^{ij}$ on $\mathcal{M}$ and $m_\alpha^{ij} = \alpha$ on $\mathbb{R}^n \setminus \mathcal{M}$ and we denote by $\widetilde{\mathfrak{h}}_\alpha$ the sesquilinear form, densely defined in $L^2(\mathbb{R}^n)$ by

$$\widetilde{\mathfrak{h}}_\alpha[\varphi,\psi] = \langle \partial_i, m_\alpha^{ij}\partial_j\psi\rangle, \quad \varphi,\psi \in \mathfrak{Q}(\widetilde{\mathfrak{h}}_\alpha) = \mathbf{H}^1(\mathbb{R}^n). \tag{26}$$

This form is non-negative and closed. It defines thus a non-negative self-adjoint operator $\widetilde{H}_\alpha \geq 0$ in $L^2(\mathbb{R}^n)$. If $c > 0$ and P denotes the orthogonal projection of $L^2(\mathbb{R}^n)$ onto $L^2(\mathcal{M})$ (i.e. the multiplication operator induced by the characteristic function of $\mathcal{M}$), the operator $\widetilde{H}_{\alpha,c} := \widetilde{H}_\alpha + c(1-P)$ is positive, self-adjoint on $\mathfrak{D}(\widetilde{H}_\alpha)$. By Proposition 3.4, there exists on $L^2(\mathbb{R}^n)$ a pseudo-selfadjoint operator $\overline{H}_\alpha = \lim_{c\to\infty} \widetilde{H}_{\alpha,c}$ (in the strong resolvent sense) and $(\overline{H}_\alpha + \nu)^{-1} \in \mathcal{B}(\mathbf{H}^{-1}(\mathbb{R}^n), \mathbf{H}^1(\mathbb{R}^n))$ for any $\nu > 0$.

Now, the closed, densely defined sesquilinear form in $L^2(\mathbb{R}^n)$ associated to $\widetilde{H}_{\alpha,c}$ is

$$\widetilde{\mathfrak{h}}_{\alpha,c}[\varphi,\psi] := \widetilde{\mathfrak{h}}_\alpha[\varphi,\psi] + c\langle (1-P)\varphi, (1-P)\psi\rangle, \quad \varphi,\psi \in \mathfrak{Q}(\widetilde{\mathfrak{h}}_{\alpha,c}) = \mathbf{H}^1(\mathbb{R}^n).$$

In view of Proposition 3.5, the operator $\overline{H}_\alpha$ is defined by the form

$$\begin{aligned} \overline{\mathfrak{h}}_{\alpha,\infty}[\varphi,\psi] &= \lim_{c\to\infty} \widetilde{\mathfrak{h}}_{\alpha,c}[\varphi,\psi], \quad \text{with} \\ \varphi,\psi \in \mathfrak{Q}(\overline{\mathfrak{h}}_{\alpha,\infty}) &:= \left\{ \varphi \in \bigcap_{c>0} \mathfrak{Q}(\widetilde{\mathfrak{h}}_{\alpha,c}) \;:\; \sup_{c>0} \widetilde{\mathfrak{h}}_{\alpha,c}[\varphi,\varphi] < \infty \right\} \end{aligned}$$

Since $\varphi \in \mathfrak{Q}(\overline{\mathfrak{h}}_{\alpha,\infty})$ if and only if $\varphi \in \mathbf{H}^1(\mathbb{R}^n)$ and $(1-P)\varphi = 0$, we deduce that $\mathfrak{Q}(\overline{\mathfrak{h}}_{\alpha,\infty}) = \mathfrak{Q}(\widetilde{\mathfrak{h}}) = \mathbf{H}_0^1(\mathcal{M})$ and $\overline{\mathfrak{h}}_{\alpha,\infty}[\varphi,\psi] = \widetilde{\mathfrak{h}}_\alpha[\varphi,\psi] = \widetilde{\mathfrak{h}}[\varphi,\psi]$, $\varphi,\psi \in \mathbf{H}_0^1(\mathcal{M})$. Hence $\overline{H}_\alpha = \widetilde{H}$ for every $\alpha > 0$.
2. Since $\mathbf{H}_0^1(\mathbb{R}^n) \subset \mathbf{H}_0^1(\mathcal{M})$ and $\mathbf{H}^{-1}(\mathcal{M}) \subset \mathbf{H}^{-1}(\mathbb{R}^n)$, we may continuously extend V such that $V \in \mathcal{B}(\mathbf{H}^1(\mathbb{R}^n), \mathbf{H}^{-1}(\mathbb{R}^n))$. Then the first part of the lemma and the resolvent equation $(H+\lambda)^{-1} = (1 + (\widetilde{H}+\lambda)^{-1}V)^{-1}(\widetilde{H}+\lambda)^{-1}$ yield the desired result. □

Lemma 5.3. *Suppose $f \in C^1(\mathbb{R}^n)$ and $\partial_j f \in L^\infty(\mathbb{R}^n)$ for every j. Let $\psi \in \mathbf{H}^{-1}(\mathbb{R}^n)$ and $f\psi \in \mathbf{H}^{-1}(\mathbb{R}^n)$. Then $f(\widetilde{H}+\nu)^{-1}\psi \in \mathbf{H}^1(\mathbb{R}^n)$ for any $\nu > 0$. Moreover, for each $\lambda > -\inf H$, one has $f(H+\lambda)^{-1}\psi \in \mathbf{H}^1(\mathbb{R}^n)$.*

Proof. Let $\varphi = (\widetilde{H}+\nu)^{-1}\psi \in \mathbf{H}^1(\mathbb{R}^n)$. Let $l \in C_0^\infty(\mathbb{R}^n)$ such that $l(x) = 1$ for $|x| \leq 1$ and $l(x) = 0$ for $|x| \geq 2$. Define $l_k(x) = l(x/k)$, $k \geq 1$. In particular, $\partial_j^n l_k(x) = k^{-n}(\partial_j^n l)(x/k)$ $(n \in \mathbb{N})$. Clearly, $\lim_{k\to\infty} l_k f\psi = f\psi \in \mathbf{H}^{-1}(\mathbb{R}^n)$. Moreover, $l_k f\varphi \in \mathbf{H}^1(\mathbb{R}^n)$. By bearing in mind the definition of the operator $\widetilde{H}_{\alpha,c}$ from the proof of Lemma 5.2(1), an application of Lemma 5.1 yields

$$\begin{aligned} (\widetilde{H}_{\alpha,c} + \nu)(l_k f\varphi) = l_k f\psi &- m_\alpha^{ij}(\partial_i(l_k f))\partial_j\varphi \\ &- \partial_i(m_\alpha^{ij}\varphi(\partial_j(l_k f))) + c(1-P)(l_k f\varphi). \end{aligned}$$

Let us denote the right-hand side by $g_{k,c}$. Since $(1-P)\varphi = 0$, the distribution $g_{k,c}$ does not depend on c and we may thus denote it by g_k. For each $v \in L^2(\mathbb{R}^n)$, $\lim_{k\to\infty} l_k v = v$

in $L^2(\mathbb{R}^n)$ and, in addition, $\lim_{k\to\infty}(\partial_j l_k) f\psi = 0$ in $L^2(\mathbb{R}^n)$ for every j. Thus we infer that

$$\lim_{k\to\infty} g_k = f\psi - m_\alpha^{ij}(\partial_i f)(\partial_j \psi) - \partial_i(m_\alpha^{ij}\psi(\partial_j f)) =: g$$

in $\mathbf{H}^{-1}(\mathbb{R}^n)$. This implies that $l_k f\varphi = (\widetilde{H}_{\alpha,c} + \nu)^{-1} g_k$ and in the limit $c \to \infty$, we get that $l_k f\varphi = (\widetilde{H} + \nu)^{-1} g_k$. Finally, by invoking Lemma 5.2(1), we deduce that $f\varphi = (\widetilde{H} + \nu)^{-1} g \in \mathbf{H}^1(\mathbb{R}^n)$. The last assertion follows from the first one, in combination with the resolvent formula $(H + \lambda)^{-1} = (\widetilde{H} + \lambda)^{-1}(1 + V(\widetilde{H} + \lambda)^{-1})^{-1}$. □

The Hamiltonians H_0 and H obey the following regularity properties:

Proposition 5.4. *Let Assumption 1.1 and Assumption 1.2 be satisfied. Then:*
1. $H_0 \in C^\infty(A)$.
2. $H \in C^{1+\beta}(A)$ *with* $\beta = \min_{j=1,2}\{\mu_j - 1, \nu_j - 1, 1\}$.

This result was established in [24, Proposition 8.1]. Although the proof of Proposition 5.4 follows a procedure well-known for the Laplace operator, the variable principle part of H requires a substantially more complicated analysis of commutators.

In addition, the Hamiltonians H and H_0 "look alike" at infinity so that the kinetic energy distribution is controlled by the total energy distribution in the following sense:

Proposition 5.5. *Let Assumption 1.1 and Assumption 1.2 be satisfied. Then* $f(H) - f(H_0)$ *is compact for any* $f \in C_\infty(\mathbb{R})$*; the continuous functions vanishing at infinity. In particular,* $\sigma_{\text{ess}}(H) = \sigma_{\text{ess}}(H_0) = [\upsilon_1, \infty)$.

For the proof we refer to [24, Proposition 9.1].

6 Principle of limiting absorption

The key ingredient in the proof of strong asymptotic completeness is a LAP valid in a framework of weighted Sobolev spaces $\mathbf{H}^s_{(\beta)}(\mathcal{M})$ (see Section 1). Evidently, for any $\beta \geq 0$, one has continuous embeddings $\mathbf{H}^{-1}_{(\beta)}(\mathcal{M}) \subset \mathbf{H}^{-1}(\mathcal{M})$ and $\mathbf{H}^1_0(\mathcal{M}) \subset \mathbf{H}^1_{(-\beta)}(\mathcal{M})$. This version of the LAP, suitable for the study of scattering theory, is the content of assertion 4 in the following theorem, wherein Υ denote the set of eigenvalues of the Dirichlet Laplacian $-\Delta_{D,Q}$ on on the bounded Lipschitz domain Q in $\mathbb{R}^{n-1}$, $n \geq 2$.

Theorem 6.1. *Let* $\mathcal{M}$, Q *and* Υ *be as above. Suppose that the matrix* M *satisfies Assumption 1.1(i), (ii)' and (iii) and that the potential* V *satisfies Assumption 1.2(i), (ii)' and (iii). Then the operator* H *in (1) has the following spectral properties:*
1. The essential spectrum of H *equals the semi-axis* $[\upsilon_1, \infty)$ *with* $\upsilon_1 = \inf \Upsilon$.
2. The set of eigenvalues of H *can accumulate only to the points of* Υ *and each eigenvalue away from* Υ *has finite multiplicity.*
3. The operator H *has no singular continuous spectrum.*

4. For any $\gamma > 1/2$, the holomorphic functions

$$\mathbb{C}_\pm \ni \zeta \mapsto (H-\zeta)^{-1} \in \mathcal{B}(\mathbf{H}^{-1}_{(\gamma)}(\mathcal{M}), \mathbf{H}^{1}_{(-\gamma)}(\mathcal{M})) \quad (27)$$

extends continuously to $\mathbb{C}_\pm \cup (\mathbb{R} \setminus [\sigma_{\mathrm{p}}(H) \cup \Upsilon\,])$ in the uniform topology.

We begin by proving assertions 1-3 in Theorem 6.1.

Proof (of Theorem 6.1, assertions 1-3)**.** In view of Proposition 5.4 H belongs to $C^{1+\beta}(A)$ for some $\beta \in (0,1]$. If follows from (11) that H is of class $\mathcal{C}^1(A)$. In Proposition 4.3 we proved that A is strictly conjugate to H_0 away from Υ. The latter, in combination with Proposition 5.5 and Theorem 3.15, implies that A is conjugate to H at $\mathbb{R}\setminus\Upsilon$. Then Assertion 1 follows from Proposition 5.5. Assertion 2 is a consequence of Theorem 3.13(1) and, assertion 3, the absence of singular continuous spectrum of H is a consequence of Theorem 3.13(2). □

Items 1-3 in Theorem 6.1 first appeared in [19]. To give a proof of Theorem 6.1(4), we first give the following version of the LAP, wherein the set $\mathcal{M}$ is replaced by the Euclidean space $\mathbb{R}^n$. This result is of considerable interest in itself. The proof of Theorem 6.2 imitates a familiar approach for the Laplace operator, but the variable principle part of H requires a refined version of Mourre's method within the context of pseudo-selfadjoint operators.

Theorem 6.2. *Let $\mathcal{M}$, Q and Υ be as above. Suppose that the matrix M satisfies Assumption 1.1 and that the potential V satisfies Assumption 1.2. If one regards H, in (1), as a pseudo-selfadjoint operator in $L^2(\mathbb{R}^n)$, then the holomorphic function*

$$\mathbb{C}_\pm \ni \zeta \mapsto (H-\zeta)^{-1} \in \mathcal{B}(\mathbf{H}^{-1}_{\frac{1}{2},1}(\mathbb{R}^n), \mathbf{H}^{1}_{-\frac{1}{2},\infty}(\mathbb{R}^n)) \quad (28)$$

extends continuously to $\mathbb{C}_\pm \cup (\mathbb{R} \setminus [\sigma_{\mathrm{p}}(H) \cup \Upsilon])$ in the weak $*$-topology.

In other words, if $u, v \in \mathbf{H}^{-1}_{\frac{1}{2},1}(\mathbb{R}^n)$, then the function $\zeta \mapsto \langle u, (H-\zeta)^{-1} v\rangle$ which is holomorphic in $\mathbb{C}_\pm$ has a continuous extension to $\mathbb{C}_\pm \cup (\mathbb{R} \setminus [\sigma_{\mathrm{p}}(H) \cup \Upsilon])$.

Proof. We shall regard H as a pseudo-selfadjoint operator in $L^2(\mathbb{R}^n)$. For this purpose we identify $L^2(\mathcal{M})$ with the closed subspace of $L^2(\mathbb{R}^n)$ consisting of elements which vanish on the complement of $\mathcal{M}$; that is, $\mathbf{H}^1_0(\mathcal{M})$ is identified with $\mathbf{H}^1(\mathbb{R}^n) \cap L^2(\mathcal{M})$.

We wish to apply Theorem 3.14. For this aim we introduce $\mathcal{A} = A_1 \otimes 1$, which is self-adjoint in $L^2(\mathbb{R}^n)$ and essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$; by writing A_1 instead of A we stress that it is the dilation operator with respect to x_1.

To apply Theorem 3.14, we need to verify several conditions. First we show that $H \in C^1(\mathcal{A})$ by means of Proposition 3.8. As usual, P denotes the orthogonal projection of $L^2(\mathbb{R}^n)$ onto $L^2(\mathcal{M})$. Evidently, $\mathfrak{D}(A) = P\mathfrak{D}(\mathcal{A})$ and $A = P\mathcal{A}P$ and one easily checks that

$$\mathcal{A}[\mathfrak{D}(\mathcal{A}) \cap L^2(\mathcal{M})] \subset L^2(\mathcal{M})$$

and

$$[\mathcal{A}, H] = [A, H] \text{ on } \mathfrak{D}(\mathcal{A}) \cap \mathfrak{D}(H).$$

whence the hypotheses (13) and (ii) in Proposition 3.8 are fulfilled; to verify (ii) we use (6), [24, Proposition 7.1] and Proposition 3.8.

Next we verify condition (i') of Proposition 3.8. Now, $\phi, x_1\phi \in L^2(\mathbb{R})$ implies that $\phi \in \mathfrak{D}(A_1)$ and thus $\phi, x_1\phi \in L^2(\mathbb{R}^n)$ implies that $\phi \in \mathfrak{D}(\mathcal{A})$. Fix $\lambda > -\inf H$. From Lemma 5.3 (with $f(x) = x_1$) we obtain that $(H+\lambda)^{-1}\phi, x_1(H+\lambda)^{-1}\phi \in L^2(\mathbb{R}^n)$ are valid for $\phi \in C_0^\infty(\mathbb{R}^n)$. The latter, in conjunction with the afore-mentioned fact implies that $(H+\lambda)^{-1}\phi, x_1(H+\lambda)^{-1}\phi \in \mathfrak{D}(\mathcal{A})$ and therefore the set

$$\{\, \phi \in C_0^\infty(\mathbb{R}^n) \,:\, (H+\lambda)^{-1}\phi \in \mathfrak{D}(\mathcal{A}) \,\}$$

is a core of $\mathcal{A}$. Then an application of Proposition 3.8 implies that $H \in C^1(\mathcal{A})$ and

$$[(H+\lambda)^{-1}, \mathcal{A}] = (H+\lambda)^{-1}[A, H](H+\lambda)^{-1} = [(H+\lambda)^{-1}, A] \tag{29}$$

Since $e^{i\tau A} = e^{i\tau A_1} \otimes 1$ on $L^2(\mathcal{M})$ and $e^{i\tau\mathcal{A}} = e^{i\tau A_1} \otimes 1$ on $L^2(\mathbb{R}^n)$, it follows that $e^{i\tau A} = Pe^{i\tau\mathcal{A}}P$ for all $\tau \in \mathbb{R}$. As seen from the proof of Proposition 5.4, we have that $[(H+\lambda)^{-1}, A] \in C^\beta(A)$ and, together with (29), this implies that $[(H+\lambda)^{-1}, \mathcal{A}] \in C^\beta(\mathcal{A})$. So $H \in C^{1+\beta}(\mathcal{A})$ and, in particular, H is $\mathcal{A}$-regular.

To apply Theorem 3.14 to H in $L^2(\mathbb{R}^n)$ it remains to select the spaces $\mathcal{K}$ and $\mathcal{K}_1$ and to specify $\mathcal{K}_{\frac{1}{2},1}$. We set $\mathcal{K} := \mathbf{H}^{-1}(\mathbb{R}^n)$ and $\mathcal{K}_1 := \mathbf{H}^{-1}_{(1)}(\mathbb{R}^n)$. Our discussion above shows that the hypotheses (i)-(iii) of Theorem 3.14 are satisfied (with $R(\lambda_0) = (H+\lambda)^{-1}$ and $A = \mathcal{A}$). Finally, we note that $\mathcal{K}_{\frac{1}{2},1} = \mathbf{H}^{-1}_{\frac{1}{2},1}(\mathbb{R}^n)$ and $\mathcal{K}^*_{\frac{1}{2},1} = \mathbf{H}^1_{-\frac{1}{2},\infty}(\mathbb{R}^n)$. □

Having established Theorem 6.2 we can give a rather short proof of the LAP formulated in Theorem 6.1(4).

Proof (of Theorem 6.1(4))**.** We recall the embeddings (6)-(7). In fact, for any $\beta > 1/2$ and $\epsilon > 0$ such that $1/2 + \epsilon < \beta$, the embeddings

$$\mathbf{H}^1_{(-\frac{1}{2}-\epsilon)}(\mathbb{R}^n) \subset \mathbf{H}^0_{(-\beta)}(\mathbb{R}^n) \text{ and } \mathbf{H}^0_{(\beta)}(\mathbb{R}^n) \subset \mathbf{H}^{-1}_{(\frac{1}{2}+\epsilon)}(\mathbb{R}^n)$$

are compact and the latter fact, together with (7), Theorem 6.2, and (6) (in this order), implies that

$$\mathbb{C}_\pm \ni \zeta \mapsto (H-\zeta)^{-1} \in \mathcal{B}(L^2_{(\beta)}(\mathbb{R}^n), L^2_{(-\beta)}(\mathbb{R}^n)), \quad \beta > 1/2, \tag{30}$$

can be continuously extended to $\mathbb{C}_\pm \cup (\mathbb{R} \setminus \Upsilon)$ in the uniform topology.

Fix $\lambda > -\inf H$. For $\zeta \in \mathbb{C}_\pm$, an iteration of the first resolvent formula yields

$$\begin{aligned}(H-\zeta)^{-1} = {} & (H+\lambda)^{-1} + (\zeta+\lambda)(H+\lambda)^{-2} \\ & + (\zeta+\lambda)^2(H+\lambda)^{-1}(H-\zeta)^{-1}(H+\lambda)^{-1}.\end{aligned} \tag{31}$$

Lemma 5.2(2) asserts that $(H+\lambda)^{-1} \in \mathcal{B}(\mathbf{H}^{-1}_{(0)}(\mathbb{R}^n), \mathbf{H}^{1}_{(0)}(\mathbb{R}^n))$ and an application of Lemma 5.3 implies that $(H+\lambda)^{-1} \in \mathcal{B}(\mathbf{H}^{-1}_{(\alpha)}(\mathbb{R}^n), \mathbf{H}^{1}_{(\alpha)}(\mathbb{R}^n))$ for every $\alpha \in [-1,1]$. Hence, in view of (31) we conlude that, for any $\beta > 1/2$, the holomorphic functions

$$\mathbb{C}_{\pm} \ni \zeta \mapsto (H-\zeta)^{-1} \in \mathcal{B}(\mathbf{H}^{-1}_{(\beta)}(\mathbb{R}^n), \mathbf{H}^{1}_{(-\beta)}(\mathbb{R}^n)) \tag{32}$$

can be continuously extended to $\mathbb{C}_{\pm} \cup (\mathbb{R} \setminus \Upsilon)$ in the uniform topology.

Finally we observe that the density of $C_0^\infty(\mathcal{M})$ in $\mathbf{H}^{-1}_{(\beta)}(\mathcal{M})$, together with the embeddings

$$\mathbf{H}^{1}_{(-\beta)}(\mathbb{R}^n) \subset \mathbf{H}^{1}_{(-\beta)}(\mathcal{M}), \quad \mathbf{H}^{-1}_{(\beta)}(\mathcal{M}) \subset \mathbf{H}^{-1}_{(\beta)}(\mathbb{R}^n),$$

yields the desired result. □

7 Scattering properties for the pair (H, H_0)

We proceed to scattering theory for the pair (H, H_0). The following classic result goes back to Lavine [21].

Theorem 7.1. *Let T_1 and T_2 be two self-adjoint operators in a separable Hilbert space $\mathcal{H}$ with spectral projections $E_{T_1}(\Omega)$ and $E_{T_2}(\Omega)$. Assume that there exist sets Ω_j, $j \in \mathbb{N}$, and operators E_k, F_k, $1 \le k \le N$, such that:*
(i) $\Omega = \cup_{j\in\mathbb{N}} \Omega_j$ where each Ω_j is a bounded open interval, and $\Omega_j \cap \Omega_k = \emptyset$ if $j \neq k$.
(ii) The operator E_k is T_1-bounded and locally T_1-smooth on Ω_j, for $1 \le k \le N$, and $j \ge 1$.
(iii) The operator F_k is T_2-bounded and locally T_2-smooth on Ω_j, for $1 \le k \le N$, and $j \ge 1$.
(iv) $T_2 - T_1 = \sum_{k=1}^N F_k^* E_k$ is valid in the sense of forms, i.e.

$$\langle T_2 u, v\rangle_{\mathcal{H}} - \langle u, T_1 v\rangle_{\mathcal{H}} = \sum_{k=1}^N \langle F_k u, E_k v\rangle_{\mathcal{H}}, \quad u \in \mathfrak{D}(T_2), \ v \in \mathfrak{D}(T_1).$$

(v) Both sets $\sigma(T_1) \setminus \Omega$ and $\sigma(T_2) \setminus \Omega$ have Lebesgue measure zero.
Then the generalized wave operators

$$\begin{aligned} W^{\pm} &= s - \lim_{t\to\pm\infty} e^{itT_2} e^{-itT_1} P_{\mathrm{ac}}(T_1) \\ \tilde{W}^{\pm} &= s - \lim_{t\to\pm\infty} e^{itT_1} e^{-itT_2} P_{\mathrm{ac}}(T_2) \end{aligned}$$

exist and are complete.

For our purpose, we choose $T_1 = H_0$ and $T_2 = H$ and $\Omega = \mathbb{R} \setminus \Upsilon$. In addition, we set $v^{ij} = m^{ij} - \delta^{ij}$. Then, for $u \in \mathfrak{D}(H)$ and $v \in \mathfrak{D}(H_0)$ we have that

$$\begin{aligned} \langle Hu, v\rangle_{L^2(\mathcal{M})} - \langle u, H_0 v\rangle_{L^2(\mathcal{M})} &= \mathfrak{h}[u,v] - \overline{\mathfrak{h}}_0[v,u] \\ &= \int_{\mathcal{M}} (\partial_i u)(x) v^{ij} \overline{(\partial_j v)(x)}\, dx \\ &\quad + \int_{\mathcal{M}} |V(x)|^{\frac{1}{2}} u(x) |V(x)|^{\frac{1}{2}} \operatorname{sign} V(x) \overline{v(x)}\, dx. \end{aligned} \tag{33}$$

where, on the right-hand side, we suppress the summations over i, j. Hence, by introducing the operators $E_{ij}, F_{ij} : \mathbf{H}^1_{(-\gamma)}(\mathcal{M}) \to L^2(\mathcal{M})$, $\gamma = \max\{\mu, \nu\}$, $1 \le i, j \le n+1$, $\mu := \max_{j=1,2}\ \mu_j/2$ (see Assumption 1.1 for the decay parameters μ_j), $\nu = \max_{j=1,2}\ \nu_j/4$ (see Assumption 1.2 for the decay parameters ν_j), defined by

$$\begin{aligned} E_{ij}u &:= \langle\cdot\rangle^{\mu} v^{ij}\partial_j u, \quad F_{ij}u := -\langle\cdot\rangle^{-\mu}\partial_i u \quad 1 \le i, j \le n, \\ E_{n+1}u &:= \langle\cdot\rangle^{\nu}|V|^{\frac{1}{2}}, \quad F_{n+1} = \langle\cdot\rangle^{-\nu}|V|^{\frac{1}{2}}\operatorname{sign} V, \end{aligned} \tag{34}$$

we have, in view of (33), that

$$H - H_0 = \sum_{1\le i,j\le n} F^*_{ij}E_{ij} + F^*_{n+1}E_{n+1} \tag{35}$$

holds in the sense of forms. In addition, we need the following auxiliary result.

Lemma 7.2. *Let $\gamma > 1/2$ and let $g \in L^\infty(\mathcal{M})$ be a function satisfying $\langle\cdot\rangle^\gamma g \in L^\infty(\mathcal{M})$. Then*
1. The operator $G : \mathbf{H}^1_{(-\gamma)}(\mathcal{M}) \to L^2(\mathcal{M})$, $Gu := g\partial^\alpha u$ (where α is a multi-index with order $|\alpha| \le 1$) is bounded.
2. The unbounded operator defined by G in $L^2(\mathcal{M})$, also denoted by G, is H-bounded.
3. The operator G is locally H-smooth on $\mathbb{R} \setminus \Upsilon$.

Proof. Evidently, the hypotheses on g and γ ensure that the first statement holds. The second statement follows from the inclusions $\mathfrak{D}(H) \subset \mathbf{H}^1_0(\mathcal{M}) \subset \mathbf{H}^1_{(-\gamma)}(\mathcal{M})$. To prove the third assertion we need to verify that for any compact set $K \subset \mathbb{R} \setminus \Upsilon$, the operator $GE_H(K)$ is A-smooth. A sufficient requirement for this is that (see, e.g., [29, Theorem XIII.30])

$$\sup_{\lambda\in K, 0<\epsilon<1} \|G(H - \lambda - i\epsilon)^{-1}G^*\|_{\mathcal{B}(L^2(\mathcal{M}))} < \infty. \tag{36}$$

From Theorem 6.1 we infer that

$$\sup_{\lambda\in K, 0<\epsilon<1} \|(H - \lambda - i\epsilon)^{-1}\|_{\mathcal{B}(\mathbf{H}^{-1}_{(\gamma)}(\mathcal{M}), \mathbf{H}^1_{(-\gamma)}(\mathcal{M}))} < \infty$$

and therefore (36) is fulfilled. This proves the third statement. □

Finally, the proof of Theorem 1.3 amounts to combining the results above:

Proof (of Theorem 1.3)**.** It follows immediately from the discussion above, Theorem 6.1, Lemma 7.2 and Theorem 7.1. □

References

[1] W. O. Amrein, A. Boutet de Monvel and V. Georgescu: *C_0-groups, commutator methods and spectral theory of N-body Hamiltonians,* Progress in Math. Ser., Vol. 135, Birkhäuser, 1996.

[2] M. Ben-Artzi and A. Devinatz: "The limiting absorption principle for partial differential operators", *Mem. Amer. Math. Soc.*, Vol. 66(364), (1987), pp. iv+70.

[3] M. Ben-Artzi, Y. Dermenjian and J.-C. Guillot: "Acoustic waves in perturbed stratified fluids: a spectral theory", *Comm. Partial Differential Equations*, Vol. 14, (1989), pp. 479–517.

[4] J. Bergh and J. Löfström: *Interpolation spaces. An introduction,* Springer-Verlag, Berlin-New York, 1976.

[5] A. Boutet de Monvel-Berthier and V. Georgescu: "Some developments and applications of the abstract Mourre theory", *Méthodes semi-classiques*, Vol. 2, Nantes, 1991; *Astérisque* Vol. 210, (1992), pp. 27–48.

[6] A. Boutet de Monvel-Berthier and V. Georgescu: "Graded C^*-algebras and many-body perturbation theory: II The Mourre estimate", *Astérisque*, Vol. 210, (1992), pp. 75–96.

[7] A. Boutet de Monvel-Berthier, V. Georgescu and A. Soffer: "N-body Hamiltonians with hard-core interactions", *Rev. Math. Phys.* Vol. 6, (1994), pp. 515–596.

[8] A. Boutet de Monvel-Berthier and D. Manda: "Spectral and scattering theory for wave propagation in perturbed stratified media", *J. Math. Anal. Appl.*, Vol. 91, (1995), pp. 137–167.

[9] A. Boutet de Monvel and R. Purice: "The conjugate operator method: application to Dirac operators and to stratified media", In: *Evolution equations, Feshbach resonances, singular Hodge theory*, Math. Top., Vol. 16, Wiley-VCH, Berlin, 1999, 243–286.

[10] J. Derezinski and C. Gérard: *Scattering theory of classical and quantum N-particle systems*, Springer-Verlag, Berlin, 1997.

[11] Y. Dermenjian, M. Durand and V. Iftimie: "Spectral analysis of an acoustic multistratified perturbed cylinder". *Comm. Partial Differential Equations*, Vol. 23(1-2), (1998), pp. 141–169.

[12] D.E. Edmunds and W.D. Evans: *Spectral theory and differential operators*, Oxford University Press, New York, 1987.

[13] D.M. Eidus: "The principle of limiting amplitude", *Uspehi Mat. Nauk*, Vol. 24(3), (1969), pp. 91–156.

[14] R. Froese and I. Herbst: "Exponential bounds and absence of positive eigenvalues for N-body Schrödinger operators", *Comm. Math. Phys.*, Vol. 87, (1982/83), pp. 429–447.

[15] C.I. Goldstein: "Eigenfunction expansions associated with the Laplacian for certain domains with infinite boundaries. I.", *Trans. Amer. Math. Soc.*, Vol. 135, (1969), pp. 1–31.

[16] E. Hille and R.S. Phillips: *Functional analysis and semi-groups* (Third printing of the revised edition of 1957), American Mathematical Society, Providence, R. I., 1974.

[17] H. Iwashita: "Spectral theory for symmetric systems in an exterior domain", *Tsukuba J. Math.*, Vol. 11, (1987), pp. 241–256.

[18] K. A. Kiers and W. van Dijk: "Scattering in one dimension: the coupled Schrödinger equation, threshold behaviour and Levinson's theorem", *J. Math. Phys.*, Vol. 37, (1996), pp. 6033–6059.

[19] D. Krejcirik and R.T. de Aldecoa: "The nature of the essential spectrum in curved quantum waveguides", *J. Phys. A*, Vol. 37, (2004), pp. 5449–5466.

[20] I. Laba: "Long-range one-particle scattering in a homogeneous magnetic field", *Duke Math. J.*, Vol. 70(2), (1993), pp. 283–303.

[21] R.B. Lavine: "Commutators and scattering theory. II. A class of one body problems", *Indiana Univ. Math. J.*, Vol. 21, (1971/72), pp. 643–656.

[22] W.C. Lyford: "Spectral analysis of the Laplacian in domains with cylinders", *Math Ann.*, Vol. 218, (1975), pp. 229–251.

[23] M. Melgaard: "Spectral properties at a threshold for two-channel Hamiltonians. II. Applications to scattering theory", *J. Math. Anal. Appl.*, Vol. 256, (2001), pp. 568–586.

[24] M. Melgaard: "Optimal limiting absorption principle for a Schrödinger type operator on a Lipschitz cylinder", *Manus. Math.*, Vol. 118, (2005), pp. 253–270.

[25] E. Mourre: "Absence of singular continuous spectrum for certain self-adjoint operators", *Comm. Math. Phys.*, Vol. 78, (1980/81), pp. 391–408.

[26] P. Perry, I.M. Sigal and B. Simon: "Spectral analysis of N-body Schrödinger operators", *Ann. of Math.*, Vol. 114(2), (1981), pp. 519–567.

[27] M. Reed and B. Simon: *Methods of modern mathematical physics, I. Functional analysis*, Academic Press, New York, 1980.

[28] M. Reed and B. Simon: *Methods of modern mathematical physics, II. Fourier analysis, self-adjointness*, Academic Press, New York, 1975.

[29] M. Reed and B. Simon: *Methods of modern mathematical physics, III. Scattering theory,* Academic Press, New York, 1979.

[30] B. Simon: "A canonical decomposition for quadratic forms with applications to monotone convergence theorems", *J. Funct. Anal.*, Vol. 28, (1978), pp. 377–385.

[31] H. Tamura: "Principle of limiting absorption for N-body Schrödinger operators – a remark on the commutator method", *Lett. Math. Phys.*, Vol. 17, (1989), pp. 31–36.

[32] H. Tamura: "Resolvent estimates at low frequencies and limiting amplitude principle for acoustic propagators", *J. Math. Soc. Japan*, Vol. 41, (1989), pp. 549–575.

[33] R. Weder: "Spectral analysis of strongly propagative systems", *J. Reine Angew. Math*, Vol. 354, (1984), pp. 95–122.