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Integrable three-dimensional coupled nonlinear dynamical systems related to centrally extended operator Lie algebras and their Lax type three-linearization

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Abstract: The Hamiltonian representation for a hierarchy of Lax type equations on a dual space to the Lie algebra of integro-differential operators with matrix coefficients, extended by evolutions for eigenfunctions and adjoint eigenfunctions of the corresponding spectral problems, is obtained via some special Bäcklund transformation. The connection of this hierarchy with integrable by Lax two-dimensional Davey-Stewartson type systems is studied.

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Since the paper of M. Adler [1], which had been treated on a one-dimensional differential operator algebra, it was understood that a wide class of Lax integrable Korteweg de Vries type nonlinear dynamical systems in partial derivatives [3, 6, 16, 33] could be described by means of Lie-algebraic techniques. It was shown that all of them can be represented as coadjoint orbits of Lie groups. The analog of the above construction for a class of matrix affine groups with central extensions was presented in [12, 16, 17, 25, 26], where its relationship with the momentum mapping, $\mathcal{R}$-matrix approach and versal deformations

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of differential operators had been stated. But, the extension problem for the Adler's construction in the case of a multi-dimensional differential operator algebra still stands open. Some preliminary results in this direction were obtained by L. Nizhnik [34] and recently by A. Samoilenko, Y. Prykarpatsky, J. Golenia and A. Prykarpatsky [18, 19, 21, 28].

In this article we suggest a new approach to the partial solving of this problem based on the notions of a Bäcklund transformation [14, 16] and a tensor product of Poisson structures on a dual space of a one-differential operator algebra [3, 16, 20]. Making use of the invariant Casimir functionals' property under the Bäcklund transformations. We construct a wide class of Lax integrable (2+1)-dimensional dynamical systems and for the first time represent them as a compatibility condition of three special linear first order differential equations, called here a triple linear Lax type representation.

As is well known, Lax type representations [6] for integrable (1+1)-dimensional nonlinear dynamical system hierarchies [3, 11, 16] on functional manifolds were first interpreted as Hamiltonian flows on the dual space to the Lie algebra of integro-differential operators in [1]. A Lie-algebraic method for constructing Lax type integrable (2+1)-dimensional nonlinear dynamical systems by means of two commuting flows from the hierarchy on the suitable coadjoint action orbit of a pseudo-differential operator with an infinite integral part was proposed in [4, 16, 30]. The connection of some Lax integrable (2+1)-dimensional systems with related hierarchies of Hamiltonian flows on the adjoint spaces to centrally extended, by means of the standard Maurer-Cartan two-cocycle pseudo-differential operator Lie algebras, was also [18, 25, 26, 32] intensively treated.

Since every Hamiltonian flow of such a type on the dual space either to an operator Lie algebra or to its central extension can be written as a compatibility condition for the corresponding isospectral problem for their eigenfunctions and their suitable evolutions, an important problem of finding the Hamiltonian representation of the Lax type hierarchy coupled with the evolutions of a finite set of eigenfunctions and their appropriate adjoints naturally arises. It was partially solved in [14, 15, 20, 27] and further developed in the [5] for the Lie algebra of integro-differential operators and its super-generalization by means of the variational Casimir functionals property under a Lie-Bäcklund transformation.

In Section 2 the general Lie-algebraic scheme of constructing Lax type integrable dynamical systems is described.

Sections 3 and 4 are devoted to Bäcklund transformations of related tensor products of Poisson structures, based on the Casimir invariance [15, 20, 27] property, and their application to constructing the Lax type integrable Davey-Stewartson equation and its triple linear representation.

Section 5 deals with a general Lie-algebraic scheme for constructing a hierarchy of Lax type integrable flows as Hamiltonian ones on the dual spaces to the centrally extended Lie algebra of integro-differential operators with matrix-valued coefficients.

In Section 6 the corresponding Hamiltonian structure for the related coupled Lax type hierarchy is obtained by means of the Bäcklund transformation technique developed in [5, 20, 27].

In Section 7 the corresponding hierarchies of additional or so called "ghost" symmetries [5, 10] for the coupled Lax type flows are also shown to be Hamiltonian. It is established that an additional hierarchy of Hamiltonian flows is generated by the Poisson structure, equal to the tensor product of the $\mathcal{R}$-deformed canonical Lie-Poisson bracket [12, 20, 27, 31] with the standard canonical Poisson bracket on a related eigenfunction space [15, 20, 27], and the corresponding integer powers of suitable eigenvalues are their Hamiltonian functions. The method for introducing one more variable into $(2+1)$-dimensional integrable dynamical systems preserving their Lax type integrability, based on the above additional symmetries, is proposed and an integrable $(3|1+1)$-dimensional analog of the Davey-Stewartson system [29, 33] is constructed.

1 The general Lie-algebraic scheme

Let $\tilde{\mathcal{G}} := C^{\infty}(\mathbb{S}^1;\mathcal{G})$ be a Lie algebra of loops, taking values in a matrix Lie algebra $\mathcal{G}$. By means of $\tilde{\mathcal{G}}$ we construct a Lie algebra $\hat{\mathcal{G}}$ for matrix integro-differential operators [1, 18]:

$$\hat{a} := \sum_{j \ll \infty} a_j \xi^j,$$

where the symbol $\xi := \partial/\partial x$ signs the differentiation with respect to the independent variable $x \in \mathbb{R}/2\pi\mathbb{Z} \simeq \mathbb{S}$. The usual Lie commutator on $\hat{\mathcal{G}}$ is defined as:

$$[\hat{a},\hat{b}] := \hat{a}\circ\hat{b} - \hat{b}\circ\hat{a}$$

for all $\hat{a},\hat{b} \in \hat{\mathcal{G}}$, where "$\circ$" is a product of integro-differential operators, taking the form:

$$\hat{a}\circ\hat{b} := \sum_{\alpha\in Z_+} \frac{1}{\alpha!}\frac{\partial^{\alpha}\hat{a}}{\partial\xi^{\alpha}}\frac{\partial^{\alpha}\hat{b}}{\partial x^{\alpha}}.$$

On the Lie algebra $\hat{\mathcal{G}}$ there exists the *ad*-invariant nondegerate symmetric bilinear form:

$$(\hat{a},\hat{b}) := \int_0^{2\pi} Tr\,(\hat{a}\circ\hat{b})\;dx, \tag{1}$$

where Tr is the operator for all $\hat{a}\in\hat{\mathcal{G}}$ given by the expression:

$$Tr\,\hat{a} := res_{\xi}\,tr\,\hat{a} = tr\,a_{-1},$$

with res_{ξ} denoting the standard residue and tr is the matrix trace. The scalar product (1) renders the Lie algebra $\hat{\mathcal{G}}$ metrizable. As a consequence, its dual linear space of matrix integro-differential operators $\hat{\mathcal{G}}^*$ is identified with the Lie algebra, that is $\hat{\mathcal{G}}^* \simeq \hat{\mathcal{G}}$.

The linear subspaces $\hat{\mathcal{G}}_+ \subset \hat{\mathcal{G}}$ and $\hat{\mathcal{G}}_- \subset \hat{\mathcal{G}}$ defined as

$$\hat{\mathcal{G}}_+ := \left\{ \hat{a} := \sum_{j=0}^{n(\hat{a})\ll\infty} a_j\xi^j :\ a_j\in\tilde{\mathcal{G}}, j=\overline{0,n(\hat{a})} \right\}, \tag{2}$$

$$\hat{\mathcal{G}}_- := \left\{ \hat{b} := \sum_{j=0}^{\infty} \xi^{-(j+1)} b_j :\ b_j\in\tilde{\mathcal{G}}, j\in Z_+ \right\},$$

are Lie subalgebras in $\hat{\mathcal{G}}$ and $\hat{\mathcal{G}} = \hat{\mathcal{G}}_+ \oplus \hat{\mathcal{G}}_-$. Because of the splitting of $\hat{\mathcal{G}}$ into the direct sum of its Lie subalgebras (2) one can construct a so called Lie-Poisson structure [1, 3, 12, 16, 25] on $\hat{\mathcal{G}}^*$, using the special linear endomorphism $\mathcal{R}$ of $\hat{\mathcal{G}}$:

$$\mathcal{R} := (P_+ - P_-)/2, \quad P_\pm \hat{\mathcal{G}} := \hat{\mathcal{G}}_\pm, \quad P_\pm \hat{\mathcal{G}}_\mp = 0.$$

For any smooth by Fréchet functionals $\gamma, \mu \in \mathcal{D}(\hat{\mathcal{G}}^*)$ the Lie-Poisson bracket on $\hat{\mathcal{G}}^*$ is given by the expression:

$$\{\gamma, \mu\}_{\mathcal{R}}(\hat{l}) = \left(\hat{l}, [\nabla\gamma(\hat{l}), \nabla\mu(\hat{l})]_{\mathcal{R}}\right), \tag{3}$$

where $\hat{l} \in \hat{\mathcal{G}}^*$ and for all $\hat{a},\ \hat{b} \in \hat{\mathcal{G}}$ the $\mathcal{R}$-commutator has the form [12, 26]:

$$[\hat{a}, \hat{b}]_{\mathcal{R}} := [\mathcal{R}\hat{a}, \hat{b}] + [\hat{a}, \mathcal{R}\hat{b}],$$

subject to which the linear space $\hat{\mathcal{G}}$ becames a Lie algebra too. The gradient $\nabla\gamma(\hat{l}) \in \hat{\mathcal{G}}$ of a functional $\gamma \in \mathcal{D}(\hat{\mathcal{G}}^*)$ at a point $\hat{l} \in \hat{\mathcal{G}}^*$ with respect to the scalar product (1) is defined as

$$\delta\gamma(\hat{l}) := \left(\nabla\gamma(\hat{l}), \delta\hat{l}\right),$$

where the linear space isomorphism $\hat{\mathcal{G}} \simeq \hat{\mathcal{G}}^*$ is taken into account.

The Lie-Poisson bracket (3) generates Hamiltonian dynamical systems on $\hat{\mathcal{G}}$ with Casimir invariants $\gamma \in I(\mathcal{G}^*)$, satisfying the condition:

$$[\nabla\gamma(\hat{l}), \hat{l}] = 0, \tag{4}$$

as the corresponding Hamiltonian functions. Owing to the expressions (3) and (4) the mentioned above Hamiltonian system takes the form:

$$d\hat{l}/dt := [\mathcal{R}\nabla\gamma(\hat{l}), \hat{l}] = [\nabla\gamma_+(\hat{l}), \hat{l}], \tag{5}$$

being equivalent to the usual commutator Lax type representation [3, 6, 16, 33]. The relationship (5) is a compatibility condition for the linear integro-differential equations:

$$\begin{aligned} \hat{l}f &= \lambda f, \\ df/dt &= \nabla\gamma_+(\hat{l})f, \end{aligned} \tag{6}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter and the vector-function $f \in W(\mathbb{S}^1; H)$ is an element of some matrix representation for the Lie algebra $\hat{\mathcal{G}}$ in some functional Hilbert space H.

Algebraic properties of the equation (5) together with (6) and the associated dynamical system on the space of adjoint functions $f^* \in W^*(\mathbb{S}^1; H)$:

$$df^*/dt = -(\nabla\gamma(\hat{l}))^*_+ f^*, \tag{7}$$

where $f^* \in W^*$ is a solution to the adjoint spectral problem:

$$\hat{l}^* f^* = \nu f^*,$$

being considered as some coupled evolution equations on the space $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ is an object of our further investigations.

2 The tensor product of Poisson structures and its Bäcklund transformation

To simplify the description below we will use the designation of the gradient vector

$$\nabla\gamma(\tilde{l},\tilde{f},\tilde{f}^*) := (\delta\gamma/\delta\tilde{l},\ \delta\gamma/\delta\tilde{f},\ \delta\gamma/\delta\tilde{f}^*)^{\intercal}$$

for any smooth functional $\gamma \in \mathcal{D}(\hat{\mathcal{G}}^* \oplus W \oplus W^*)$. On the spaces $\hat{\mathcal{G}}^*$ and $W \oplus W^*$ there exists canonical Poisson structures [3, 14, 27]

$$\delta\gamma/\delta\tilde{l} :\xrightarrow{\tilde{\theta}} [(\delta\gamma/\delta\tilde{l})_+,\tilde{l}] - [\delta\gamma/\delta\tilde{l},\tilde{l}]_+ \tag{8}$$

at a point $\tilde{l} \in \hat{\mathcal{G}}^*$ and

$$(\delta\gamma/\delta\tilde{f},\ \delta\gamma/\delta\tilde{f}^*)^{\intercal} :\xrightarrow{\tilde{J}} (\delta\gamma/\delta\tilde{f}^*,\ -\delta\gamma/\delta\tilde{f})^{\intercal} \tag{9}$$

at a point $(\tilde{f},\tilde{f}^*) \in W \oplus W^*$ correspondingly. It should be noted that the Poisson structure (8)) is transformed into (5) for any Casimir functional $\gamma \in I(\hat{\mathcal{G}}^*)$. Thus, on the augmented space $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ one can obtain a Poisson structure as the tensor product $\tilde{\Theta} := \tilde{\theta} \otimes \tilde{J}$ of the structures (8) and (9) .

Let us consider the following Bäcklund transformation [14, 16]:

$$(\hat{l}, f, f^*) :\xrightarrow{B} (\tilde{l}(\hat{l}, f, f^*), \tilde{f} = f, \tilde{f}^* = f^*), \tag{10}$$

generating on $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ a Poisson structure Θ with respect to variables $(\hat{l}, f, f^*)$ of the coupled evolution equations (5)- (7).

The main condition for the mapping (10) to be defined is the coincidence of the dynamical system

$$(d\hat{l}/dt,\ df/dt,\ df^*/dt)^{\intercal} := -\Theta\nabla\gamma(\hat{l}, f, f^*) \tag{11}$$

with (5)- (7) in the case of $\gamma \in I(\hat{\mathcal{G}}^*)$, i.e. when the functional γ is taken to be not dependent of variables $(f, f^*) \in W \oplus W^*$. To satisfy that condition, one should find a variation of some Casimir functional $\gamma \in I(\hat{\mathcal{G}}^*)$ at $\delta\tilde{l} = 0$, taking into account flows (6), (7) and the Bäcklund transformation (10):

$$\begin{aligned} \delta\gamma(\tilde{l},\tilde{f},\tilde{f}^*)\Big|_{\delta\tilde{l}=0} &= (<\delta\gamma/\delta\tilde{f},\delta\tilde{f}>) + (<\delta\gamma/\delta\tilde{f}^*,\delta\tilde{f}^*>) = \\ &(<-d\tilde{f}^*/dt,\delta\tilde{f}>) + (<d\tilde{f}/dt,\delta\tilde{f}^*>)\Big|_{\tilde{f}=f,\,\tilde{f}^*=f^*} = \\ &(<(\delta\gamma/\delta\hat{l})_+^* f^*,\delta f>) + (<(\delta\gamma/\delta\hat{l})_+ f,\delta f^*>) = \\ &(<f^*,(\delta\gamma/\delta\hat{l})_+\delta f>) + (<(\delta\gamma/\delta\hat{l})_+ f,\delta f^*>) = \\ &(\delta\gamma/\delta\hat{l},\delta f\xi^{-1}\otimes f^*) + (\delta\gamma/\delta\hat{l}, f\xi^{-1}\otimes\delta f^*) = \\ &(\delta\gamma/\delta\hat{l},\delta(f\xi^{-1}\otimes f^*)) := (\delta\gamma/\delta\hat{l},\delta\hat{l}), \end{aligned} \tag{12}$$

where $\gamma \in I(\hat{\mathcal{G}}^*)$. As a result of the expression (12)) one obtains the relationships:

$$\delta\hat{l}\Big|_{\delta\tilde{l}=0} = \delta(f\xi^{-1}\otimes f^*),$$

or having assumed the linear dependence of $\hat{l}$ and $\tilde{l} \in \hat{\mathcal{G}}^*$ one gets right away that

$$\hat{l} = \tilde{l} + f\xi^{-1} \otimes f^*. \tag{13}$$

Thus, the Bäcklund transformation (10) can be now written as

$$(\hat{l}, f, f^*) :\stackrel{B}{\rightarrow} (\tilde{l} = \hat{l} - f\xi^{-1} \otimes f^*, f, f^*). \tag{14}$$

The expression (14) generalizes the result, obtained in the papers [20, 27, 35] for the Lie algebra $\hat{\mathcal{G}}$ of integro-differential operators with scalar coefficients. The existence of the Bäcklund transformation (10) makes it possible to formulate the following theorem.

Theorem 2.1. *The dynamical system on $\hat{\mathcal{G}}^* \oplus W \oplus W^*$, being the Hamiltonian with respect to the canonical Poisson structure $\tilde{\Theta} : T^*(\hat{\mathcal{G}}^* \oplus W \oplus W^*) \to T(\hat{\mathcal{G}}^* \oplus W \oplus W^*)$ and generated by the evolution equations:*

$$d\tilde{l}/dt = [\nabla\gamma_+(\tilde{l}), \tilde{l}] - [\nabla\gamma(\tilde{l}), \tilde{l}]_+, \quad d\tilde{f}/dt = \delta\gamma/\delta\tilde{f}^*, \quad d\tilde{f}^*/dt = -\delta\gamma/\delta\tilde{f},$$

where $\gamma \in I(\mathcal{G}^*)$ is a Casimir functional at $\hat{l} \in \hat{\mathcal{G}}^*$, connected with $\tilde{l} \in \hat{\mathcal{G}}^*$ by (11), is equivalent to the system (5), (6) and (7) via the constructed above Bäcklund transformation (14).

By means of simple calculations via the formula (see [16, 27])

$$\tilde{\Theta} = B'\Theta B'^*,$$

where $B' : T(\hat{\mathcal{G}}^* \oplus W \oplus W^*) \to T(\hat{\mathcal{G}}^* \oplus W \oplus W^*)$ is the Fréchet derivative of (14), one brings about the following form of the Poisson structure Θ on an element $(\hat{l}, f, f^*) \in \mathcal{G}^* \oplus W \oplus W^*$

$$\nabla\gamma(\hat{l}, f, f^*) :\stackrel{\Theta}{\rightarrow} \begin{pmatrix} \left[\hat{l}, (\delta\gamma/\delta\hat{l})_+\right] - \left[\hat{l}, \delta\gamma/\delta\hat{l}\right]_+ - \\ -f\xi^{-1} \otimes \delta\gamma/\delta f + \delta\gamma/\delta f^* \xi^{-1} \otimes f^* \;\; \delta\gamma/\delta f^* - (\delta\gamma/\delta\hat{l})_+ f \;\; -\delta\gamma/\delta f + (\delta\gamma/\delta\hat{l})_+^* f \end{pmatrix} \tag{15}$$

that makes it possible to formulate the theorem.

Theorem 2.2. *The dynamical system (11), being Hamiltonian with respect to the Poisson structure Θ in the form (15) and a function $\gamma \in I(\hat{\mathcal{G}}^*)$, gives the inherited Hamiltonian representation for the coupled evolution equations (5)- (7).*

By means of the expression (13) one can construct Hamiltonian evolution equations, describing commutative flows on the augmented space $\hat{\mathcal{G}}^* \oplus W \oplus W^*$ at a fixed element $\tilde{l} \in \hat{\mathcal{G}}^*$. Owing to (15) such an equation is equivalent to the system

$$\begin{cases} d\hat{l}/d\tau_n = [\hat{l}^n_+, \hat{l}], \\ df/d\tau_n = \hat{l}^n_+ f, \\ df^*/d\tau_n = -(\hat{l}^*)^n_+ f^*, \end{cases} \tag{16}$$

generated by involutive with respect to the Poisson bracket (8) Casimir invariants $\gamma_n \in I(\hat{\mathcal{G}}^*)$, $n \in \mathbb{N}$, taking the standard form:

$$\gamma_n = 1/(n+1)(\hat{l}^n, \hat{l})$$

at $\hat{l} \in \hat{\mathcal{G}}^*$.

The compatibility conditions of the Hamiltonian systems (16) for different $n \in \mathbb{Z}_+$ can be used for obtaining Lax integrable equations on the usual spaces of smooth 2π-periodic multivariable functions that will be done in the next section.

3 The Lax type integrable Davey-Stewartson equation and its triple linear representation

Choose the element $\tilde{l} \in \hat{\mathcal{G}}^*$ in an exact form such as

$$\tilde{l} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xi - \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix},$$

where $u, \bar{u} \in \mathcal{S}(\mathbb{S}^1;\mathbb{C})$ and $\mathcal{G} = gl(2;\mathbb{C})$. Then

$$\hat{l} = \tilde{l} + \begin{pmatrix} f_1\xi^{-1}f_1^* & f_1\xi^{-1}f_2^* + u \\ f_2\xi^{-1}f_1^* + \bar{u} & f_2\xi^{-1}f_2^* \end{pmatrix}, \tag{17}$$

where $f = (f_1, f_2)^\intercal$, $f^* = (f_1^*, f_2^*)^\intercal$ and "$-$" can mean the complex or related with it conjugation. Below we will study the evolutions (16) of vector-functions $(f, f^*) \in W(\mathbb{S}^1;\mathbb{C}^2) \oplus W^*(\mathbb{S}^1;\mathbb{C}^2)$ with respect to the variables $y = \tau_1$ and $t = \tau_2$ at the point (17). They can be obtained from the second and third equations in (16), letting $n = 1$ and $n = 2$, as well as from the first one. The latter is the compatibility condition of the spectral problem

$$\hat{l}\Phi = \lambda\Phi, \tag{18}$$

where $\Phi = (\Phi_1, \Phi_2)^\intercal \in W(\mathbb{S}^1;\mathbb{C}^2)$, $\lambda \in \mathbb{C}$ is some complex parameter, with the following linear equations:

$$d\Phi/dy = \hat{l}_+\Phi, \tag{19}$$
$$d\Phi/dt = \hat{l}_+^2\Phi, \tag{20}$$

arising from (16) at $n = 1$ and $n = 2$ correspondingly. The compatibility of equations (18) and (19) lead to the relationships:

$$\begin{aligned} &\partial u/\partial y = -2f_1f_2^*, \quad \partial\bar{u}/\partial y = -2f_1^*f_2, \\ &\partial f_1/\partial y = \partial f_1/\partial x - uf_2, \quad \partial f_1^*/\partial y = \partial f_1^*/\partial x - \bar{u}f_2^*, \\ &\partial f_2/\partial y = -\partial f_2/\partial x + \bar{u}f_1, \quad \partial f_2^*/\partial y = -\partial f_2^*/\partial x + uf_1^*. \end{aligned} \tag{21}$$

Analogously, replacing $t \in \mathbb{R}$ by $it \in i\mathbb{R}$, $i^2 = -1$, one gets from (18) and (20):

$$\begin{aligned} du/dt &= i(\partial^2 u/\partial x \partial y + 2u(f_1 f_1^* + f_2 f_2^*)), \\ d\bar{u}/dt &= -i(\partial^2 \bar{u}/\partial x \partial y + 2\bar{u}(f_1 f_1^* + f_2 f_2^*)), \\ &\partial(f_1 f_1^*)/\partial y - \partial(f_1 f_1^*)/\partial x = 1/2\partial(u\bar{u})/\partial y \\ &= -(\partial(f_2 f_2^*)/\partial x + \partial(f_2 f_2^*)/\partial y) \end{aligned} \tag{22}$$

and

$$\begin{aligned} df_1/dt &= i(\partial^2 f_1/\partial x^2 + (2f_1 f_1^* - u\bar{u})f_1 - \partial u/\partial x f_2), \\ df_1^*/dt &= -i(\partial^2 f_1^*/\partial x^2 + (2f_1 f_1^* - u\bar{u})f_1^* - \partial \bar{u}/\partial x f_2^*), \\ df_2/dt &= i(\partial^2 f_2/\partial x^2 - (2f_2 f_2^* + u\bar{u})f_2 - \partial \bar{u}/\partial x f_1), \\ df_2^*/dt &= -i(\partial^2 f_2^*/\partial x^2 - (2f_2 f_2^* + u\bar{u})f_2^* - \partial u/\partial x f_1^*). \end{aligned} \tag{23}$$

The relationships (22), (23) take the well known form of the Davey-Stewartson equation [3, 11, 36] at $\bar{u} \in \mathcal{S}(\mathbb{S}^1;\mathbb{C})$ being the complex conjugate to $u \in \mathcal{S}(\mathbb{S}^1;\mathbb{C})$. The compatibility for every pair of equations (18), (19) and (20), can be rewritten as the first order linear ordinary differential equations as follows:

$$d\Phi/dx = \begin{pmatrix} \lambda & u & -f_1 \\ \bar{u} & -\lambda & f_2 \\ f_1^* & f_2^* & 0 \end{pmatrix} \Phi, \tag{24}$$

$$d\Phi/dy = \begin{pmatrix} \lambda & 0 & -f_1 \\ 0 & \lambda & -f_2 \\ f_1^* & f_2^* & 0 \end{pmatrix} \Phi, \tag{25}$$

$$d\Phi/dt = i \begin{pmatrix} \lambda^2 + f_1 f_1^* & 1/2\partial u/\partial y & -\lambda f_1 - \partial f_1/\partial y \\ -1/2\partial \bar{u}/\partial y & -\lambda^2 - f_2 f_2^* & -\lambda f_2 - \partial f_1/\partial y \\ \lambda f_1^* + \partial f_1^*/\partial y & \lambda f_2^* + \partial f_2^*/\partial y & f_2 f_2^* - f_1 f_1^* \end{pmatrix} \Phi, \tag{26}$$

where $\Phi = (\Phi_1, \Phi_2, \Phi_3)^\intercal \in W(\mathbb{S}^1;\mathbb{C}^3)$, provide its Lax type integrability. Thus, the following theorem holds.

Theorem 3.1. *The Davey-Stewartson equation (22), (23) possesses the Lax representation as the compatibility condition for (24) and (26) under the additional constraint (21), arising naturally from the equations (24) and (25) .*

In fact, one has found above a triple linearization for a (2+1)-dimensional dynamical system, that is a new important ingredient of the Lie algebraic approach to Lax type integrable flows, based on the Bäcklund type transformation (14) developed in this work. It is clear that the similar construction of a triple linearization like (24) - (26) can be done for many other both old and new (2+1)-dimensional dynamical systems. Another paper is being prepared on these dynamical systems.

4 The centrally extended Lie-algebraic structure

Let $\tilde{\mathcal{G}} := C^\infty(\mathbb{S}\times\mathbb{S};\mathcal{G})$ which is a current Lie algebra of mappings taking values in a semi-simple matrix Lie algebra $\mathcal{G}$. By means of this algebra $\tilde{\mathcal{G}}$ one constructs the Lie algebra $\hat{\mathcal{G}}$ of the following matrix integro-differential operators:

$$a := \mathbf{I}\xi^m + \sum_{j<m} a_j \xi^j,$$

$a_j \in \hat{\mathcal{G}}$, $j<m$, $j\in\mathbb{Z}$, $m\in\mathbb{N}$, and as before, the symbol $\xi := \partial/\partial x$ denotes differentiation with respect to the independent variable $x\in\mathbb{R}/2\pi\mathbb{Z}\simeq\mathbb{S}$. The related central extended Lie commutator on $\hat{\mathcal{G}}_c := \hat{\mathcal{G}}\oplus\mathbb{C}$ is given as [5, 11, 26, 32]:

$$[(a,\alpha),(b,\beta)] := ([a,b],\omega(\hat{a},\hat{b}))\ , \tag{27}$$

where $\alpha,\beta\in\mathbb{C}$, being generated by means of the standard Maurer-Cartan two-cocycle on $\hat{\mathcal{G}}$:

$$\omega(a,b) := (a,[\partial/\partial y, b])\ ,$$

where $\partial/\partial y$ is the differentiation with respect to the independent variable $y\in S$ and $[\partial/\partial y, b] := \partial b/\partial y$. The commutator (27) can be deformed by means of the above defined endomorphism $\mathcal{R}$ of $\hat{\mathcal{G}}$:

$$[(a,\alpha),(b,\beta)]_{\mathcal{R}} := ([a,b]_{\mathcal{R}},\omega_{\mathcal{R}}(a,b))\ , \tag{28}$$

where the $\mathcal{R}$-commutator takes the form:

$$[a,b]_{\mathcal{R}} := [\mathcal{R}a,b]+[a,\mathcal{R}b]\ ,$$

and the $\mathcal{R}$-deformed two-cocycle is determined in the following way:

$$\omega(a,b)_{\mathcal{R}} := \omega(\mathcal{R}a,b)+\omega(a,\mathcal{R}b)\ .$$

For any Fréchet smooth functionals $\gamma,\mu\in D(\hat{\mathcal{G}}_c^*)$ the Lie-Poisson bracket on $\hat{\mathcal{G}}_c^*$ related with the commutator (28) and the extended scalar product:

$$((a,\alpha),(b,\beta)) := (a,b)+\alpha\beta\ ,$$

where $a,b\in\hat{\mathcal{G}}$ and $\alpha,\beta\in\mathbb{C}$, is given as

$$\{\gamma,\mu\}_{\mathcal{R}}(l) = (l,[\nabla\gamma(l),\nabla\mu(l)]_{\mathcal{R}})+c\omega_{\mathcal{R}}(\nabla\gamma(l),\nabla\mu(l))\ , \tag{29}$$

where $l \in \hat{\mathcal{G}}^*$ and $c \in \mathbb{C}$. Based on the scalar product (1) the gradient $\nabla\gamma(l) \in \hat{\mathcal{G}}_c$ of some functional $\gamma \in D(\hat{\mathcal{G}}_c^*)$ at the point $l \in \hat{\mathcal{G}}_c^*$ is naturally defined as

$$\delta\gamma(l) := (\nabla\gamma(l), \delta l) \ .$$

Construct now the Casimir functionals $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, as

$$\gamma_n(l) := \int_0^{2\pi}\int_0^{2\pi} Tr\,(\xi^n \hat{l}_0)dxdy\ , \tag{30}$$

being invariant with respect to Ad^*-action of the corresponding to $\hat{\mathcal{G}}_c^*$ abstract Lie group $\hat{G}_c$ and satisfying the following condition [26]

$$(l - c\partial/\partial y) \circ \Phi = \Phi \circ (l_0 - c\partial/\partial y) \tag{31}$$

at a point $l \in \hat{\mathcal{G}}^*$. Here we have in (31)

$$\hat{l}_0 := \xi^m + \sum_{j<m} c_j \xi^j \in \hat{\mathcal{G}}^*,$$

with $c_j \in \tilde{\mathcal{G}}$, $[\xi, c_j] = 0$, $j < m$, $j \in \mathbb{Z}$, $m \in \mathbb{N}$, and

$$\Phi = \mathbf{1} + \sum_{r\in\mathbb{N}} \Phi_r \xi^{-r} \in \hat{G}_-,$$

$\Phi \in \hat{G}_-$ and $\hat{G}_-$ is the suitable abstract Lie group [4, 15, 26, 27, 32], generated by the Lie subalgebra $\hat{\mathcal{G}}_-$. Just as in [15, 20, 27], it can be shown that condition (31) is equivalent to the following relationship

$$[l - c\partial/\partial y, \nabla\gamma_n(l)] = 0, \tag{32}$$

for all $n \in \mathbb{N}$. In the case of $c = 0$ the Casimir functionals take the usual Adler's form [1, 15].

The Lie-Poisson bracket (29) generates the hierarchy of Hamiltonian dynamical systems on $\hat{\mathcal{G}}_c^*$ with Casimir funtionals $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, as the corresponding Hamiltonian functions, taking the form:

$$d\hat{l}/dt_n := [\mathcal{R}\nabla\gamma_n(l),\ l - c\partial/\partial y] = [(\nabla\gamma_n(l))_+,\ l - c\partial/\partial y]. \tag{33}$$

where the lower index "+" sign a differential part of the corresponding integro-differential operator. This equation is equivalent to the usual commutator Lax type representation. It is easy to verify that for every $n \in \mathbb{N}$ the above relationship is the compatibility condition for the following systems of linear integro-differential equations:

$$(l - c\partial/\partial y)f = \lambda f\ , \tag{34}$$

and

$$df/dt_n = (\nabla\gamma_n(l))_+ f\ , \tag{35}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $f \in W := W(\mathbb{S} \times \mathbb{S}; H)$ and H is a matrix representation space of the Lie algebra $\mathcal{G}$. The related to (35) dynamical system on the adjoint function space $W^* := W^*(\mathbb{S} \times \mathbb{S}; H)$ takes the form:

$$df^*/dt_n = -(\nabla\gamma_n(l))^*_+ f^* \ , \tag{36}$$

where $f^* \in W^*$ is a solution of the adjoint spectral relationship

$$(l^* + c\partial/\partial y) f^* = \nu f^* \ , \tag{37}$$

with a spectral parameter $\nu \in \mathbb{C}$.

Further, one will assume that the spectral relationship (34) admits $N \in N$ different eigenvalues $\lambda_i \in \mathbb{C}$, $i = \overline{1, N}$, and the study of algebraic properties of the equation (33) combined with $N \in \mathbb{N}$ copies of (35):

$$df_i/dt_n = (\nabla\gamma_n(\hat{l}))_+ f_i \ , \tag{38}$$

for the corresponding eigenfunctions $f_i \in W(\mathbb{S} \times \mathbb{S}; H)$, $i = \overline{1, N}$, and the same number of copies of (36):

$$df_i^*/dt_n = -(\nabla\gamma_n(\hat{l}))^*_+ f_i^* \ , \tag{39}$$

for the suitable adjoint eigenfunctions $f_i^* \in W^*(\mathbb{S} \times \mathbb{S}; H)$ related with $N \in \mathbb{N}$ different eigenvalues $\nu_i \in \mathbb{C}$, $i = \overline{1, N}$, of (37), being considered as a coupled evolution system on the space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$. The same problem at $c = 0$ and $N = 1$ has been studied before in the papers [14, 15, 27].

5 The centrally extended Poisson bracket on the augmented phase space

To simplify the description below we shall use the following notation of the gradient vector:

$$\nabla\gamma(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*) := (\delta\gamma/\delta\tilde{l},\ \delta\gamma/\delta\tilde{\mathrm{f}},\ \delta\gamma/\delta\tilde{\mathrm{f}}^*)^\top,$$

where $\tilde{\mathrm{f}} := (\tilde{f}_1, \ldots, \tilde{f}_N)$, $\tilde{\mathrm{f}}^* := (\tilde{f}_1^*, \ldots, \tilde{f}_N^*)$ and $\delta\gamma/\delta\tilde{\mathrm{f}} := (\delta\gamma/\delta\tilde{f}_1, \ldots, \delta\gamma/\delta\tilde{f}_N)$, $\delta\gamma/\delta\tilde{\mathrm{f}}^* := (\delta\gamma/\delta f_1^*, \ldots, \delta\gamma/\delta\tilde{f}_N^*)$, at a point $(\tilde{l}, \tilde{\mathrm{f}}, f^*)^\top \in \hat{\mathcal{G}}^* \oplus W^N \oplus W^{*N}$ for any smooth functional $\gamma \in D(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$.

On the spaces $\hat{\mathcal{G}}_c^*$ and $W^N \oplus W^{*N}$ there exist canonical Poisson structures such as

$$\delta\gamma/\delta\tilde{l} :\xrightarrow{\tilde{\theta}} [\tilde{l} - c\partial/\partial y, (\delta\gamma/\delta\tilde{l})_+] - [\tilde{l} - c\partial/\partial y, \delta\gamma/\delta\tilde{l}]_+ \ , \tag{40}$$

where $\tilde{\theta} : T^*(\hat{\mathcal{G}}_c^*) \to T(\hat{\mathcal{G}}_c^*)$ is an implectic operator corresponding to (29) at a point $\tilde{l} \in \hat{\mathcal{G}}^*$ and

$$(\delta\gamma/\delta\tilde{\mathrm{f}},\ \delta\gamma/\delta\tilde{\mathrm{f}}^*)^\top :\xrightarrow{\tilde{J}} (-\delta\gamma/\delta\tilde{\mathrm{f}}^*,\ \delta\gamma/\delta\tilde{\mathrm{f}})^\top \ , \tag{41}$$

where $\tilde{J} : T^*(W^N \oplus W^{*N}) \to T(W^N \oplus W^{*N})$ is an implectic operator corresponding to the symplectic form $\omega^{(2)} = \sum_{i=1}^N d\tilde{f}_i^* \wedge d\tilde{f}_i$ at a point $(\tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*) \in W^N \oplus W^{*N}$. It should be noted here that the Poisson structure (40) generates the equation (33) for any Casimir functional $\gamma \in I(\hat{\mathcal{G}}_c^*)$. Therefore, on the augmented phase space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ one can obtain a Poisson structure as the tensor product $\tilde{\Theta} := \tilde{\theta} \otimes \tilde{J}$ of (40) and (41).

Consider now the following [15, 20, 27] Bäcklund transformation:

$$(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*)^\top \stackrel{B}{:\to} (l(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*),\, \mathrm{f} = \tilde{\mathrm{f}},\, \mathrm{f}^* = \tilde{\mathrm{f}}^*)^\top\ , \tag{42}$$

generating on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ a Poisson structure $\Theta : T^*(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}) \to T(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$. The main condition imposed on the mapping (42) is the coincidence of the resulting dynamical system

$$(dl/dt_n,\, d\mathrm{f}/dt_n,\, d\mathrm{f}^*/dt_n)^\top := -\Theta\, \nabla \overline{\gamma}_n(l, \mathrm{f}, \mathrm{f}^*) \tag{43}$$

with the equations (33), (38) and (39) for the case when $\overline{\gamma}_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in N$, are not dependent on variables $(f, f^*) \in W^N \oplus W^{*N}$.

To satisfy that condition we will find a variation of a Casimir functional $\overline{\gamma}_n := \gamma_n|_{l=l(\tilde{l},\mathrm{f},\mathrm{f}^*)} \in D(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$, $n \in \mathbb{N}$, under the constraint $\delta\tilde{l} = 0$, taking into account the evolutions (38), (39) and the definition of the Bäcklund transformation (42). Hence we have

$$\begin{aligned}
\delta\overline{\gamma}_n(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*)\Big|_{\delta\tilde{l}=0} &= \sum_{i=1}^{N} \Big(< \delta\overline{\gamma}_n/\delta\tilde{f}_i, \delta\tilde{f}_i > + < \delta\overline{\gamma}_n/\delta\tilde{f}_i^*, \delta\tilde{f}_i^* >\Big) \\
&= \sum_{i=1}^{N} \Big(< -d\tilde{f}_i^*/dt_n, \delta\tilde{f}_i > + < d\tilde{f}_i/dt_n, \delta\tilde{f}_i^* >\Big)\Big|_{\tilde{\mathrm{f}}=\mathrm{f},\, \tilde{\mathrm{f}}^*=\mathrm{f}^*} \\
&= \sum_{i=1}^{N} (< (\delta\gamma_n/\delta l)_+^* f_i^*, \delta f_i > + < (\delta\gamma_n/\delta l)_+ f_i, \delta f_i^* >) \\
&= \sum_{i=1}^{N} (< f_i^*, (\delta\gamma_n/\delta l)_+ \delta f_i > + < (\delta\gamma_n/\delta l)_+ f_i, \delta f_i^* >) \\
&= \sum_{i=1}^{N} ((\delta\gamma_n/\delta l, (\delta f_i)\xi^{-1} \otimes f_i^*) + (\delta\gamma_n/\delta l, f_i\xi^{-1} \otimes \delta f_i^*)) \\
&= \left(\delta\gamma_n/\delta l, \delta \sum_{i=1}^{N} f_i\xi^{-1} \otimes f_i^*\right) := (\delta\gamma_n/\delta l, \delta l)\ ,
\end{aligned} \tag{44}$$

where $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in N$ and the brackets $< .,. >$ denotes the standard paring of the spaces W^* and W.

As a result of the expression (44) one obtains the relationship:

$$\delta l|_{\delta\tilde{l}=0} = \sum_{i=1}^{N} \delta(f_i\xi^{-1} \otimes f_i^*)\ . \tag{45}$$

Having assumed the linear dependence of l on $\tilde{l} \in \hat{\mathcal{G}}^*$ one gets right away from (45) that

$$l = \tilde{l} + \sum_{i=1}^{N} f_i\xi^{-1} \otimes f_i^*\ . \tag{46}$$

Thus, the Bäcklund transformation (42) can be written as

$$(\tilde{l}, \tilde{\mathrm{f}}, \tilde{\mathrm{f}}^*)^\top :\xrightarrow{B} (l = \tilde{l} + \sum_{i=1}^{N} f_i \xi^{-1} \otimes f_i^*, \mathrm{f}, \mathrm{f}^*)^\top \ . \tag{47}$$

The expression (47) generalizes results obtained both for the scalar of Lie algebra of integro-differential operators in [20, 26, 27] and for the matrix one in [15, 27]. The existence of the Bäcklund transformation (47) provides validity of the following theorem.

Theorem 5.1. *The dynamical system (43) on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ is equivalent to the following system of evolution equations:*

$$\begin{aligned} d\tilde{l}/dt_n &= [(\nabla \overline{\gamma}_n(\tilde{l}))_+, \tilde{l}] - [\nabla \overline{\gamma}_n(\tilde{l}), \tilde{l}]_+ \ , \\ d\tilde{\mathrm{f}}/dt_n &= \delta \overline{\gamma}_n / \delta \tilde{\mathrm{f}}^* \ , \quad d\tilde{\mathrm{f}}^*/dt_n = -\delta \overline{\gamma}_n / \delta \tilde{\mathrm{f}} \ , \end{aligned}$$

where $\overline{\gamma}_n := \gamma_n|_{l=l(\tilde{l},\mathrm{f},\mathrm{f}^*)} \in D(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$ and $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$ is a Casimir functional at a point $l \in \hat{\mathcal{G}}^*$ for every $n \in \mathbb{N}$, under the Bäcklund transformation (47).

Now by means of simple calculations via formula:

$$\Theta = B' \tilde{\Theta} B'^* \ ,$$

where $B' : T(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}) \rightarrow T(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$ is a Fréchet derivative of (47), one finds easily the following form of the Bäcklund transformed Poisson structure Θ on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$:

$$\nabla \gamma(l, \mathrm{f}, \mathrm{f}^*) :\xrightarrow{\Theta} \begin{pmatrix} [l - c\partial/\partial y, (\delta\gamma/\delta l)_+] - [l - c\partial/\partial y, \delta\gamma/\delta l]_+ + \\ \sum_{i=1}^N (f_i \xi^{-1} \otimes (\delta\gamma/\delta f_i) - (\delta\gamma/\delta f_i^*) \xi^{-1} \otimes f_i^*) \\ -\delta\gamma/\delta \mathrm{f}^* - (\delta\gamma/\delta l)_+ \mathrm{f} \\ \delta\gamma/\delta \mathrm{f} + (\delta\gamma/\delta l)_+^* \mathrm{f}^* \end{pmatrix} , \tag{48}$$

where $\gamma \in D(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$ is an arbitrary smooth functional. Thereby, one can formulate the follwing theorem.

Theorem 5.2. *The hierarchy of dynamical systems (33), (38) and (39) is Hamiltonian one with respect to the Poisson structure Θ in the form (48) and the functionals $\overline{\gamma}_n := \gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, being Casimir invariants on $\hat{\mathcal{G}}_c^*$.*

Based on the expression (43) one can construct a new hierarchy of Hamiltonian evolution equations describing commutative flows generated by involutive with respect to the Poisson bracket (29) Casimir invariants $\gamma_n \in I(\hat{\mathcal{G}}_c^*)$, $n \in \mathbb{N}$, on the augmented phase space $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$.

6 The hierarchies of additional symmetries

The hierarchy (33), (38) and (39) of evolution equations possesses another natural set of invariants including all higher powers of the eigenvalues λ_k, $k=\overline{1,N}$. The latter can be considered as Fréchet smooth functionals on the augmented phase space $\hat{\mathcal{G}}_c^*\oplus W^N\oplus W^{*N}$ owing to the evident representations:

$$\lambda_k^s =< f_k^*, (l-c\partial/\partial y)^s f_k >, \tag{49}$$

where $s\in\mathbb{N}$, holding under the normalizing constraints

$$< f_k^*, f_k >= 1 \ .$$

In the case of the Bäcklund transformation (46), where

$$l := l_+ + \sum_{i=1}^{N} f_i\xi^{-1}\otimes f_i^*, \tag{50}$$

the formula (49) gives rise to the following variation of the functionals $\lambda_k^s\in D(\hat{\mathcal{G}}_c^*\oplus W^N\oplus W^{*N})$, $k=\overline{1,N}$, $s\in\mathbb{N}:$

$$\begin{aligned}
\delta\lambda_k^s &=< \delta f_k^*, (l-c\partial/\partial y)^s f_k > \\
&+ < (f_k^*, \delta(l-c\partial/\partial y)^s) f_k > + < f_k^*, (l-c\partial/\partial y)^s(\delta f_k) > \\
&= (M_k^s, \delta l_+) + \sum_{i=1}^{N}\left(< (-M_k^s+\delta_k^i(l-c\partial/\partial y)^s)^* f_i^*, \delta f_i >\right) \\
&+ \sum_{i=1}^{N}\left(< (-M_k^s+\delta_k^i(l-c\partial/\partial y)^s) f_i, \delta f_i^* >\right) \ ,
\end{aligned}$$

where δ_k^i, $i,k=\overline{1,N}$, is the Kronecker symbol and the operators M_k^s, $k=\overline{1,N}$, $s\in\mathbb{N}$, are determined as

$$M_k^s := \sum_{p=0}^{s-1}((l-c\partial/\partial y)^p f_k)\xi^{-1}\otimes((l^*+c\partial/\partial y)^{s-1-p} f_k^*) \ .$$

Thus, one obtains the exact forms of gradients for the functionals $\lambda_k^s\in D(\hat{\mathcal{G}}_s^*\oplus W^N\oplus W^{*N})$, $k=\overline{1,N}:$

$$\begin{aligned}
\nabla\lambda_k^s(l_+,\mathrm{f},\mathrm{f}^*) &= (M_k^s, (-M_k^s+\delta_k^i(l-c\partial/\partial y)^s)^* f_i^*, \\
(-M_k^s+\delta_k^i(l-c\partial/\partial y)^s) f_i &: \ i=\overline{1,N})^\top \ .
\end{aligned} \tag{51}$$

By means of the expressions (51), (40) and (41) one finds a new hierarchy of coupled evolution equations on $\hat{\mathcal{G}}_c^*\oplus W^N\oplus W^{*N}:$

$$dl_+/d\tau_{s,k} = -[M_k^s, l_+ - c\partial/\partial y]_+ \ , \tag{52}$$

$$df_i/d\tau_{s,k} = (-M_k^s+\delta_k^i(l-c\partial/\partial y)^s) f_i \ , \tag{53}$$

$$df_i^*/d\tau_{s,k} = (M_k^s-\delta_k^i(l-c\partial/\partial y)^s)^* f_i^* \ , \tag{54}$$

where $i = \overline{1, N}$ and $\tau_{s,k} \in \mathbb{R}$, $s \in \mathbb{N}$, are evolution parameters. Owing to the Bäcklund transformation (50) the equation (52) can be rewritten equivalently in the following commutator form:

$$\begin{aligned} dl/d\tau_{s,k} &= -[M_k^s, l - c\partial/\partial y] \\ &= -\lambda_k^p \nu_k^{s-1-p}[M_k^1, l - c\partial/\partial y] = \lambda_k^p \nu_k^{s-1-p} dl/d\tau_{1,k} \ , \end{aligned} \tag{55}$$

where $p = \overline{0, s-1}$. Thereby, one can formulate the following theorem.

Theorem 6.1. *For $k = \overline{1, N}$ and $s \in \mathbb{N}$ the dynamical systems (55), (53) and (54) and Hamiltonian ones with respect to the Poisson structure Θ in the form (48) and the invariant functionals $\overline{\gamma}_s := \lambda_k^s \in D(\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N})$. The dynamical systems (55), (53) and (54) describe flows on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ commuting both one with each other and with the hierarchy of Lax type dynamical systems (33), (38) and (39).*

Proof. To prove the theorem it is sufficient to show that

$$[d/dt_n, d/d\tau_{1,k}] = 0 \ , \quad [d/d\tau_{1,k}, d/d\tau_{1,q}] = 0 \ , \tag{56}$$

where $k, q = \overline{1, N}$ and $n \in \mathbb{N}$. The first equality in the formula (56) follows from the identities:

$$d(\nabla\gamma_n(l))_+/d\tau_{1,k} = [(\nabla\gamma_n(l))_+, M_1^1]_+ \ , \quad dM_1^1/dt_n = [(\nabla\gamma_n(l))_+, M_1^1]_- \ ,$$

the second one being a consequence of the following relationship:

$$dM_k^1/d\tau_{1,q} - dM_q^1/d\tau_{1,k} = [M_k^1, M_q^1] \ ,$$

that proves the theorem. □

Thereby, for every $k = \overline{1, N}$ and all $s \in \mathbb{N}$ the dynamical systems (55), (53) and (54) on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ form a hierarchy of additional homogeneous or so called "ghost" symmetries for the Lax type flows (33), (38) and (39) on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$. The additional symmetry hierarchy for Lax type integrable one-dimensional dynamical systems associated with the Lie algebra $\hat{\mathcal{G}}^*$ of integro-differential operators was first described as some infinitely graded algebra in [10, 15]. It has been widely used for constructing Lax type integrable two-dimensional dynamical systems in [3, 10, 15, 36].

If $N \geq 2$, one can obtain a new class of nontrivial Hamiltonian flows $d/dT_n := d/dt_n + \sum_{k=1}^{N-1} d/d\tau_{n,k}$, $n \in \mathbb{N}$, on $\hat{\mathcal{G}}_c^* \oplus W^N \oplus W^{*N}$ in the Lax type form by use of the invariants considered above for the centrally extended Lie algebra $\hat{\mathcal{G}}_c^*$ of integro-differential operators. Acting on the eigenfunctions $(f_i, f_i^*) \in W \oplus W^*$, $i = \overline{1, N}$, these flows generate some integrable $(N+1)$-dimensional nonlinear dynamical systems.

For example, in the case of the element $l := \partial/\partial x + f_1\xi^{-1} \otimes f_1^* + f_2\xi^{-1} \otimes f_2^* \in \hat{\mathcal{G}}^*$ with $(f_1, f_2, f_1^*, f_2^*) \in W^2(\mathbb{S} \times \mathbb{S}; H) \times W^{*2}(\mathbb{S} \times \mathbb{S}; H)$, the flows $d/d\tau := d/d\tau_{1,1}$ and

$d/dT := d/dT_2 = d/dt_2 + d/d\tau_{2,1}$ on $\hat{\mathcal{G}}_c^* \oplus W^2 \oplus W^{*2}$ acting on the functions f_i, f_i^*, $i = \overline{1,2}$, give rise to such dynamical systems as

$$\begin{aligned} &f_{1,\tau} = f_{1,x} - cf_{1,y} + f_2 u\ , \quad f_{1,\tau}^* = f_{1,x}^* - cf_{1,y}^* + f_2^*\bar{u}\ , \\ &f_{2,\tau} = -f_1\bar{u}\ , \quad f_{2,\tau}^* = -f_1^* u\ , \end{aligned} \tag{57}$$

and

$$\begin{aligned} &f_{1,T} = f_{1,xx} + f_{1,\tau\tau} + wf_1 + 2f_1 v_\tau\ , \\ &f_{1,T}^* = -f_{1,xx}^* - f_{1,\tau\tau}^* - wf_1^* - 2f_1^* v_\tau\ , \\ &f_{2,T} = f_{2,xx} + wf_2 - f_{1,\tau}\bar{u} + f_1\bar{u}_\tau\ , \\ &f_{2,T}^* = -f_{2,xx}^* - wf_2^* + f_{1,\tau}^* u - f_1^* u_\tau\ , \\ &cw_y = w_x - 2(f_1 \otimes f_1^* + f_2 \otimes f_2^*)_x\ , \\ &u_x = f_1^T f_2^*\ , \quad \bar{u}_x = f_1^{*T} f_2\ , \quad v_x = f_1^T f_1^*\ , \end{aligned} \tag{58}$$

where one puts $(\nabla\gamma_2(l))_+ := \partial^2/\partial x^2 + w$ for some function $w \in \tilde{\mathcal{G}}$ depending parametrically on variables $\tau, T \in \mathbb{R}$. The systems (57) and (58) represent a Lax type integrable (3+1)-dimensional generalization of the (2+1)-dimensional system being equivalent to the Davey-Stewartson one [11, 29, 33] with an infinite sequence of conservation laws which can be found by the formula (30) in the form

$$\gamma_n(l) := tr \int_0^{2\pi}\int_0^{2\pi} (f_1\partial^{n-1} f_1^*/\partial x^{n-1} + f_2\partial^{n-1} f_2^*/\partial x^{n-1})dxdy\ ,$$

where $n \in \mathbb{N}$. The suitable Lax type linearization is given by the spectral problem (34) augmented by the set of evolution equations:

$$f_\tau = -M_1^1 f\ , \tag{59}$$

$$f_T = ((\nabla\gamma_2(l))_+ - M_1^2)f\ , \tag{60}$$

for an arbitrary eigenfunction $f \in W(\mathbb{S}\times\mathbb{S}; H)$. The relationships (59) and (60) give rise to the additional nonlinear constraint:

$$w_\tau = 2(f_1 \otimes f_1^*)_x. \tag{61}$$

In the case $\dim H = 1$ the Lax type representation (34), (59) and (60) for the mentioned above (3+1)-dimensional generalization (57), (58) and (61) of the Davey-Stewartson system [15, 29, 33] has equivalent matrix form:

$$\frac{dF}{dx} = \begin{pmatrix} 0 & 0 & f_1^* \\ 0 & 0 & f_2^* \\ -f_1 & -f_2 & \lambda + c\partial/\partial y \end{pmatrix} F\ ,$$

$$\frac{dF}{d\tau} = \begin{pmatrix} -(\lambda + c\partial/\partial y) & \bar{u} & f_1^* \\ -u & 0 & 0 \\ -f_1 & 0 & 0 \end{pmatrix} F\ , \quad \frac{dF}{dT} = CF\ ,$$

where $F = (F^1, F^2, F^3 = f)^\top \in W(\mathbb{S}\times\mathbb{S};\mathbb{C}^3)$, $C := \{C_{mn} \in gl(3;\mathbb{C}) : \; m,n = \overline{1,3}\}$, and

$$\begin{aligned}
C_{11} &= -(\lambda + c\partial/\partial y)^2 - u\bar{u} - 2f_1 f_1^* \ , \\
C_{12} &= -f_1 f_2^* - (\lambda + c\partial/\partial y)\bar{u} - \bar{u}_\tau \ , \\
C_{13} &= 2((\lambda + c\partial/\partial y)f_1^* - f_{1,x}^*) - \bar{u} f_2^* \ , \\
C_{21} &= -(\lambda + c\partial/\partial y)u - u_\tau - f_1 f_2^* \ , \\
C_{22} &= -f_2 f_2^* + u\bar{u} \ , \\
C_{23} &= (\lambda + c\partial/\partial y)f_2^* - f_{2,x}^* + u f_1^* \ , \\
C_{31} &= -(\lambda + c\partial/\partial y)f_1 - f_{1,x} - f_{1,\tau} \ , \\
C_{32} &= -(\lambda + c\partial/\partial y)f_2 - f_{2,x} + \bar{u} f_1 \ , \\
C_{33} &= (\lambda + c\partial/\partial y)^2 + w - f_2 f_2^* \ ,
\end{aligned}$$

to which one can apply the standard inverse spectral transform method [7, 11, 33].

The results obtained above can be also used for constructing a wide class of integrable (3+1)-dimensional nonlinear dynamical systems with triple Lax type linearizations [15].

7 Conclusions

As it is well known, there existed by now only two regular enough algorithmic approaches [3, 16, 16, 34] to constructing integrable multi-dimensional (mainly 2+1) dynamical systems on functional spaces. Our approach, devised in this work, is substantially based on the results previously done in [14, 20], explains completely the analytical properties of three-dimensional flows before delivered in works [3, 7, 33, 36]. As the key points of our approach we used the canonical Hamiltonian structures naturally existing on the augmented phase space and related with them the Bäcklund transformation which saves Casimir invariants of a chosen matrix integro-differential Lie algebra. The latter gives rise to some additional Hamiltonian properties of considered augmented evolution flows before studied in [3, 16, 20] making use of the standard inverse scattering transform [6, 11, 33] and the formal symmetry reduction for the KP-hierarchy [11, 36] of commuting operator flows.

As one can convince ourselves analyzing the structure of the Bäcklund type transformation (14), that it strongly depends on the type of an *ad*-invariant scalar product chosen on an operator Lie algebra $\hat{\mathcal{G}}$ and its Lie algebras decomposition like (2). Since there exist in general other possibilities of choosing such decompositions and *ad*-invariant scalar products in $\hat{\mathcal{G}}$, they give rise naturally to another resulting type of corresponding Bäcklund transformations, which can be a subject of another special investigation. Let us here only mention the choice of a scalar product related with the operator Lie algebra $\hat{\mathcal{G}}$ centrally extended by means of the standard Maurer-Cartan two-cocycle [16, 18, 25], bringing about new types of multi-dimensional integrable flows.

The last aspect of the Bäcklund approach to constructing Lax type integrable flows and their partial solutions, which is worth mentioning, is related with Darboux-Bäcklund type transformations [9, 19, 21, 23, 24, 28] and their new generalization recently developed in [15, 37]. They give rise to very effective procedures of constructing multi-dimensional integrable flows on functional spaces with an arbitrary number of independent variables simultaneously delivering a wide class of exact analytical solutions, depending on many constant parameters, which can appear to be useful for diverse applications in applied sciences. All of the above Bäcklund type transformation aspects can be treated as special investigations, giving rise to new directions in the theory of multi-dimensional evolution flows and their integrability.

Several Lie-algebraic approaches [5, 15, 18, 26, 30] to constructing Lax integrable multi-dimensional (mainly 2+1) nonlinear dynamical systems on functional manifolds and their supersymmetric generalizations have been well known. In this paper we developed a method of introducing one more commuting variable into Lax type integrable (2+1)-dimensional dynamical systems arising on a dual space to the centrally extended matrix Lie-algebra of integro-differential operators. It is based on the natural hierarchy of additional symmetries [10, 13, 15, 20, 27]. The resulting integrable (3+1)-dimensional dynamical systems obtained by means of this method possess an infinite sequence of conservation laws and related triple Lax type linearizations. Owing to the latter property their soliton type solutions can be found by means of either the standard inverse spectral transform method [7, 11] or Darboux-Bäcklund transformations [9, 19, 21, 22, 28].

The structure of the constructed Lie-Bäcklund transformation (47), being a key point of the devised approach, strongly depends on an *ad*-invariant scalar product chosen for an operator Lie algebra $\hat{\mathcal{G}}$ and on a suitable Lie algebra decomposition (see [3, 16]). Since there exist other possibilities of choosing the corresponding *ad*-invariant scalar products on $\hat{\mathcal{G}}$, such decompositions will give rise naturally to another Bäcklund transformations. In further work the method is planned to be developed for some special centrally extended Lie algebra of super-integro-differential operators [8, 13].

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