Central European Journal of Mathematics
DOI: 10.2478/s11533-006-0039-0
Research article
CEJM 5(1) 2007 164–180

Slice modules over minimal 2-fundamental algebras*

Zygmunt Pogorzały†, Karolina Szmyt

Faculty of Mathematics and Computer Science, Nicholaus Copernicus University, 87-100 Toruń, Poland

Received 21 June 2006; accepted 25 September 2006

Abstract: We consider a class of algebras whose Auslander-Reiten quivers have starting components that are not generalized standard. For these components we introduce a generalization of a slice and show that only in finitely many cases (up to isomorphism) a slice module is a tilting module.

Keywords: Minimal 2-fundamental algebra, Auslander-Reiten quiver, slice module, tilting module
MSC (2000): 16 G 20, 16 G 70

Introduction

Let K be a fixed algebraically closed field. All algebras we consider will be finite dimensional associative K-algebras with a unit element. They will also be assumed to be basic and connected.

For a given algebra A we shall denote by $\mathrm{mod}(A)$ the category of the finite dimensional right A-modules. We shall consider the Auslander-Reiten quiver Γ_A of the algebra A [3].

We are interested in minimal 2-fundamental algebras as introduced in [13]. It happens quite frequently that if A is a minimal 2-fundamental algebra then its Auslander-Reiten quiver Γ_A contains a component at the beginning that is not generalized standard in the sense of Skowroński [16] and contains the projective vertices. Thus it is reasonable to generalize a notion of a slice introduced in [9] (see also [14]) and study when a slice module is a tilting module. We shall define a postprojective (respectively, preinjective) slice $\mathcal{S}$ and consider a slice module $M_{\mathcal{S}}$. Tilting modules over representation-finite algebras were studied in this way in [9].

* The first named author was supported by the Polish Scientific Grant KBN No 1 P03A 018 27
† E-mail: zypo@mat.uni.torun.pl

In this paper we shall characterize postprojective (respectively, preinjective) slice modules that are tilting (respectively, cotilting) modules. Contrary to tame hereditary algebras there are only finitely many postprojective (respectively, preinjective) slices $\mathcal{S}$ whose slice modules $M_{\mathcal{S}}$ are tilting (respectively, cotilting) modules.

1 Preliminaries

Consider a finite dimensional basic K-algebra A. Following Gabriel [8] one can associate to A a bound quiver (Q_A, I_A) in such a way that $A \cong KQ_A/I_A$, where KQ_A is the path algebra of the quiver Q_A, and I_A is a two-sided ideal in KQ_A contained in the square of the two-sided ideal generated by the arrows. The algebra A is called *triangular* if Q_A has no oriented cycles.

An algebra A is said to be special biserial if there exists a bound quiver (Q_A, I_A) with $A \cong KQ_A/I_A$ such that:

(1) Every vertex of Q_A is the source of at most two arrows.

(2) Every vertex of Q_A is the sink of at most two arrows.

(3) For every arrow α in Q_A there exists at most one arrow β (respectively, γ) such that $\alpha\beta \notin I_A$ (resp., $\gamma\alpha \notin I_A$).

Throughout the paper we shall always consider special biserial algebras of the form KQ/I with (Q, I) satisfying the above conditions.

Let $A = KQ_A/I_A$ be special biserial. Then A is called a *string algebra* (see [6]) if I_A is generated only by paths. There is a full classification of indecomposable finite dimensional right A-modules [7, 18]. For every finite dimensional right A-module M we have two cases. In the first case M is induced by a walk w for which we shall often use the notation $M(w)$ and say that M is a *string module* [18]. In the other case M is the so-called bound module [18] which we will not consider.

We shall use an algorithm for computing Auslander-Reiten sequences for string modules due to Skowroński and Waschbüsch [17].

A triangular string algebra $A = KQ_A/I_A$ is said to be $\tilde{\mathbb{A}}_m$-*separated* provided that for any two subquivers Q', Q'' in Q_A of type $\tilde{\mathbb{A}}_m$ such that $KQ' \cap I_A = 0 = KQ'' \cap I_A$ we have $Q'_0 \cap Q''_0 = \emptyset$, where Q'_0, Q''_0 denote the sets of vertices of Q', Q'', respectively.

A triangular string $\tilde{\mathbb{A}}_m$-separated algebra $A = KQ_A/I_A$ is said to be 2-*fundamental* [13] if it is connected and the following conditions are satisfied:

(i) There exist exactly two full subquivers Q', Q'' of type $\tilde{\mathbb{A}}_m$ in (Q_A, I_A) such that $KQ' \cap I_A = 0 = KQ'' \cap I_A$ and the quiver $\bar{Q}_A$ obtained from Q_A by removing the arrows from Q' and Q'' and identifying the vertices of Q' with a vertex $0'$ and the vertices of Q'' with a vertex $0''$ is a tree.

(ii) For any 0^j of $0'$, $0''$ there exists either a maximal path v in $\bar{Q}_A$ starting at 0^j such that $v \notin I_A$, or a maximal path u in $\bar{Q}_A$ ending at 0^j such that $u \notin I_A$. If v (treated as a path in Q_A) starts at some vertex x in Q^j that is a sink of two maximal paths v_1, v_2 in Q^j then $v_1v \notin I_A$ or $v_2v \notin I_A$. If u (treated as a path in Q_A) ends at some vertex y in Q^j that is a source of two maximal paths u_1, u_2 in Q^j then $uu_1 \notin I_A$ or $uu_2 \notin I_A$.

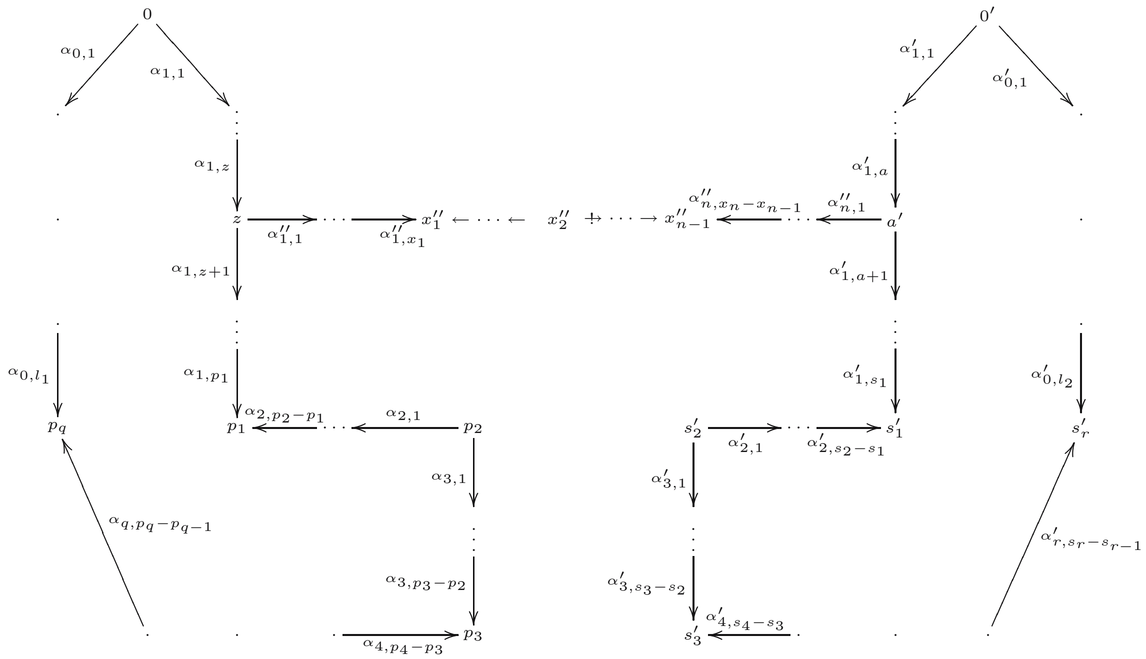
A 2-fundamental algebra A is said to be *minimal* if the graph obtained from the quiver $\bar{Q}_A$ by forgetting orientations of the arrows is of the form $0' \text{ —— } \cdots \text{ —— } 0''$.

It is well-known that we can attach to any K-algebra A its Auslander-Reiten quiver Γ_A (see [3, 4]). We shall not distinguish between indecomposable A-modules and vertices of Γ_A. A component in Γ_A will always mean a connected component.

Following [13] a component $\mathcal{C}$ of Γ_A is said to be *starting* (resp., *ending*) if there is no nonzero morphism $f : X \to Y$ between indecomposable modules X, Y such that $Y \in \mathcal{C}$ and $X \notin \mathcal{C}$ (resp., $X \in \mathcal{C}$ and $Y \notin \mathcal{C}$).

Following Skowroński [16] we say that a component $\mathcal{C}$ of Γ_A is *generalized standard* if $\mathrm{rad}^\infty(X, Y) = 0$ for any indecomposable right A-modules $X, Y \in \mathcal{C}$. Recall that $\mathrm{rad}^\infty(\mathrm{mod}(A))$ is the intersection of all positive powers of the Jacobson radical $\mathrm{rad}(\mathrm{mod}(A))$.

Consider the following three strictly increasing sequences of positive integers $\underline{p} = (p_1, \ldots, p_q)$, $\underline{s} = (s_1, \ldots, s_r)$ and $\underline{x} = (x_1, \ldots, x_n)$ with $q, r, n \geq 1$. Let $l_1, l_2 \geq 1$ be two integers. Then we can consider a quiver $Q^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ of the following form



Let $I^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ denote a two-sided ideal in the path algebra $KQ^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ generated by the paths $\alpha_{1,z}\alpha''_{1,1}$, $\alpha'_{1,a}\alpha''_{n-1,1}$. Denote by $A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ the algebra $KQ^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}/I^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$.

Under the above assumptions and notations we can consider a quiver $Q^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ that is dual to $Q^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$. Let $I^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ denote a two-sided ideal in the path algebra $KQ^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ generated by the paths $\alpha''_{1,1}\alpha_{1,z}$, $\alpha''_{n-1,1}\alpha'_{1,a}$. Denote by $A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ the algebra $KQ^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}/I^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$.

These two families of algebras will be of great importance throughout the paper.

Lemma 1.1. *Let A be a minimal 2-fundamental algebra whose Auslander-Reiten quiver Γ_A contains a starting component that is not generalized standard. Then there are three* *strictly increasing sequences of positive integers* $\underline{p}, \underline{x}, \underline{s}$ *and two integers* $l_1, l_2 \geq 1$ *such that* $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$.

Proof. This lemma is a direct consequence of [13, Theorem 5.7]; and the proof of Lemma 5.6 in [13]. □

Lemma 1.2. *Let* A *be a minimal* 2*-fundamental algebra whose Auslander-Reiten quiver* Γ_A *contains an ending component that is not generalized standard. Then there are three strictly increasing sequences of positive integers* $\underline{p}, \underline{x}, \underline{s}$ *and two integers* $l_1, l_2 \geq 1$ *such that* $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$.

Proof. This lemma is a direct consequence of [13, Theorem 5.7] and the proof of Lemma 5.6 in [13]. □

2 Slice modules and their projective dimensions

Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$, and $\mathcal{C}$ be the only starting component in Γ_A. It is easy to see that $\mathcal{C}$ contains all indecomposable projective right A-modules. Moreover, we know from [13, Theorem 5.7] that $\mathcal{C}$ is not generalized standard. A *postprojective slice* in $\mathcal{C}$ is defined to be a set $\mathcal{S} = \{N_1, N_2, \dots, N_t\}$ of vertices of $\mathcal{C}$ such that the following conditions are satisfied:

(0) $\mathcal{S}$ consists only of postprojective modules.

(1) There is no oriented cycle in $\mathcal{C}$ consisting of modules from $\mathcal{S}$.

(2) If $M_0 \to M_1 \to \cdots \to M_m$ is a path in $\mathcal{C}$ such that $M_0, M_m \in \mathcal{S}$ then $M_1, \dots, M_{m-1} \in \mathcal{S}$.

(3) $\mathcal{S}$ contains exactly one representative of every τ-orbit of the projective A-modules.

For an algebra $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ one can define a preinjective slice in the unique ending component of Γ_A dually.

Let $\mathcal{S}$ be a postprojective (resp., preinjective) slice in $\mathcal{C}$. Then a module $M_{\mathcal{S}}$ that is isomorphic to a direct sum $\bigoplus_{i=1}^{t} N_i$ is called a *postprojective* (resp., *preinjective*) slice module of $\mathcal{S}$. Our next aim is to compute projective (resp., injective) dimensions of postprojective (resp., preinjective) slice modules. We shall use a special technique in the considered cases rather than applying the results of [10–12].

For an algebra $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ or $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ consider a string module $M(w)$ for a walk w in $(Q^{(j)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}, I^{(j)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)})$, $j = 1, 2$. Then there are nonzero paths u_i, v_i in the above bound quiver such that u_1 starts at b_1, both u_i, v_i end at c_i, $i = 1, \dots, d$, v_i, u_{i+1} start at b_{i+1}, $i = 1, \dots, d$, and u_{d+1} ends at c_{d+1}. Furthermore, $w \in \{w_1, w_2, w_3, w_4\}$ for the walks $w_1 = u_1 v_1^{-1} u_2 v_2^{-1} \cdots u_d v_d^{-1}$, $w_2 = u_1 v_1^{-1} u_2 v_2^{-1} \cdots u_d v_d^{-1} u_{d+1}$, $w_3 = v_1^{-1} u_2 v_2^{-1} \cdots u_d v_d^{-1}$, $w_4 = v_1^{-1} u_2 v_2^{-1} \cdots u_d v_d^{-1} u_{d+1}$.

Lemma 2.1. *For an algebra* $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ *consider a string module* $M(w)$ *with* $w \in \{w_1, w_2, w_3, w_4\}$. *Then the following conditions are satisfied:*

(1) If $w = u_1$ *and* $c_1 = (a-1)'$ *or* $c_1 = z-1$ *then* $\text{proj.dim}(M(w)) = 2$.

(2) If $w \in \{w_2, w_4\}$, $c_{d+1} = z-1$ *or* $c_{d+1} = (a-1)'$ *then* $\text{proj.dim}(M(w)) = 2$.

(3) If $w \in \{w_1, w_3\}$, $v_d = \alpha_{0,1}v'_d$ *and* $b_{d+1} = z-1$ *or* $v_d = \alpha'_{0,1}v'_d$ *and* $b_{d+1} = (a-1)'$ *then* $\text{proj.dim}(M(w)) = 2$.

(4) If $w \in \{w_1, w_2\}$, $u_1 = \alpha_{0,1}u'_1$ *and* $b_1 = z-1$ *or* $u_1 = \alpha'_{0,1}u'_1$ *and* $b_1 = (a-1)'$ *then* $\text{proj.dim}(M(w)) = 2$.

(5) If $w \in \{w_3, w_4\}$, *and* $c_1 = z-1$ *or* $c_1 = (a-1)'$ *then* $\text{proj.dim}(M(w)) = 2$.

Proof. We start by considering the case when $w = u_1$. Suppose that $c_1 = z-1$. Then we have the following minimal projective resolution of $M(w)$:

$$0 \to P'' \to P' \oplus P_z \to P_{b_1} \to M(w) \to 0,$$

where P' is zero if $b_1 \neq 0$ and P' is a nonzero projective direct summand in $\text{rad}(P_0)$ if $b_1 = 0$. Furthermore, if $z = p_1$ then $P'' \cong \text{rad}(P_z)$ and if $z \neq p_1$ then P'' is the direct summand in $\text{rad}(P_z)$ whose socle is isomorphic to $S_{x''_1}$. Thus $\text{proj.dim}(M(w)) = 2$. If $c_1 = (a-1)'$ then similar arguments show that $\text{proj.dim}(M(w)) = 2$. Consequently, condition (1) is proved.

If $w = w_2$, $c_{d+1} = z-1$ then we have the following minimal projective resolution of $M(w)$:

$$0 \to \bar{P}' \oplus \bar{P}'' \to P' \oplus P'' \oplus \bigoplus_{j=2}^{d} \tilde{S}_{c_j} \to \bigoplus_{i=1}^{d+1} P_{b_i} \to M(w) \to 0,$$

where $\tilde{S}_{c_i} \cong S_{c_i}$ if $c_i \neq z, a'$ and $\tilde{S}_{c_i} \cong P_{c_i}$ if $c_i = z = p_1$ or $c_i = a' = s'_1$ and where P' is zero if P_{b_1} is uniserial, P' is a nonzero projective direct summand in $\text{rad}(P_{b_1})$ if P_{b_1} is nonuniserial, or else P' is isomorphic to P_z or to $P_{a'}$ if $b_1 = z-1$ and $u_1 = \alpha_{0,1}u'_1$ or $b_1 = (a-1)'$ and $u_1 = \alpha'_{0,1}u'_1$. Moreover, $P'' \cong P_z$ and $\bar{P}''$ is either $\text{rad}(P_z)$ which is projective if $z = p_1$ or $\bar{P}''$ is the nonzero projective direct summand in $\text{rad}(P_z)$ whose socle is $S_{x''_1}$ if $z \neq p_1$. Furthermore, $\bar{P}'$ is zero or $\bar{P}'$ is a uniserial projective A-module if $P' \cong P_z$ or $P' \cong P_{a'}$. Thus $\text{proj.dim}(M(w)) = 2$.

If $w = w_4$, $c_{d+1} = z-1$ then we have the following minimal projective resolution of $M(w)$:

$$0 \to \bar{P}' \oplus \bar{P}'' \to P' \oplus P'' \oplus \bigoplus_{j=2}^{d} \tilde{S}_{c_j} \to \bigoplus_{i=2}^{d+1} P_{b_i} \to M(w) \to 0,$$

where $\tilde{S}_{c_i} \cong S_{c_i}$ if $c_i \neq z, a'$ and $\tilde{S}_{c_i} \cong P_{c_i}$ if $c_i = z = p_1$ or $c_i = a' = s'_1$ and where P' is zero or a nonzero projective submodule of P_{b_2} or $P' \cong P_z$ if $c_1 = z-1$ or else $P' \cong P_{a'}$ if $c_1 = (a-1)'$. Moreover, $P'' \cong P_z$ and $\bar{P}''$ is either $\text{rad}(P_z)$ which is projective if $z = p_1$ or $\bar{P}''$ is the projective direct summand in $\text{rad}(P_z)$ whose socle is $S_{x''_1}$ if $z \neq p_1$. Furthermore, $\bar{P}'$ is zero or $\bar{P}'$ is a uniserial projective A-module if $P' \cong P_z$ or $P' \cong P_{a'}$. Thus $\text{proj.dim}(M(w)) = 2$.

If $w \in \{w_2, w_4\}$ and $c_{d+1} = (a-1)'$ then similar arguments show that $\text{proj.dim}(M(w)) = 2$. Consequently, condition (2) is proved.

If $w = w_1$, $v_d = \alpha_{0,1}v'_d$ and $b_{d+1} = z-1$ then we have the following minimal projective resolution of $M(w)$:

$$0 \to \bar{P}' \oplus \bar{P}'' \to P' \oplus P'' \oplus \bigoplus_{j=1}^{d} \tilde{S}_{c_j} \to \bigoplus_{i=1}^{d+1} P_{b_i} \to M(w) \to 0,$$

where $\tilde{S}_{c_j} \cong S_{c_j}$ if $c_j \neq z, a'$ and $\tilde{S}_{c_j} \cong P_{c_j}$ if $c_j = z = p_1$ or $c_j = a' = s'_1$. Moreover, $P'' \cong P_z$ and P' is zero or a nonzero projective uniserial module isomorphic to $\mathrm{rad}(P_{c_1})$, $P' \cong P_z$ if $c_1 = z-1$, or else $P' \cong P_{a'}$ if $c_1 = (a-1)'$. Furthermore, $\bar{P}''$ is either $\mathrm{rad}(P_z)$ which is projective if $z = p_1$ or $\bar{P}''$ is the projective direct summand in $\mathrm{rad}(P_z)$ whose socle is $S_{x''_1}$ if $z \neq p_1$. Further $\bar{P}'$ is zero or $\bar{P}'$ is a uniserial projective A-module if $P' \cong P_z$ or $P' \cong P_{a'}$. Thus $\mathrm{proj.dim}(M(w)) = 2$.

If $w \in \{w_1, w_2\}$, $v_d = \alpha'_{0,1}v'_d$ and $b_{d+1} = (a-1)'$ then similarly we can show that $\mathrm{proj.dim}(M(w)) = 2$. Consequently, condition (3) is proved.

If $w = w_1, w_2$, $w_1 = \alpha_{0,1}u'_1$ and $b_1 = z-1$ or $u_1 = \alpha'_{0,1}u'_1$ and $b_1 = (a-1)'$ then we get $\mathrm{proj.dim}(M(w)) = 2$, because we can use the left-right symmetry of conditions (3) and (4). Thus condition (4) is verified.

It is also clear that conditions (2) and (5) are left-right symmetric. Thus $\mathrm{proj.dim}(M(w)) = 2$ in the case and thus condition (5) is verified. □

Lemma 2.2. *For an algebra* $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ *consider a string module* $M(w)$ *with* $w \in \{w_1, w_2, w_3, w_4\}$. *Then the following conditions are satisfied:*

(1) If $w = u_1$ *and* $b_1 = (a-1)'$ *or* $b_1 = z-1$ *then* $\mathrm{inj.dim}(M(w)) = 2$.

(2) If $w \in \{w_1, w_3\}$, $b_{d+1} = z-1$ *or* $b_{d+1} = (a-1)'$ *then* $\mathrm{inj.dim}(M(w)) = 2$.

(3) If $w \in \{w_2, w_4\}$, $u_{d+1} = u'_{d+1}\alpha_{0,1}$ *and* $c_{d+1} = z-1$ *or* $u_{d+1} = u'_{d+1}\alpha'_{0,1}$ *and* $c_{d+1} = (a-1)'$ *then* $\mathrm{inj.dim}(M(w)) = 2$.

(4) If $w \in \{w_3, w_4\}$, $v_1 = v'_1\alpha_{0,1}$ *and* $c_1 = z-1$ *or* $v_1 = v'_1\alpha'_{0,1}$ *and* $c_1 = (a-1)'$ *then* $\mathrm{inj.dim}(M(w)) = 2$.

(5) If $w \in \{w_1, w_2\}$ *and* $b_1 = z-1$ *or* $b_1 = (a-1)'$ *then* $\mathrm{inj.dim}(M(w)) = 2$.

Proof. Dual arguments to those used in the proof of Lemma 2.1 prove the lemma. We leave details to the reader. □

Proposition 2.3. *For an algebra* $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ *consider a string module* $M(w)$ *with* $w \in \{w_1, w_2, w_3, w_4\}$. *If* w *does not satisfy any of the conditions (1) - (5) in Lemma 2.1 then* $\mathrm{proj.dim}(M(w)) \leq 1$.

Proof. Let $w = w_1$. Then it is easily seen that there exists an epimorphism $f : \bigoplus_{i=1}^{d+1} P_{b_i} \to M(w_1)$. Notice that if P_{b_1} and $P_{b_{d+1}}$ are uniserial modules then $\ker(f) \cong \bigoplus_{j=1}^{d} \tilde{S}_{c_j}$, where every $\tilde{S}_{c_j} \cong S_{c_j}$ is a simple projective module except when $c_j = z = p_1$ or $c_j = a' = s'_1$, or when $w_1 = u_1$ and c_1 is a sink of exactly one arrow. In the case $c_j = z = p_1$ or $c_j = a' = s'_1$ we have $\tilde{S}_{c_j} \cong P_{c_j}$ is projective. In the case $w_1 = u_1$ we have $\ker(f) \cong \mathrm{rad}(P_{c_1})$ which is projective if $c_1 \neq z-1$ and $c_1 \neq (a-1)'$. Thus we have either proj.dim$(M(w)) \leq 1$ or $w_1 = u_1$ and $c_1 = z - 1$ or $c_1 = (a-1)'$ which is precisely condition (1) of Lemma 2.1.

If P_{b_1}, $P_{b_{d+1}}$ are not uniserial then $\ker(f) \cong \bigoplus_{j=1}^{d} \tilde{S}_{c_j} \oplus P' \oplus P''$, where every $\tilde{S}_{c_j} \cong S_{c_j}$ is a simple projective module unless $c_j = z = p_1$ or $c_j = a' = s_1'$ in which case $\tilde{S}_{c_j} \cong P_{c_j}$. Moreover, P' is either a projective direct summand in $\mathrm{rad}(P_{b_1})$ if $b_1 \neq z-1, (a-1)'$, or is nonprojective otherwise. If P' is not projective then $b_1 = z - 1$ and $u_1 = \alpha_{0,1} u_1'$, because P_{b_1} is not uniserial, or $b_1 = (a-1)'$ and $u_1 = \alpha_{0,1}' u_1'$, and this is stated in condition (4) of Lemma 2.1. A similar analysis shows that if P'' is a direct summand of $\mathrm{rad}(P_{b_{d+1}})$ which is nonprojective then we obtain the case of w_1 stated in condition (3) of Lemma 2.1. It is also clear that if one of P_{b_1}, $P_{b_{d+1}}$ is nonuniserial then we can use the same arguments. Consequently, if $w = w_1$ then the required condition holds.

Let $w = w_2$. Then we have an epimorphism $f : \bigoplus_{i=1}^{d+1} P_{b_i} \to M(w_2)$. It is easy to see that if P_{b_1} is uniserial and $c_{d+1} \neq z-1, (a-1)'$ then $\ker(f) \cong \bigoplus_{j=1}^{d} \tilde{S}_{c_j} \oplus \mathrm{rad}(P_{c_{d+1}})$, where every $\tilde{S}_{c_j} \cong S_{c_j}$ is a simple projective module unless $c_j = z = p_1$ or $c_j = a' = s_1'$ in which case we have $\tilde{S}_{c_j} \cong P_{c_j}$. Therefore $\ker(f)$ is a projective module in this case.

If P_{b_1} is nonuniserial then similar considerations to those used in the case $w = w_1$ allow us to conclude that $\ker(f)$ is not projective provided that $b_1 = z - 1$ and $u_1 = \alpha_{0,1} u_1'$ or $b_1 = (a-1)'$ and $u_1 = \alpha_{0,1}' u_1'$ and this is stated in condition (4) of Lemma 2.1.

If $c_{d+1} = z - 1$ or $c_{d+1} = (a-1)'$ then $\mathrm{rad}(P_{c_{d+1}})$ is not projective, and this is stated in condition (2) of Lemma 2.1. Consequently, the required condition holds in the case $w = w_2$.

Since the case $w = w_3$ is left-right symmetric to $w = w_2$, we obtain that either condition (3) or (5) of Lemma 2.1 is satisfied, or proj.dim$(M(w)) \leq 1$.

Let $w = w_4$. Then we have an epimorphism $f : \bigoplus_{i=2}^{d+1} P_{b_i} \to M(w_4)$. Using the same arguments as in the case $w = w_2$ for $\mathrm{rad}(P_{c_{d+1}})$ we obtain that $\mathrm{rad}(P_{c_{d+1}})$ is not projective if $c_{d+1} = z - 1$ or $c_{d+1} = (a-1)'$. But this is stated in condition (2) of Lemma 2.1. Similarly, if $\mathrm{rad}(P_{c_1})$ is not projective then $c_1 = z - 1$ or $c_1 = (a-1)'$ which is stated in condition (5) of Lemma 2.1. Consequently, the required condition holds in the case $w = w_4$ and the proposition is proved. □

Proposition 2.4. *For an algebra* $A \cong A^{(2)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$ *consider a string module* $M(w)$ *with* $w \in \{w_1, w_2, w_3, w_4\}$. *If* w *does not satisfy any of the conditions (1) - (5) in Lemma 2.2 then* inj.dim$(M(w)) \leq 1$.

Proof. This is dual to the proof of Proposition 2.3. □

Corollary 2.5. *For an algebra* $A \cong A^{(1)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$ *let* $\mathcal{C}$ *be a starting component in* Γ_A *that is not generalized standard. If* $\mathcal{S}$ *is a postprojective slice in* $\mathcal{C}$ *then we have* proj.dim$(M_{\mathcal{S}}) \leq 1$.

Proof. We shall prove the corollary for every module $M(w) \in \mathcal{S}$ such that there is a chain

$$P \to X_1 \to X_2 \to \cdots \to X_n = M(w)$$

of indecomposable right A-modules and irreducible morphisms, where P is a projective module. Then $P \in \mathcal{C}$ by 2.1. We infer by the Skowroński-Waschbüsch algorithm that, in the considered case, the walk w does not satisfy any of the conditions (1) - (5) in Lemma 2.1. Thus we deduce from Proposition 2.3 that proj.dim$(M(w)) \leq 1$.

Finally, it is clear that for for each $M \in \mathcal{S}$ there is a walk w such that $M \cong M(w)$ and there is the above chain of indecomposable right A-modules and irreducible morphisms. Therefore proj.dim$(M)_{\mathcal{S}}) \leq 1$. □

Corollary 2.6. *For an algebra* $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ *let* $\mathcal{C}$ *be an ending component in* Γ_A *that is not generalized standard. If* $\mathcal{S}$ *is a preinjective slice in* $\mathcal{C}$ *then we have* inj.dim$(M_{\mathcal{S}}) \leq 1$.

Proof. This is dual to the proof of Corollary 2.5. □

3 Homomorphisms between slice modules

We have the following algorithm for computing Auslander-Reiten sequences for string modules, determined by Skowroński and Waschbüsch in [17]. If $w \in \{w_1, w_2, w_3, w_4\}$ then $w_R = w\kappa^{-1}u$, where κ is an arrow such that κv_d is a nonzero path in case $w \in \{w_1, w_3\}$ and κ is an arrow in case $w \in \{w_2, w_4\}$; u is a maximal nonzero path starting at the source of κ (u may be trivial) provided that the above arrow κ exists. If the arrow κ does not exist then $w_R = w'$, where w' is as follows. If $w \in \{w_2, w_4\}$ and $u_{d+1} = u'_{d+1}\delta$ for an arrow δ then $w' = u_1v_1^{-1}u_2v_2^{-1}\dots u_dv_d^{-1}u'_{d+1}$ or $w' = v_1^{-1}u_2v_2^{-1}\dots u_dv_d^{-1}u'_{d+1}$, respectively. If $w \in \{w_1, w_3\}$ and $u_d = u'_d\delta$ for an arrow δ then $w' = u_1v_1^{-1}u_2v_2^{-1}\dots u'_d$ or $w' = v_1^{-1}u_2v_2^{-1}\dots u'_d$, respectively.

In a similar way we can construct a walk w_L using the ideas on the other end of the walk w. Then we can compose the above constructions and obtain a walk $w_{RL} = w_{LR}$. Thus for a noninjective A-module $M(w)$ we have the following Auslander-Reiten sequence in mod(A)

$$0 \to M(w) \to M(w_R) \oplus M(w_L) \to M(w_{RL}) \to 0.$$

Dually one constructs three walks $w_{L^{-1}}$, $w_{R^{-1}}$, $w_{L^{-1}R^{-1}} = w_{R^{-1}L^{-1}}$ such that for a nonprojective A-module $M(w)$ there is the following Auslander-Reiten sequence in mod(A)

$$0 \to M(w_{L^{-1}R^{-1}}) \to M(w_{L^{-1}}) \oplus M(w_{R^{-1}}) \to M(w) \to 0.$$

Lemma 3.1. *Let* $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$. *If* $\mathcal{C}$ *is the starting component in* Γ_A *then there is no finite oriented cycle in* $\mathcal{C}$ *of irreducible morphisms.*

Proof. Since $\mathcal{C}$ is a connected component in Γ_A that contains all indecomposable projective vertices, we have for every vertex $M \in \mathcal{C}$ a sequence

$$S = X_0 \,—\!\!—\, X_1 \,—\!\!—\, \cdots \,—\!\!—\, X_{n-1} \,—\!\!—\, X_n = M$$

of indecomposable A-modules and irreducible morphisms, where S is a simple projective A-module and $X_i \,—\!\!—\, X_{i+1}$ denotes either $X_i \longrightarrow X_{i+1}$ or $X_{i+1} \longrightarrow X_i$ which is an irreducible morphism. We shall prove the lemma inductively on n.

If $n = 0$ then $M = S$ and S cannot lie on a finite oriented cycle in $\mathcal{C}$, because S is simple projective.

Now assume that for every module $N \in \mathcal{C}$ such that there is a sequence

$$S = X_0 \,—\!\!—\, X_1 \,—\!\!—\, \cdots \,—\!\!—\, X_{n-1} \,—\!\!—\, X_n = N$$

there is no finite oriented cycle in $\mathcal{C}$ containing N.

Suppose that $M \in \mathcal{C}$ and there is a sequence

$$S = X_0 \,—\!\!—\, X_1 \,—\!\!—\, \cdots \,—\!\!—\, X_{n-1} \,—\!\!—\, X_n \,—\!\!—\, X_{n+1} = M$$

and M lies on a finite oriented cycle in $\mathcal{C}$. Consider the case when $X_n \longrightarrow X_{n+1}$. If the oriented cycle $M \to Y_1 \to \cdots \to Y_m \to M$ satisfies $Y_m \cong X_n$ then we have that X_n lies on a finite oriented cycle in $\mathcal{C}$ which contradicts the inductive assumption. Thus we get that $Y_m \ncong X_n$ and there is an Auslander-Reiten sequence of the form $0 \to \tau M \to X_n \oplus Y_m \to M \to 0$ or M is projective. If each Y_j, $j = 1, \ldots, m$, and M is nonprojective then we have the following cycle $\tau(M) \to \tau(Y_1) \to \cdots \to \tau)(Y_m) \to \tau(M)$. If $\tau(Y_1) \cong X$ then we get a contradiction to the inductive assumption. Thus $\tau(Y_1) \cong Y_m$. In this case we have the following cycle $\tau(M) \to X_n \to M \to Y_1 \to \cdots \to Y_m \to \tau(Y_2) \to \cdots \to \tau(Y_m) \to \tau(M)$ in $\mathcal{C}$ which also contradicts the inductive assumption.

Now consider the case when Y_{j_0} is projective for some $j_0 \in \{1, \ldots, m\}$. Then Y_{j_0-1} is also a projective direct summand in $\mathrm{rad}(Y_{j_0})$ except the case $Y_{j_0} \cong P_{z-1}$ or $Y_{j_0} \cong P_{(a-1)'}$. It is clear that if none of Y_j is isomorphic to P_{z-1} or $P_{(a-1)'}$ then a simple projective A-module lies on a finite oriented cycle in $\mathcal{C}$ and this is impossible. Thus we can assume without loss of generality that $Y_{j_0} \cong P_{z-1}$. Then applying the Skowroński-Waschbüsch algorithm for computing irreducible morphisms we get that if there is a chain of irreducible homomorphisms $Y_{j_0} \cong Y_{j_0}(w(1)) \to X(w(2)) \to \cdots \to X(w(t))$ then we have $l(w(1)) < l(w(2)) < \cdots < l(w(t))$, where $l(w)$ stands for the length of the walk w. Thus Y_{j_0} cannot lie on a finite oriented cycle of irreducible morphisms.

In the case $Y_{j_0} \cong P_{(a-1)'}$ we repeat the above arguments. Consequently, we have that M cannot lie on a finite oriented cycle.

Now consider the case $X_{n+1} \to X_n$. Similar arguments to the above show that M cannot lie on a finite oriented cycle and this finishes the proof. □

Lemma 3.2. *Let $A \cong A^{(2)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$. If $\mathcal{C}$ is the ending component in Γ_A then there is no finite oriented cycle in $\mathcal{C}$.*

Proof. This is dual to the proof of Lemma 3.1. □

Proposition 3.3. *Let* $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ *and* $\mathcal{C}$ *be the starting component in* Γ_A. *If* $\mathcal{S}$ *is a postprojective slice in* $\mathcal{C}$ *and* M, N *are elements of* $\mathcal{S}$ *then for every* $0 \neq f : \tau^{-1}(N) \to M$ *we have* $f \in \operatorname{rad}^{\infty}(\tau^{-1}(N), M)$.

Proof. Since M, N belong to a postprojective slice $\mathcal{S}$ in $\mathcal{C}$, we have that there is a finite chain

$$M \cong X_0 \frac{\quad}{} X_1 \frac{\quad}{} \cdots \frac{\quad}{} X_n \cong N$$

with $X_0, X_1, \ldots, X_n \in \mathcal{S}$ and $X_i \frac{\quad}{} X_{i+1}$ is either $X_i \longrightarrow X_{i+1}$ or $X_{i+1} \longrightarrow X_i$ in $\mathcal{C}$. We shall prove inductively on n that there is not a nonzero homomorphism from $\tau^{-1}(N)$ to M that is a composition of finitely many irreducible homomorphisms.

If $n = 0$ then $M = N$. Suppose that there is a nonzero homomorphism $f : \tau^{-1}(M) \to M$ that is not contained in $\operatorname{rad}^{\infty}(\tau^{-1}(M), M)$. Then there exists a finite sequence $\tau^{-1}(M) \to Y_1 \to Y_2 \to \cdots \to Y_m \to M$ of indecomposable A-modules and irreducible homomorphisms. But there are also irreducible homomorphisms $M \to Z$ and $Z \to \tau^{-1}(M)$ with Z indecomposable. Thus we obtain that $\tau^{-1}(M)$ lies on a finite oriented cycle in $\mathcal{C}$ which contradicts Lemma 3.1. Therefore $f \in \operatorname{rad}^{\infty}(\tau^{-1}(M), M)$.

Now assume that if M, N satisfy the above assumptions and there is a sequence

$$M \cong X_0 \frac{\quad}{} X_1 \frac{\quad}{} \cdots \frac{\quad}{} X_n \cong N$$

with $n \leq n_0$ then for every $0 \neq f : \tau^{-1}(N) \to M$ we have f is not a composition of finitely many irreducible homomorphisms. Consider M, N as above with a sequence

$$M \cong X_0 \frac{\quad}{} X_1 \frac{\quad}{} \cdots \frac{\quad}{} X_{n_0} \frac{\quad}{} X_{n_0+1} \cong N$$

and let $0 \neq f : \tau^{-1}(N) \to M$. Suppose that f is a composition of finitely many irreducible homomorphisms..

Let $X_{n_0} \frac{\quad}{} X_{n_0+1}$ be an irreducible homomorphism $X_{n_0} \longrightarrow X_{n_0+1}$. We deduce from the fact that f is a composition of finitely many irreducible homomorphisms that there exists a finite sequence $\tau^{-1}(N) \to Y_1 \to \cdots \to Y_m \to M$ of indecomposable A-modules and irreducible homomorphisms whose composition is not zero. Thus we deduce from the Skowroński-Waschbüsch algorithm that for A-modules in $\mathcal{C}$ every irreducible homomorphism $g : U \to V$ between postprojective U, V is a monomorphism. Thus $\tau^{-1}(N)$ is isomorphic to a submodule of M. But we have a sequence of irreducible homomorphisms $X_{n_0} \to N \to \tau^{-1}(X_{n_0}) \to \tau^{-1}(N)$ whose composition is also a monomorphism. Thus we have a sequence of irreducible monomorphisms $\tau^{-1}(X_{n_0}) \to \tau^{-1}(N) \to Y_1 \to \cdots \to Y_m \to M$, which contradicts the inductive assumption. Thus f is not a composition of finitely many irreducible homomorphisms in this case.

Let $X_{n_0} \frac{\quad}{} X_{n_0+1}$ be an irreducible homomorphism $X_{n_0+1} \longrightarrow X_{n_0}$. We deduce again from the fact that f is a composition of finitely many irreducible homomorphisms that there exists a finite sequence $\tau^{-1}(N) \to Y_1 \to \cdots \to Y_m \to M$ of indecomposable

A-modules and irreducible homomorphisms whose composition is not zero. Then using similar reasoning as above we get a contradiction. Consequently, f is not a composition of finitely many irreducible homomorphisms and so $f \in \operatorname{rad}^{\infty}(\tau^{-1}(N), M)$ and this finishes the proof. □

Proposition 3.4. *Let $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the ending component in Γ_A. If $\mathcal{S}$ is a preinjective slice in $\mathcal{C}$ and M, N belong to $\mathcal{S}$ then for every $0 \neq f : M \to \tau(N)$ we have $f \in \operatorname{rad}^{\infty}(M, \tau(N))$.*

Proof. Dual arguments to those used in the proof of Proposition 3.3 show the proposition. □

Lemma 3.5. *Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the starting component in Γ_A. Let $u = \alpha_{1,z+1}\alpha_{1,z+2}\cdots\alpha_{1,p_1}$ if $z \neq p_1$ and $u = p_1$ if $z = p_1$. Let $v = \alpha'_{1,a+1}\alpha'_{1,a+2}\cdots\alpha'_{1,s_1}$ if $a \neq s_1$ and $v = s'_1$ if $a = s_1$. If $X_j = X(u^{-1}_{L^j})$ and $Y_j = Y(v_{R^j})$, $j = 0, 1, 2, \ldots$, then for every finite sequence $X_j \to Z_1 \to Z_2 \to \cdots \to Z_t$ or $Y_j \to Z_1 \to Z_2 \to \cdots \to Z_t$, $t \geq 1$, of indecomposable A-modules and irreducible homomorphisms we have that every irreducible homomorphism in the sequence is a monomorphism.*

Proof. Applying the Skowroński-Waschbüsch algorithm for computing irreducible homomorphisms, it is easy to see that for every j, e, d we have $l((u^{-1}_{L^j})_{L^d R^e}) < l((u^{-1}_{L^j})_{L^{d+1} R^e})$ and $l((u^{-1}_{L^j})_{L^d R^e}) < l((u^{-1}_{L^j})_{L^d R^{e+1}})$. Since an irreducible homomorphism between indecomposable A-modules is either mono or epic, we infer by the above inequalities that the required condition holds. □

A similar argument holds for v_{R^j}.

Lemma 3.6. *Let $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the ending component in Γ_A. Let $u = \alpha_{1,p_1}\cdots\alpha_{1,z+1}$ if $z \neq p_1$ and $u = p_1$ if $z = p_1$. Let $v = \alpha'_{1,s_1}\cdots\alpha'_{1,a+1}$ if $a \neq s_1$ and $v = s'_1$ if $a = s_1$. If $X^j = X(u_{L^{-j}})$ and $Y^j = Y(v^{-1}_{R^{-j}})$, $j = 0, 1, 2, \ldots$, then for every finite sequence $X^j \leftarrow Z^1 \leftarrow Z^2 \leftarrow \cdots \leftarrow Z^t$ or $Y^j \leftarrow Z^1 \leftarrow Z^2 \leftarrow \cdots \leftarrow Z^t$, $t \geq 1$, of indecomposable A-modules and irreducible homomorphisms we have that every irreducible homomorphism in the sequence is an epimorphism.*

Proof. This is dual to the proof of Lemma 3.5. □

Lemma 3.7. *(1) Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the starting component in Γ_A. Let u, v be as in Lemma 3.5. Then there are the following sequences of irreducible epimorphisms in $\mathcal{C}$:*

$$\cdots \to X(u^{-1}_{L^j R^{-d}}) \to X(u^{-1}_{L^j R^{-d+1}}) \to \cdots \to X(u^{-1}_{L^j R^{-2}}) \to X(u^{-1}_{L^j R^{-1}}) \to X(u^{-1}_{L^j}),$$

$$\cdots \to Y(v_{R^j L^{-d}}) \to Y(v_{R^j L^{-d+1}}) \to \cdots \to Y(v_{R^j L^{-2}}) \to Y(v_{R^j L^{-1}}) \to Y(v_{R^j}),$$

$j = 0, 1, 2, \ldots$.

(2) Let $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the ending component in Γ_A. Let u, v be as in Lemma 3.6. Then there are the following sequences of irreducible monomorphisms in $\mathcal{C}$:

$$X(u_{L^{-j}}) \to X(u_{L^{-j}R}) \to X(u_{L^{-j}R^2}) \to \cdots \to X(u_{L^{-j}R^d}) \to \cdots,$$

$$Y(v^{-1}_{R^{-j}}) \to Y(v^{-1}_{R^{-j}L}) \to Y(v^{-1}_{R^{-j}L^2}) \to \cdots \to Y(v^{-1}_{R^{-j}L^d}) \to \cdots,$$

$j = 0, 1, 2, \ldots$.

Proof. A direct application of the Skowroński-Waschbüsch algorithm proves the lemma. □

Lemma 3.8. *Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the starting component in Γ_A. Then the following conditions are satisfied:*

(1) If $\operatorname{rad}^\infty(P_c, P_d) \neq 0$ then $c \in \{z, z+1, \ldots, p_1\}$ and $d \in \{0, 1, \ldots, z-1\}$ or $c \in \{a', (a+1)', \ldots, s'_1\}$ and $d \in \{0', 1', \ldots, (a-1)'\}$.

(2) If $p = p_2 - p_1 + p_4 - p_3 + \cdots + p_{q-2} = p_{q-1} + l_1$ then for $w = u^{-1}\alpha''_{1,1}\cdots\alpha''_{1,x_1}$ we have $M(w) \cong P_z$ and $\operatorname{rad}^\infty(M(w)), M(w_{L^cR^d})) \neq 0$ if and only if $c \geq p$, where u is as in Lemma 3.5.

(3) If $s = s_2 - s_1 + s_4 - s_3 + \cdots + s_{r-2} - s_{r-1} + l_2$ then for $w = \alpha''^{-1}_{n-1,x_n-x_{n-1}}\cdots\alpha''^{-1}_{n-1,1}v$ we have $M(w) \cong P_{a'}$ and $\operatorname{rad}^\infty(M(w), M(w_{L^cR^d})) \neq 0$ if and only if $d \geq s$, where v is as in Lemma 3.5.

Proof. We have the following irreducible homomorphisms in $\mathcal{C}$: $P_{p_1} \to P_{p_1-1} \to \cdots \to P_{z+1} \to P_z$ and $X(u^{-1}) \to P_{z-1} \to P_{z-2} \to \cdots \to P_0$. Since every composition of a finite sequence of irreducible homomorphisms from P_z to an indecomposable A-module Y is a monomorphism, we have that the epimorphism $P_z \to X(u^{-1})$ belongs to $\operatorname{rad}^\infty(P_z, X(u^{-1}))$. Thus we get $c \in \{z, \ldots, p_1\}$ and $d \in \{0, \ldots, z-1\}$ if $\operatorname{rad}^\infty(P_c, P_d) \neq 0$ or similarly $c \in \{a', \ldots, s'_1\}$ and $d \in \{0', \ldots, (a-1)'\}$, proving (1).

In order to prove condition (2) we have that $M(w_{L^p}) \cong X((u^{-1})_{R^t})$ for some $t \geq 2$. Then applying Lemmas 3.5, 3.7(1), we have that $\operatorname{rad}^\infty(P_z, M(w_{L^cR^d})) \neq 0$ if and only if $c \geq p$. □

Similarly one can prove condition (3).

Lemma 3.9. *Let $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the ending component in Γ_A. Then the following conditions are satisfied:*

(1) If $\operatorname{rad}^\infty(E_c, E_d) \neq 0$ then $c \in \{0, 1, \ldots, z-1\}$ and $d \in \{z, z+1, \ldots, p_1\}$ or $c \in \{0', 1', \ldots, (a-1)'\}$ and $d \in \{a', (a+1)', \ldots, s'_1\}$.

(2) If $p = p_2 - p_1 + p_4 - p_3 + \cdots + p_{q-2} - p_{q-1} + l_1$ then for $w = u\alpha''^{-1}_{1,1}\cdots\alpha''^{-1}_{1,x_1}$ we have $M(w) \cong E_z$ and $\operatorname{rad}^\infty(M(w_{L^{-c}R^{-d}}), M(w)) \neq 0$ if and only if $c \geq p$.

(3) em If $s = s_2 - s_1 + s_4 - s_3 + \cdots + s_{r-2} - s_{r-1} + l_2$ then for $w = \alpha''_{n-1,x_n-x_{n-1}}\cdots\alpha''_{n-1,1}v^{-1}$ we have $M(w) \cong E_{a'}$ and $\operatorname{rad}^\infty(M(w_{R^{-d}L^{-c}}), M(w)) \neq 0$ if and only if $d \geq s$.

Proof. Dual arguments to those used in the proof of Lemma 3.8 prove the lemma. □

Proposition 3.10. *(1) Let* $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ *and* $\mathcal{C}$ *be the starting component in* Γ_A. *Let* P, P' *be indecomposable projective* A*-modules. If for some nonnegative integers* l, t *we have* $\tau^{-l}(P) \cong M(w)$ *and* $\tau^{-t}(P') \cong M(\bar{w})$ *and* $\mathrm{rad}^{\infty}(M(w), M(\bar{w})) \neq 0$ *then there are integers* c, d *such that* $\bar{w} = w_{L^c R^d}$ *and* $c \geq p$ *or* $d \geq s$.

(2) Let $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ *and* $\mathcal{C}$ *be the ending component in* Γ_A. *Let* E, E' *be indecomposable injective* A*-modules. If for some nonnegative integers* l, t *we have* $\tau^{l}(E) \cong M(w)$ *and* $\tau^{t}(E') \cong M(\bar{w})$ *and* $\mathrm{rad}^{\infty}(M(\bar{w}), M(w)) \neq 0$ *then there are integers* c, d *such that* $\bar{w} = w_{L^{-c} R^{-d}}$ *and* $c \geq p$ *or* $d \geq s$.

Proof. Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$. Consider two indecomposable postprojective A-modules $\tau^{-l}(P) \cong M(w)$ and $\tau^{-t}(P') \cong M(\bar{w})$. Then we deduce from the Skowroński-Waschbüsch algorithm that $w = u^{-1}_{L^j} w'$, for some $j \geq 0$, or $w = w'' v_{R^i}$, for some $i \geq 0$, because only in these cases could $\mathrm{rad}^{\infty}(M(w), M(\bar{w})) \neq 0$. Consider the case $w = u^{-1}_{L^j} w'$. Observe that for $(u^{-1}_{L^j} w')_{L^c}$ with $c < p$ we have $\mathrm{rad}^{\infty}(M(u^{-1}_{L^j} w'), M((u^{-1}_{L^j} w')_{L^c})) = 0$. Indeed, since $M(w)$ is postprojective, we have that w' is nontrivial. Furthermore, $w' = \alpha''_{1,1} \tilde{w}'$, because we have an epimorphism from $M(w)$ onto $X(u^{-1}_{L^j})$. Moreover, it is clear that $(u^{-1}_{L^j} w')_{L^c} = u^{-1}_{L^{j+c}} w'$. If $c < p$ then the nonzero homomorphism is a composition of an epimorphism $f : M(w) \to X(u^{-1}_{L^j})$ with a monomorphism $g : X(u^{-1}_{L^j}) \to M(u^{-1}_{L^{c+j}} w')$. But $u^{-1}_{L^{c+j}} = (u^{-1}_{L^{p+j}})_{L^{c'}}$ if and only if $c \geq p$, where $c' + p = c$. Thus we get that $\bar{w} = w_{L^c R^d}$ and $c \geq p$.

A similar analysis in the case $w = w'' v_{R^i}$ shows that $\bar{w} = w_{L^c R^d}$ and $d \geq s$, and this finishes the proof of condition (1). □

In order to prove condition (2) we apply dual arguments.

4 Tilting modules

Following Happel and Ringel [9] (see also [5] we shall call a finitely generated A-module T_A a *tilting module* (respectively, *cotilting module*) if it satisfies the following conditions

(1) $\mathrm{proj.dim}(T_A) \leq 1$ (respectively, $\mathrm{inj.dim}(T_A) \leq 1$)

(2) $\mathrm{Ext}^1_A(T, T) = 0$

(3) The number of the nonisomorphic indecomposable direct summands of T_A is equal to the rank of the Grothendieck group $K_0(A)$ of A.

Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the starting component in Γ_A. Let $\mathcal{S}$ be a postprojective slice in $\mathcal{C}$. Assume that $M_{\mathcal{S}} = \bigoplus_{i=1}^{t} M(w_i)$ and the indecomposable modules $M(w_i)$ are enumerated in such a way that $M(w_1)$ belongs to the τ-orbit of P_{z-1} and $M(w_t)$ belongs to the τ-orbit of $P_{(a-1)'}$. Then $\mathcal{S}$ can be presented as

$$M(w_1) \;\text{——}\; M(w_2) \;\text{——}\cdots\text{——}\; M(w_t),$$

where $M(w_i) \text{——} M(w_{i+1})$ is a left arrow $M(w_i) \leftarrow M(w_{i+1})$ or a right arrow $M(w_i) \to M(w_{i+1})$, $i = 1, \ldots, t-1$. Then we can attach to every element $M(w_i)$ of $\mathcal{S}$ two numbers:

$l_{\mathcal{S}}(M(w_i))$ is the number of the left arrows in $M(w_1)$ ——$M(w_2)$ ——$\cdots$ ——$M(w_i)$ and $r_{\mathcal{S}}(M(w_i))$ is the number of the right arrows in $M(w_i)$ ——$M(w_{i+1})$ ——$\cdots$ ——$M(w_t)$.

If $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ then we can dually define $l_{\mathcal{S}}(M(w_i))$ and $r_{\mathcal{S}}(M(w_i))$ for a preinjective slice $\mathcal{S}$ in the ending component $\mathcal{C}$.

Theorem 4.1. *(1) Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the starting component in Γ_A. Let $\mathcal{S}$ be a postprojective slice in $\mathcal{C}$. Then the following conditions hold:*

(1i) If for every $M(w_i) \in \mathcal{S}$, $i = 1,\dots,t$, we have $l_{\mathcal{S}}(M(w_I)) \leq p$ and $r_{\mathcal{S}}(M(w_i)) \leq s$ then the slice module $M_{\mathcal{S}} = \bigoplus_{i=1}^t M(w_i)$ is a tilting A-module.

(1ii) If for every $M(w_i) \in \mathcal{S}$, $i = 1,\dots,t$, such that $l_{\mathcal{S}}(M(w_i)) > p$ there is no epimorphism $f : \tau^{-1}(M(w_i)) \to X(u^{-1}_{L^j})$, $j \geq 1$, and for every $M(w_i) \in \mathcal{S}$, $i = 1,\dots,t$, such that $r_{\mathcal{S}}(M(w_i)) > s$ there is no epimorphism $g : \tau^{-1}(M(w_i)) \to Y(v_{R^j})$, $j \geq 1$, then the slice module $M_{\mathcal{S}} = \bigoplus_{i=1}^t M(w_i)$ is a tilting A-module.

(1iii) If there exists $i_0 \in \{1,\dots,t\}$ such that $l_{\mathcal{S}}(M(w_{i_0})) > p$ and there is an epimorphism $f : \tau^{-1}(M(w_{i_0})) \to X(u^{-1}_{L^j})$ for some $j \geq 1$ then the slice module $M_{\mathcal{S}} = \bigoplus_{i=1}^t M(w_i)$ is not a tilting A-module.

(1iv) If there exists $i_0 \in \{1,\dots,t\}$ such that $r_{\mathcal{S}}(M(w_{i_0})) > s$ and there is an epimorphism $g : \tau^{-1}(M(w_{i_0})) \to Y(v_{R^j})$ for some $j \geq 1$ then the slice module $M_{\mathcal{S}} = \bigoplus_{i=1}^t M(w_i)$ is not a tilting A-module.

(2) Let $A \cong A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ and $\mathcal{C}$ be the ending component in Γ_A. Let $\mathcal{S}$ be a preinjective slice in $\mathcal{C}$. Then the following conditions hold:

(2i) If for every $M(w_i) \in \mathcal{S}$, $i = 1,\dots,t$, we have $l_{\mathcal{S}}(M(w_i)) \leq s$ and $r_{\mathcal{S}}(M(w_i)) \leq p$ then the slice module $M_{\mathcal{S}} = \bigoplus_{i=1}^t M(w_i)$ is a cotilting A-module.

(2ii) If for every $M(w_i) \in \mathcal{S}$, $i = 1,\dots,t$, such that $l_{\mathcal{S}}(M(w_i)) > s$ there is no monomorphism $f : Y(v^{-1}_{R^{-j}}) \to \tau(M(w_i))$, $j \geq 1$, and for every $M(w_i) \in \mathcal{S}$, $i = 1,\dots,t$, such that $r_{\mathcal{S}}(M(w_i)) > p$ there is no monomorphism $g : X(u_{L^{-j}}) \to \tau(M(w_i))$, $j \geq 1$, then the slice module $M_{\mathcal{S}} = \bigoplus_{i=1}^t M(w_i)$ is a cotilting A-module.

(2iii) If there exists $i_0 \in \{1,\dots,t\}$ such that $l_{\mathcal{S}}(M(w_{i_0})) > s$ and there is a monomorphism $f : Y(v^{-1}_{R^{-j}}) \to \tau(M(w_{i_0}))$ for some $j \geq 1$ then the slice module $M_{\mathcal{S}} = \bigoplus_{i=1}^t M(w_i)$ is not a cotilting A-module.

(2iv) If there exists $i_0 \in \{1,\dots,t\}$ such that $r_{\mathcal{S}}(M(w_{i_0})) > p$ and there is a monomorphism $g : X(u_{L^{-j}}) \to \tau(M(w_{i_0}))$ for some $j \geq 1$ then the slice module $M_{\mathcal{S}} = \bigoplus_{i=1}^t M(w_i)$ is not a cotilting A-module.

Proof. Let $A \cong A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$. Consider the starting component $\mathcal{C}$ in Γ_A. Let $\mathcal{S} = \{M(w_i)\}_{i=1}^t$ be a postprojective slice in $\mathcal{C}$.

Suppose that for every $M(w_i) \in \mathcal{S}$, $i = 1,\dots,t$, we have $l_{\mathcal{S}}(M(w_i)) \leq p$ and $r_{\mathcal{S}}(M(w_i)) \leq s$. Then for every two $M(w_i)$, $M(w_j)$ we have that $\operatorname{rad}^\infty(\tau^{-1}(M(w_i)), M(w_j)) = 0$. Indeed, if there is a nonzero homomorphism $f : \tau^{-1}(M(w_i)) \to M(w_j)$ then we infer by Proposition 3.10(1) that $w_j = ((w_i)_{LR})_{L^c R^d}$ and $c \geq p$ or $d \geq s$. But if $l_{\mathcal{S}}(M(w_i) \leq p$ then we replace $\mathcal{S}$ onto $\mathcal{S}'$ in such a way that we consider $\tau^{-1}(M(w_i))$ instead of $M(w_i)$ and the other elements are not changed. Then we have $l_{\mathcal{S}'}(\tau^{-1}(M(w_i))) \leq p-1$. Thus

$w_j \neq ((w_i)_{LR})_{L^cR^d}$ for $c \geq p$. Similarly we obtain that $w_j \neq ((w_i)_{LR})_{L^cR^d}$ for some $d \geq s$. Consequently, $\mathrm{rad}^\infty(\tau^{-1}(M(w_i)), M(w_j)) = 0$ for each pair i, j. Then we infer by the Auslander-Reiten formulae that $D\underline{\mathrm{Hom}}_A(\tau^{-1}(M_\mathcal{S}), M_\mathcal{S}) \cong \mathrm{Ext}^1_A(M_\mathcal{S}, M_\mathcal{S}) = 0$. Moreover, we know from Corollary 2.5 that $\mathrm{proj.dim}(M_\mathcal{S}) \leq 1$. Therefore $M_\mathcal{S}$ is a tilting A-module which proves (1i).

Now assume that for every $i = 1, \ldots, t$ we have that either $l_\mathcal{S}(M(w_i)) \leq p$ or $l_\mathcal{S}(M(w_i)) > p$ and there is no epimorphism $f : \tau^{-1}(M(w_i)) \to X(u^{-1}_{L^j})$, $j \geq 1$. The same condition is assumed on $M(w_i)$ with respect to $r_\mathcal{S}(M(w_i))$. Then for each pair i, j of integers we have either $l_\mathcal{S}(M(w_i)) \leq p$ and then, as in the proof of (1i), we get $\underline{\mathrm{Hom}}_A(\tau^{-1}(M(w_i)), M(w_j)) = 0$ or $l_\mathcal{S}(M(w_i)) > p$ and there is no epimorphism $f : \tau^{-1}(M(w_i)) \to X(u^{-1}_{L^j})$, $j \geq 1$. Thus replacing $M(w_i)$ by $\tau^{-1}(M(w_i))$ we get a slice $\mathcal{S}'$ with $l_{\mathcal{S}'}(M(w_i)) \geq p$. If $\mathrm{rad}^\infty(\tau^{-1}(M(w_i)), M(w_j)) \neq 0$ then we have a basis homomorphism that must be a composition of an epimorphism $f : \tau^{-1}(M(w_i)) \to X(u^{-1}_{L^j})$, $j \geq 0$, with a monomorphism $g : X(u^{-1}_{L^j}) \to M(w_j)$. Thus we deduce from our assumptions that the only possibility for this case is that $j = 0$. But in this case $g : X(u^{-1}) \to M(w_j)$ factorizes through P_{z-1}. Therefore we have $\underline{\mathrm{Hom}}_A(\tau^{-1}(M(w_i)), M(w_j)) = 0$.

A similar analysis with $r_\mathcal{S}(M(w_i))$ implies that $0 = D\underline{\mathrm{Hom}}_A(\tau^{-1}(M_\mathcal{S}), M_\mathcal{S}) = \mathrm{Ext}^1_A (M_\mathcal{S}, M_\mathcal{S})$. Consequently, $M_\mathcal{S}$ is a tilting A-module, by of Proposition 3.3 so (1ii) is proved.

If there exists $1 \leq i_0 \leq t$ such that $l_\mathcal{S}(M(w_{i_0})) > p$ and there is an epimorphism $f : \tau^{-1}(M(w_{i_0})) \to X(u^{-1}_{L^{j_0}})$ for some $j_0 \geq 1$ then the composed homomorphism $gf \neq 0$ for a monomorphism $g : X(u^{-1}_{L^{j_0}}) \to M((w_{i_0})_{L^{l_\mathcal{S}(M(w_{i_0}))}})$. Moreover, there is an integer d such that $M((w_{i_0})_{L^{l_\mathcal{S}(M(w_{i_0}))}R^d}) \cong M(w_1)$ and we have a monomorphism $g' : X(u^{-1}_{L^{j_0}}) \to M(w_1)$. Thus $g'f \neq 0$ and also $\underline{g'f} \neq 0$, because $j_0 \geq 1$. Therefore $\underline{\mathrm{Hom}}_A(\tau^{-1}(M(w_{i_0})), M(w_1)) \neq 0$ and we deduce from the Auslander-Reiten formulae that $\mathrm{Ext}^1_A(M(w_1), M(w_{i_0})) \neq 0$. Therefore $\mathrm{Ext}^1_A(M_\mathcal{S}, M_\mathcal{S}) \neq 0$ and $M_\mathcal{S}$ is not a tilting A-module. Consequently, condition (1iii) is proved.

A similar analysis to the above used in the proof of (1iii) shows condition (1iv), and thus (1) is proved. □

Dual arguments prove (2).

Theorem 4.2. *(1) Let $A \cong A^{(1)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$ and $\mathcal{C}$ be the starting component in Γ_A. Let $\mathcal{S}$ be a postprojective slice in $\mathcal{C}$. If $M_\mathcal{S} = \bigoplus_{i=1}^t M(w_i)$ and for each $i = 1, \ldots, t$ there are positive integers b_i, c_i, d_i, e_i such that $M(w_i) \cong X(u^{-1}_{L^{b_i}R^{c_i}}) \cong Y(v_{L^{d_i}R^{e_i}})$ then the slice module $M_\mathcal{S}$ is not tilting.*

(2) Let $A \cong A^{(2)}_{(\underline{p}, l_1, \underline{x}, \underline{s}, l_2)}$ and $\mathcal{C}$ be the ending component in Γ_A. Let $\mathcal{S}$ be a preinjective slice in $\mathcal{C}$. If $M_\mathcal{S} = \bigoplus_{i=1}^t M(w_i)$ and for each $i = 1, \ldots, t$ there are positive integers b_i, c_i, d_i, e_i such that $M(w_i) \cong X(u_{L^{-b_i}R^{-c_i}}) \cong Y(v^{-1}_{L^{-d_i}R^{-e_i}})$ then the slice module $M_\mathcal{S}$ is not cotilting.

Proof. If the elements of $\mathcal{S}$ form the following subquiver $M(w_1) \text{---} M(w_2) \text{---} \cdots \text{---} M(w_t)$

in $\mathcal{C}$ then we have that $l_{\mathcal{S}}(M(w_t)) > p$ or $r_{\mathcal{S}}(M(w_1)) > s$. Indeed, if $l_{\mathcal{S}}(M(w_t)) \leq p$ and $r_{\mathcal{S}}(M(w_1)) \leq s$ then the quiver $Q^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$ has at most $p+s$ arrows and this is impossible by the definitions of p and s.

Now consider the case $l_{\mathcal{S}}(M(w_t)) > p$. Since $M(w_t) \cong X(u^{-1}_{L^{b_i}R^{c_i}})$ by the assumption, we have an epimorphism $f : \tau^{-1}(M(w_t)) \to X(u^{-1}_{L^{b_i+1}})$, $b_i \geq 1$. Thus we infer by Theorem 4.1(1iii) that $M_{\mathcal{S}}$ is not tilting.

If $r_{\mathcal{S}}(M(w_1)) > s$ then similarly it follows that $M_{\mathcal{S}}$ is not tilting. Consequently, condition (1) is proved. □

Dually one can prove (2).

Corollary 4.3. *(1) There are only finitely many (up to isomorphism) postprojective slice $A^{(1)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$-modules that are tilting.*

(2) There are only finitely many (up to isomorphism) preinjective slice $A^{(2)}_{(\underline{p},l_1,\underline{x},\underline{s},l_2)}$-modules that are cotilting.

Proof. In order to prove (1) it is easy to see that there are only finitely many (up to isomorphism) postprojective slices $\mathcal{S}$ in the starting component $\mathcal{C}$ that do not satisfy the assumptions of Theorem 4.2(1) and dually with condition (2). □

References

[1] I. Assem: "Tilting theory - an introduction", In: *Topics in Algebras*, Banach Center Publications, Vol. 26, Part I, PWN, Warszawa, 1990, pp. 127–180.

[2] M. Auslander and I. Reiten: "Representation theory of artin algebras III", *Comm. Alg.*, Vol. 3, (1975), pp. 239–294.

[3] M. Auslander and I. Reiten: "Representation theory of artin algebras IV", *Comm. Alg.*, Vol. 5, (1977), pp. 443–518.

[4] M. Auslander, I. Reiten and S.O. Smalø: *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math., Vol. 36, Cambridge Univ. Press, Cambridge, 1995.

[5] K. Bongartz: *Tilted algebras*, LNM 903, Springer, Berlin, 1981, pp. 26-38.

[6] M.C.R. Butler and C.M. Ringel: "Auslander-Reiten sequences with few middle terms and applications to string algebras", *Comm. Alg.*, Vol. 15, (1987), pp. 145–179.

[7] P. Dowbor and A. Skowroński: "Galois coverings of representation-infinite algebras", *Comment. Math. Helv.*, Vol. 62, (1987), pp. 311–337.

[8] P. Gabriel: *Auslander-Reiten sequences and representation-finite algebras*, LNM 831, Springer, Berlin, 1980, pp. 1–71.

[9] D. Happel and C.M. Ringel: "Tilted algebras", *Trans. Amer. Math. Soc.*, Vol. 274, (1982), pp. 399–443.

[10] F. Huard: "Tilted gentle algebras", *Comm. Alg.*, Vol. 26(1), (1998), pp. 63-72.

[11] F. Huard and Sh. Liu: "Tilted special biserial algebras", *J. Algebra*, Vol. 217, (1999), pp. 679–700.

[12] F. Huard and Sh. Liu: "Tilted string algebras", *J. Pure Appl. Algebra*, Vol. 153, (2000), pp. 151–164.

[13] Z. Pogorzały and M. Sufranek: "Starting and ending components of the Auslander-Reiten quivers of a class of special biserial algebras", *Colloq. Math.*, Vol. 99(1), (2004), pp. 111–144.

[14] C.M. Ringel: *Tame algebras and integral quadratic forms*, LNM 1099, Springer, Berlin, 1984.

[15] J. Schröer: "Modules without self-extensions over gentle algebras", *J. Algebra*, Vol. 216, (1999), pp. 178–189.

[16] A. Skowroński: "Generalized standard Auslander-Reiten components", *J. Math. Soc. Japan*, Vol. 46, (1994), pp. 517-543.

[17] A. Skowroński and J. Waschbüsch: "Representation-finite biserial algebras", *J. Reine Angew. Math.*, Vol. 345, (1983), pp. 172–181.

[18] B. Wald and J. Waschbüsch: "Tame biserial algebras", *J. Algebra*, Vol. 95, (1985), pp. 480–500.