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Differential invariants of generic hyperbolic Monge–Ampère equations

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Abstract: In this paper basic differential invariants of generic hyperbolic Monge–Ampère equations with respect to contact transformations are constructed and the equivalence problem for these equations is solved.

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1 Introduction

With this paper we start a systematic study of differential invariants of Monge–Ampère equations, with our objective being the classification problem, methods of integration, and other applications. Complete proofs of the results announced in [16] are presented. We are interested in the classical case of two independent variables. The Monge–Ampère

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equations merit special attention due to a large spectrum of various applications, first of all, in differential geometry and mathematical physics. Moreover, they form a natural testing area for new methods emerging in the modern theory of nonlinear PDE's.

In spite of more than 200 years of history of Monge–Ampère equations and numerous publications devoted to them, it would be an exaggeration to say that their nature is well understood. An important success was establishing the existence and uniqueness theorems by Lewy and others (see [3, 10] for local aspects and [22] for global ones). The classical Monge integration method was modernized by Matsuda [17, 18] and Morimoto [20], etc. Our interest in differential invariants is motivated not only by the classification problem but, no less, by hopes that they could illuminate many aspects of the theory of Monge–Ampère equations.

According to [24] (see also [1]) scalar differential invariants provide a key to solving the classification problem for any kind of geometrical structures. In fact, geometrical structures of a given type are classified by solutions of a naturally associated *classifying* (differential) equation, which describes "family ties" connecting the corresponding scalar differential invariants. More exactly, scalar differential invariants are smooth functions on the *classifying diffiety*, which is the infinite prolongation of the classifying equation. This diffiety generally has singularities and its singular strata classify those geometrical structures that possess nontrivial symmetries. Each of these strata is also an infinitely prolonged differential equation in a lesser number of independent variables. For instance, homogeneous structures correspond to the zero-dimensional case. So the classification problem consists of a complete description of all strata composing the classifying diffiety, and therefore involves a complete symmetry analysis of the geometric structures under consideration. The interested reader will find an illustration of the above said in [25] where plane 3-webs, a rather simple geometrical structure, is considered.

The classification problem for Monge–Ampère equations dates back to Sophus Lie. For modern proofs of Lie's theorems, classification problems for various strata of Monge-Ampère equations see, e.g., [6–9, 13–15, 21, 23] and references therein. Directly using the geometry of jet bundles, in this paper we interpret a hyperbolic Monge–Ampère equation as a pair of 2-dimensional, skew-orthogonal, non-lagrangian subdistributions of the contact distribution on a 5-dimensional contact manifold. This pair of subdistributions was considered by other authors from a different point of view. See, for instance, [12, 14, 19]. We look for more than just scalar differential invariants of Monge–Ampère equations with respect to the group of contact transformations. Here, we limit ourself to the case of generic hyperbolic equations, which is motivated by two reasons. First, the study of singular strata very much benefits from the knowledge of the generic one. Second, for the hyperbolic equations, differential invariants are more easily visible due to the existence of bicharacteristics.

Differential invariants found in this paper give a solution of the classification problem for generic hyperbolic equations. This solution requires substantial computer support in the analysis of concrete cases and further work is necessary to improve its efficiency.

Differential invariants for elliptic and parabolic Monge–Ampère equations can be obtained more or less straightforwardly by following the approach developed in this paper. This idea and the study of singular strata will be the subject of subsequent publications.

2 Preliminaries

Below, all manifolds and maps are supposed to be smooth. By $[f]^k_p$, $k = 0, 1, 2, \dots, \infty$, we denote the k-jet of a map f at a point p. $\mathbb{R}$ stands for the field of real numbers, and $\mathbb{R}^n$ for the n-dimensional arithmetic space.

2.1 Jet bundles

Here we recall necessary definitions and facts about jet bundles, see [4, 5].

Let M be an n-dimensional manifold, E an $n+m$-dimensional manifold and

$$\pi : E \longrightarrow M .$$

a fiber bundle. By

$$\pi_k : J^k\pi \to M , \quad \pi_k : [S]^k_p \mapsto p , \quad k = 0, 1, 2, \dots$$

we denote the bundle of all k-jets of sections of π. For any $l > m \geq 0$, the natural projection is defined as

$$\pi_{l,m} : J^l\pi \to J^m\pi , \quad \pi_{l,m} : [S]^l_p \mapsto [S]^m_p .$$

Any section S of π generates the section j_kS of the bundle π_k by the formula

$$j_kS : p \mapsto [S]^k_p .$$

Put

$$L^k_S = \operatorname{Im} j_kS .$$

Let θ_{k+1} be an arbitrary point of $J^{k+1}\pi$, $\theta_k = \pi_{k+1,k}(\theta_{k+1})$, and $T_{\theta_k}(J^k\pi)$ the tangent space to $J^k\pi$ at the point θ_k. Then θ_{k+1} defines the subspace $K_{\theta_{k+1}} \subset T_{\theta_k}(J^k\pi)$ by the formula

$$K_{\theta_{k+1}} = T_{\theta_k}(L^k_S) .$$

Clearly, θ_{k+1} is identified with $K_{\theta_{k+1}}$. It is easy to prove that

$$T_{\theta_k}(J^k\pi) = K_{\theta_{k+1}} \oplus T_{\theta_k}(\pi_k^{-1}(p)) . \tag{1}$$

Consider all submanifolds of the form L^k_S containing θ_k. The subspace spanned by their tangent spaces $T_{\theta_k}(L^k_S)$ is denoted by $\mathcal{C}(\theta_k)$ and it is called the *Cartan plane at* θ_k. The distribution

$$\mathcal{C}_k : \theta_k \mapsto \mathcal{C}(\theta_k)$$

is called the *Cartan distribution on* $J^k\pi$. The distribution $\mathcal{C}_k$, $k \geq 1$, can be defined as the kernel of the *Cartan form*

$$U_k = \mathrm{pr}_2 \circ (\pi_{k,k-1})_* ,$$

where $\mathrm{pr}_2 : T_{\theta_{k-1}}(J^{k-1}\pi) \to T_{\theta_{k-1}}(\pi^{-1}_{k-1}(p))$ is the projection generated by direct sum decomposition (1).

2.2 The contact structure

Consider the trivial bundle

$$\tau : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}^2 , \quad \tau : (\, x, y, z \,) \mapsto (\, x, y \,) .$$

By x, y, z, $p = z_x$, $q = z_y$, $r = z_{xx}$, $s = z_{xy}$, $t = z_{yy}$ we denote the standard coordinates in $J^2\tau$.

The Cartan distribution C_1 on $J^1\tau$ is identical to the *contact structure* on $J^1\tau$. The corresponding contact 1-form U_1 has the canonical form

$$U_1 = dz - p\,dx - q\,dy .$$

in the standard coordinates.

A diffeomorphism $\varphi : J^1\tau \to J^1\tau$ is called a *contact transformation* if it preserves the Cartan distribution. Obviously, a diffeomorphism φ is a contact transformation iff there exist a nowhere vanishing function λ such that

$$\varphi^*(U_1) = \lambda\, U_1 .$$

Any contact transformation φ can be lifted to the diffeomorphism

$$\varphi_\tau^{(1)} : J^2\tau \longrightarrow J^2\tau$$

by the formula

$$\varphi_\tau^{(1)} : \theta_2 \equiv K_{\theta_2} \mapsto \varphi_*(K_{\theta_2}) \equiv \tilde{\theta}_2 = \varphi_\tau^{(1)}(\theta_2) .$$

If φ is defined on an open set $V \subset J^1\tau$, then $\varphi_\tau^{(1)}$ is defined on an open, everywhere dense subset of $\tau_{2,1}^{-1}(V)$.

A vector field Z in $J^1\tau$ is a *contact vector field* if its flow φ_t consists of contact transformations. Clearly, Z is a contact vector field iff there exist a function λ such that

$$L_Z(U_1) = \lambda\, U_1 ,$$

where L_Z is the Lie derivative with respect to Z.

There exists a natural one-to-one correspondence between the set of all contact vector fields in $J^1\tau$ and the set of all functions in $J^1\tau$. It is defined by the formula

$$Z \mapsto f = Z \lrcorner\, U_1 .$$

The function $f = Z \lrcorner\, U_1$ is called the *generating function of the contact vector field* Z. The contact vector field Z corresponding to f is denoted by Z_f. In standard coordinates, the field Z_f is given by the formula

$$Z_f = -f_p \frac{\partial}{\partial x} - f_q \frac{\partial}{\partial y} + (f - p f_p - q f_q) \frac{\partial}{\partial z} + (f_x + p f_z) \frac{\partial}{\partial p} + (f_y + q f_z) \frac{\partial}{\partial q} . \qquad (2)$$

2.3 Operations over vector-valued forms

Let M be a smooth n-dimensional manifold, $\Lambda^i(M)$ the $C^\infty(M)$-module of i-forms on M and $D(M)$ the $C^\infty(M)$-module of vector fields on M. Let $\alpha \in \Lambda^k(M)$, $\beta \in \Lambda^r(M)$, and $X, Y \in D(M)$. Then the Frölicher–Nijenhuis bracket $[\![\cdot\,,\cdot]\!]$ of the vector-valued forms $\alpha \otimes X$ and $\beta \otimes Y$ is defined by the formula

$$\begin{aligned}[\![\alpha \otimes X,\, \beta \otimes Y]\!] = \alpha \wedge \beta \otimes \big[X, Y\big] + \alpha \wedge X(\beta) \otimes Y - Y(\alpha) \wedge \beta \otimes X \\ + (-1)^k d\alpha \wedge (X \lrcorner \beta) \otimes Y - (-1)^k (Y \lrcorner \alpha) \wedge d\beta \otimes X\,,\end{aligned}$$

see [2].

The contraction $\lrcorner$ of forms $\alpha \otimes X$ and $\beta \otimes Y$ is defined by the formula

$$(\alpha \otimes X) \lrcorner (\beta \otimes Y) = \alpha \wedge (X \lrcorner \beta) \otimes Y\,.$$

2.4 Projectors and their curvatures

The following simple construction allows one to associate a vector valued 2-form with a projector. Namely, let $P, Q \in D(M)$ be endomorphisms of the $C^\infty(M)$-module $D(M)$ such that $QP = 0$. Then

$$\Omega_{Q,P}(X, Y) = Q[P(X), P(Y)], \quad X, Y \in D(M), \tag{3}$$

obviously, is skew-symmetric and $C^\infty(M)$-bilinear, i.e., a vector valued form. More precisely, it takes values in $\operatorname{Im} Q \subset D(M)$. If $P : D(M) \to D(M)$ is a projector, i.e., $P^2 = P$, then the associated *curvature form* of P is defined to be

$$\mathcal{R}_P = \Omega_{I-P,P} \tag{4}$$

with $I = \mathrm{id}_{D(M)}$.

3 Hyperbolic Monge–Ampère equations

3.1 Monge–Ampère equations

The Monge–Ampère equation is a partial differential equation of the form

$$N(z_{xx} z_{yy} - z_{xy}^2) + A z_{xx} + B z_{xy} + C z_{yy} + D = 0\,, \tag{5}$$

where x, y are independent variables, z is a dependent variable, $z_{xx} = \partial^2 z/\partial x^2$, $z_{xy} = \partial^2 z/\partial x\,\partial y$, $z_{yy} = \partial^2 z/\partial y^2$, and coefficients N, A, B, C, D are functions of x, y, z, $z_x = \partial z/\partial x$ and $z_y = \partial z/\partial y$.

We identify equation (5) with the submanifold $\mathcal{E}$ of the jet bundle $J^2\tau$ determined by the equation

$$N(rt - s^2) + Ar + Bs + Ct + D = 0\,. \tag{6}$$

Obviously,

$$\tau_{2,1}(\mathcal{E}) = J^1\tau\,.$$

Let $\theta_2 \in \mathcal{E}$, $\tau_{2,1}(\theta_2) = \theta_1$, and F_{θ_1} be the fiber of the projection $\tau_{2,1}$ over the point $\theta_1 \in J^1\tau$. Then the subspace

$$\mathrm{Smbl}_{\theta_2}\,\mathcal{E} = T_{\theta_2}\mathcal{E} \cap T_{\theta_2}F_{\theta_1},$$

where $T_{\theta_2}\mathcal{E}$ is the tangent space to $\mathcal{E}$ at θ_2 is called the *symbol of the equation* $\mathcal{E}$ *at the point* $\theta_2 \in \mathcal{E}$. In terms of standard coordinates, $\mathrm{Smbl}_{\theta_2}\,\mathcal{E}$ is described by the linear equation

$$N(t\tilde{r} + r\tilde{t} - 2s\tilde{s}) + A\tilde{r} + B\tilde{s} + C\tilde{t} = 0, \tag{7}$$

where $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ are the standard coordinates in $T_{\theta_2}F_{\theta_1}$ generated by the standard coordinates on $J^2\tau$.

A point $\theta_2 \in \mathcal{E}$ can be elliptic, parabolic, or hyperbolic. To introduce these notions, let us consider a one-dimensional subspace $P \subset \mathcal{C}(\theta_1)$ such that $(\tau_1)_*P \neq 0$. By definition, put

$$l(P) = \{\, \theta_2 \in F_{\theta_1} \bigm| P \subset K_{\theta_2} \,\}\,.$$

The submanifold $l(P)$ is called a *1-ray*. In terms of standard coordinates, let $\theta_1 = (x, y, z, p, q)$, $P = \langle v \rangle$ and

$$v = \zeta_1\frac{\partial}{\partial x} + \zeta_2\frac{\partial}{\partial y} + \mu\frac{\partial}{\partial z} + \eta_1\frac{\partial}{\partial p} + \eta_2\frac{\partial}{\partial q}\,. \tag{8}$$

Then $(\tau_1)_*P \neq 0$ means that

$$(\zeta_1, \zeta_2) \neq (0, 0)\,, \tag{9}$$

$v \in \mathcal{C}(\theta_1)$ means that

$$\mu = \zeta_1 p + \zeta_2 q\,, \tag{10}$$

and $P \subset K_{\theta_2}$ means that

$$\begin{cases} \eta_1 = \zeta_1 r + \zeta_2 s\,, \\ \eta_2 = \zeta_1 s + \zeta_2 t\,, \end{cases} \tag{11}$$

where r, s, t are the standard coordinates of θ_2 in the fiber F_{θ_1}. From system (11), we see that $l(P)$ is an affine straight line in F_{θ_1}. By $\ell_{\theta_2}(P)$ we denote the tangent space $T_{\theta_2}l(P)$ to $l(P)$ at the point $\theta_2 \in l(P)$. We call it a *1-ray subspace*. In terms of the standard coordinates $\tilde{r}, \tilde{s}, \tilde{t}$ in $T_{\theta_2}F_{\theta_1}$, vectors of $\ell_{\theta_2}(P)$ satisfy

$$\begin{cases} \zeta_1\tilde{r} + \zeta_2\tilde{s} = 0\,, \\ \zeta_1\tilde{s} + \zeta_2\tilde{t} = 0\,, \end{cases} \tag{12}$$

Obviously, $\ell_{\theta_2}(P)$ is spanned by the vector

$$(\,\tilde{r},\ \tilde{s},\ \tilde{t}\,) = (\,\zeta_2^2,\ -\zeta_1\zeta_2,\ \zeta_1^2\,)\,. \tag{13}$$

Taking into account (9), we observe that all 1-ray subspaces form the cone

$$\mathcal{V}_{\theta_2} = \{\, \tilde{r}\tilde{t} - \tilde{s}^2 = 0 \,\}$$

in the tangent space $T_{\theta_2}F_{\theta_1}$. This cone is called the *cone of singular square forms*. Obviously, the intersection $\mathrm{Smbl}_{\theta_2}\,\mathcal{E} \cap \mathcal{V}_{\theta_2}$ is either zero, or a single 1-ray subspace, or two 1-ray subspaces. Correspondingly, the point $\theta_2 \in \mathcal{E}$ is then called *elliptic*, *parabolic* or *hyperbolic*. It is not difficult to prove that a contact transformation takes an elliptic, parabolic, or hyperbolic point to an elliptic, parabolic, or hyperbolic point, respectively. The equation $\mathcal{E}$ is called *elliptic*, *parabolic* or *hyperbolic* if all its points are elliptic, parabolic or hyperbolic, respectively. In this work, we consider hyperbolic Monge–Ampère equations only. It is easy to see that $\mathcal{E}$ is hyperbolic iff its coefficients satisfy the condition

$$\Delta = B^2 - 4AC + 4ND > 0\,. \tag{14}$$

3.2 Skew-orthogonal distributions

Directly from geometry of jet bundles we draw out the interpretation of a hyperbolic Monge–Ampère equation as a pair of skew-orthogonal two-dimensional distributions in the Cartan distribution on $J^1\tau$. See [12, 14, 19] for an alternative approach.

Let θ_1 be an arbitrary point of $J^1\tau$. By $\mathcal{Q}_{\theta_1}$ we denote the union of all one-dimensional subspaces P of $\mathcal{C}(\theta_1)$ such that $\tau_* P \neq 0$ and the 1-ray $l(P)$ is tangent to $\mathcal{E}$ at least at one point.

Proposition 3.1. *Let $\mathcal{E}$ be a hyperbolic Monge–Ampère equation. Then $\mathcal{Q}_{\theta_1}$ is the union of two-dimensional subspaces $\mathcal{D}^1_{\mathcal{E}}(\theta_1)$ and $\mathcal{D}^2_{\mathcal{E}}(\theta_1)$ of the Cartan plane $\mathcal{C}(\theta_1)$, so that*

(1) $\mathcal{C}(\theta_1) = \mathcal{D}^1_{\mathcal{E}}(\theta_1) \oplus \mathcal{D}^2_{\mathcal{E}}(\theta_1)$,

(2) $\mathcal{D}^1_{\mathcal{E}}(\theta_1)$ and $\mathcal{D}^2_{\mathcal{E}}(\theta_1)$ are skew-orthogonal with respect to the symplectic form $dU_1 = dx \wedge dp + dy \wedge dq$ on $\mathcal{C}$.

Proof. We prove this proposition for Monge–Ampère equations such that $N \neq 0$. The proof for $N = 0$ follows from the fact that every Monge–Ampère equation can be transformed to one with $N \neq 0$ by an appropriate contact transformation.

Let $v \in \mathcal{Q}_{\theta_1}$ and $P = \langle v \rangle$. The condition for $l(P)$ to be tangent to $\mathcal{E}$ can be written in the following way. We can assume that v is of the form (8). Then the vector of fiber coordinates $(\zeta_2^2, -\zeta_1\zeta_2, \zeta_1^2)$ is tangent to $l(P)$. Now using (7) we deduce that $l(P)$ is tangent to $\mathcal{E}$ iff

$$N(r\zeta_1^2 + 2s\zeta_1\zeta_2 + t\zeta_2^2) + A\zeta_2^2 - B\zeta_1\zeta_2 + C\zeta_1^2 = 0\,.$$

Taking into account that the coordinates ζ_i and η_i of v are connected by equations (11), we reduce this equation to the form

$$N(\zeta_1\eta_1 + \zeta_2\eta_2) + A\zeta_2^2 - B\zeta_1\zeta_2 + C\zeta_1^2 = 0\,. \tag{15}$$

Then in view of (9) we assume that $\zeta_1 \neq 0$ (the case $\zeta_2 \neq 0$ is analogous). Then from (11) we get

$$r = \frac{1}{\zeta_1^2}(\eta_1\zeta_1 - \eta_2\zeta_2 + \zeta_2^2 t)\,, \quad s = \frac{1}{\zeta_1}(\eta_2 - \zeta_2 t)\,.$$

Substituting these expressions for r and s in equation (6) and taking into account equation (15), we obtain the equation

$$N\eta_2^2 + (A\zeta_2 - B\zeta_1)\eta_2 - A\zeta_1\eta_1 - D\zeta_1^2 = 0. \tag{16}$$

Solving the system of equations (15) and (16) with respect to η_1 and η_2, we find

$$\eta_1 = \frac{(B \mp \sqrt{\Delta})\zeta_2 - 2C\zeta_1}{2N}\,, \quad \eta_2 = \frac{(B \pm \sqrt{\Delta})\zeta_1 - 2A\zeta_2}{2N}\,.$$

Finally, in view of (10), we see that

$$v = \zeta_1\left(\frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - \frac{C}{N}\frac{\partial}{\partial p} + \frac{B \pm \sqrt{\Delta}}{2N}\frac{\partial}{\partial q}\right) \\ + \zeta_2\left(\frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + \frac{B \mp \sqrt{\Delta}}{2N}\frac{\partial}{\partial p} - \frac{A}{N}\frac{\partial}{\partial q}\right). \tag{17}$$

This proves that $\mathcal{Q}_{\theta_1} = \langle X_1, X_2\rangle \cup \langle X_3, X_4\rangle$ with

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - \frac{C}{N}\frac{\partial}{\partial p} + \frac{B - \sqrt{\Delta}}{2N}\frac{\partial}{\partial q}\,, \\ X_2 &= \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + \frac{B + \sqrt{\Delta}}{2N}\frac{\partial}{\partial p} - \frac{A}{N}\frac{\partial}{\partial q}, \\ X_3 &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} - \frac{C}{N}\frac{\partial}{\partial p} + \frac{B + \sqrt{\Delta}}{2N}\frac{\partial}{\partial q}\,, \\ X_4 &= \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + \frac{B - \sqrt{\Delta}}{2N}\frac{\partial}{\partial p} - \frac{A}{N}\frac{\partial}{\partial q}\,. \end{aligned} \tag{18}$$

Put

$$\mathcal{D}^1_{\mathcal{E}}(\theta_1) = \langle X_1, X_2\rangle\,, \quad \mathcal{D}^2_{\mathcal{E}}(\theta_1) = \langle X_3, X_4\rangle\,.$$

Now it is straightforward to verify that subspaces $\mathcal{D}^1_{\mathcal{E}}(\theta_1)$ and $\mathcal{D}^2_{\mathcal{E}}(\theta_1)$ are skew-orthogonal and $\mathcal{D}^1_{\mathcal{E}}(\theta_1) \cap \mathcal{D}^2_{\mathcal{E}}(\theta_1) = \{0\}$. This completes the proof.

From (18) we see that for a Monge–Ampère equation such that $N \neq 0$, the map τ_{1*} projects $\mathcal{D}^1_{\mathcal{E}}(\theta_1)$ and $\mathcal{D}^2_{\mathcal{E}}(\theta_1)$ onto the tangent space to the base of the bundle τ without degeneration.

It should be noted that if $N = 0$ (that is, if $\mathcal{E}$ is a quasilinear second order PDE), then the projections $\tau_{1*}\big(\mathcal{D}^1_{\mathcal{E}}(\theta_1)\big)$ and $\tau_{1*}\big(\mathcal{D}^2_{\mathcal{E}}(\theta_1)\big)$ are one-dimensional.

Thus an arbitrary hyperbolic Monge–Ampère equation generates two 2-dimensional skew-orthogonal subdistributions of the Cartan distribution $\mathcal{C}_1$ in $J^1\tau$.

Proposition 3.2. *([12]) Let $\mathcal{E}$ be a hyperbolic Monge–Ampère equation. Then $\theta_2 \in \mathcal{E}$ if and only if one of the following equivalent conditions holds:*
(1) $K_{\theta_2} \cap \mathcal{D}^1_{\mathcal{E}}(\theta_1)$ is 1-dimensional,
(2) $K_{\theta_2} \cap \mathcal{D}^2_{\mathcal{E}}(\theta_1)$ is 1-dimensional.

Proof. As in the proof of Proposition 3.1 one can assume that $N \neq 0$.

Let $\theta_2 \in \mathcal{E}$. Then $\mathrm{Smbl}_{\theta_2} \mathcal{E} \cap \mathcal{V}_{\theta_2} = \ell_{\theta_2}(\langle v \rangle) \cup \ell_{\theta_2}(\langle \tilde{v} \rangle)$, where $\ell_{\theta_2}(\langle v \rangle)$ and $\ell_{\theta_2}(\langle \tilde{v} \rangle)$ are different straight lines and, so, vectors v and $\tilde{v}$ are independent. They are skew-orthogonal, since K_{θ_2} is a Lagrangian plane in $\mathcal{C}(\theta_1)$ and, by definition of $\mathcal{Q}_{\theta_1}$, v, $\tilde{v} \in \mathcal{Q}_{\theta_1}$. This means that K_{θ_2} intersects planes $\mathcal{D}^1_{\mathcal{E}}(\theta_1)$ and $\mathcal{D}^2_{\mathcal{E}}(\theta_1)$ along $\langle v \rangle$ and $\langle \tilde{v} \rangle$, respectively.

Let θ_2 be a point of $J^2\tau$ such that K_{θ_2} intersects the plane $\mathcal{D}^1_{\mathcal{E}}(\theta_1)$ along a straight line, that is, $K_{\theta_2} \cap \mathcal{D}^1_{\mathcal{E}}(\theta_1) = \langle v \rangle$. By substituting coordinates η_1, η_2 of the vector v given by formula (17) into eq. (11), we obtain

$$\left(r + \frac{C}{N}\right)\zeta_1 + \left(s - \frac{B - \sqrt{\Delta}}{2N}\right)\zeta_2 = 0,$$
$$\left(s - \frac{B + \sqrt{\Delta}}{2N}\right)\zeta_1 + \left(r + \frac{A}{N}\right)\zeta_2 = 0.$$

By hypothesis this system is of rank 1 (cf. (9)) and hence its determinant is zero. Now it remains to note that this is exactly equation (6) and, so, $\theta_2 \in \mathcal{E}$. The case of $\mathcal{D}^2_{\mathcal{E}}(\theta_1)$ differs only by the sign at $\sqrt{\Delta}$.

An important consequence of this proposition is that a hyperbolic Monge–Ampère equation $\mathcal{E}$ is completely determined by one of the associated distributions $\mathcal{D}^i_{\mathcal{E}}$, $i = 1, 2$.

Thus, every hyperbolic Monge–Ampère equation $\mathcal{E}$ is naturally equivalent to a pair of 2-dimensional, skew-othogonal non-lagrangian subdistributions $\mathcal{D}^1_{\mathcal{E}}$, $\mathcal{D}^2_{\mathcal{E}}$ of the Cartan distribution $\mathcal{C}_1$ in $J^1\tau$. In particular, the equivalence problem for hyperbolic Monge–Ampère equations with respect to contact transformations may be interpreted as the equivalence problem for pairs of 2-dimensional, skew-orthogonal non-lagrangian subdistributions of $\mathcal{C}_1$ with respect to contact transformations.

3.3 Bundles of Monge–Ampère equations

From now on we put $M = J^1\tau$.

3.3.1 Bundles of hyperbolic Monge–Ampère equations

Let $\mathcal{E}$ be a Monge–Ampère equation (5). It is identified with the section

$$S_{\mathcal{E}} : \epsilon \mapsto \big[N(\epsilon) : A(\epsilon) : B(\epsilon) : C(\epsilon) : D(\epsilon)\big]$$

of the trivial bundle

$$\rho : \mathbb{R}\mathrm{P}^4 \times M \longrightarrow M\,, \quad \big([v^0 : v^1 : v^2 : v^3 : v^4], \epsilon\big) \mapsto \epsilon\,,$$

where $\mathbb{RP}^4$ is the 4-dimensional projective space. Obviously, this identification is a bijection of the set of all Monge–Ampère equations onto the set of all sections of ρ.

Consider the open subset E of the total space of ρ defined by the condition (14), i.e.,

$$(v^2)^2 - 4v^1v^3 + 4v^4v^0 > 0\,.$$

Clearly, the section $S_{\mathcal{E}}$ corresponding to a hyperbolic Monge–Ampère equation $\mathcal{E}$ takes values in E. Thus we can define the bundle of hyperbolic Monge–Ampère equations by the formula

$$\pi = \rho\big|_E \colon E \longrightarrow M\,, \quad \big([v^0 : v^1 : v^2 : v^3 : v^4],\, \epsilon\big) \mapsto \epsilon\,. \tag{19}$$

We use local coordinates $x, y, z, p, q,\ u^1, \ldots, u^4$ in the total space E of π, where x, y, z, p, q are the standard coordinates on M, while the coordinates $u^1, \ldots, u^4$ on the fibres of π are defined as follows. Consider the affine hyperplane in $\mathbb{R}^5$ defined by the equation $v^0 = 1$. It generates the local chart in E

$$[1 : v^1 : v^2 : v^3 : v^4] \mapsto (v^1, v^2, v^3, v^4)\,.$$

Following formulas (18), we introduce the local coordinates $u^1, \ldots, u^4$ along the fibres of π by

$$u^1 = -v^3\,, \quad u^2 = \frac{v^2 - \sqrt{\Delta}}{2}\,, \quad u^3 = \frac{v^2 + \sqrt{\Delta}}{2}\,, \quad u^4 = -v^1\,, \tag{20}$$

where $\Delta = (v^2)^2 - 4v^1v^3 + 4v^4$.

These coordinates extend to the standard coordinates $x, y, z, p, q, u^i, u^i_x, u^i_y, u^i_z, u^i_p, u^i_q, \ldots, u^i_\sigma, \ldots$, on $J^k\pi$, used in this paper until we replace them with a more convenient set in Sect. 4.3.

3.3.2 The lifting of contact transformations

Let φ be a contact transformation defined in M. Then φ transforms any Monge–Ampère equation $\mathcal{E}$ to another Monge–Ampère equation $\tilde{\mathcal{E}}$. In other words, φ induces a transformation of the corresponding sections $S_{\mathcal{E}} \mapsto S_{\tilde{\mathcal{E}}}$ and, consequently, a diffeomorphism $\varphi^{(0)}$ of the total space of π such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \varphi^{(0)}\ } & E \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ M & \xrightarrow[\ \varphi\ ]{} & M \end{array}$$

is commutative (in the domain of $\varphi^{(0)}$). The diffeomorphism $\varphi^{(0)}$ is called the *lifting of φ to the bundle π*.

The diffeomorphism $\varphi^{(0)}$, in its turn, can be lifted to a diffeomorphism $\varphi^{(k)}$ of $J^k\pi$ by the formula

$$\varphi^{(k)}(\,[S]^k_\epsilon\,) = \big[\,\varphi^{(0)} \circ S \circ \varphi^{-1}\,\big]^k_{\varphi(\epsilon)}\,.$$

Obviously, for any $l > m$, the diagram

$$\begin{array}{ccc} J^l\pi & \xrightarrow{\varphi^{(l)}} & J^l\pi \\ {\scriptstyle \pi_{l,m}}\downarrow & & \downarrow{\scriptstyle \pi_{l,m}} \\ J^m\pi & \xrightarrow[\varphi^{(m)}]{} & J^m\pi \end{array}$$

is commutative (in the domains of $\varphi^{(l)}$). The diffeomorphism $\varphi^{(k)}$ is called the *lifting of* φ *to the jet bundle* $J^k\pi$.

3.3.3 The lifting of contact vector fields

Let Z be a contact vector field in M and let φ_t be its flow. Then $\varphi_t^{(k)}$ defines a vector field $Z^{(k)}$ in $J^k\pi$. This field is called the *lifting of* Z *to* $J^k\pi$. Obviously,

$$(\,\pi_{l,m}\,)_*\big(\,Z^{(l)}\,\big) = Z^{(m)}\,, \quad \infty \geq l > m \geq -1\,,$$

where $Z^{(-1)} = Z$.

It is not difficult to see that the map

$$Z \longmapsto Z^{(k)}$$

is a homomorphism of the Lie algebra of all contact vector fields on M into the Lie algebra off all vector fields on $J^k\pi$.

The local expression of $Z^{(k)}$ can be found as follows. First, change the notation by putting $x^1 = x$, $x^2 = y$, $x^3 = z$, $x^4 = p$, $x^5 = q$. Recall that the operator D_j of total derivative with respect to x^j in J_∞ is given by the formula

$$D_j = \frac{\partial}{\partial x^j} + \sum_{|\sigma|\geq 0}\sum_{i=1}^{4} u^i_{\sigma j}\frac{\partial}{\partial u^i_\sigma}\,,\ j = 1, 2, \ldots, 5\,,$$

The operator of evolution differentiation corresponding to a generating function $\psi(Z) = (\psi^1(Z), \ldots, \psi^4(Z))^t$ is defined by the formula

$$\ni_{\psi(Z)} = \sum_{|\sigma|\geq 0}\sum_{i=1}^{4} D_\sigma\big(\,\psi^i(Z)\,\big)\frac{\partial}{\partial u^i_\sigma}\,,$$

where $\sigma = \{j_1 \ldots j_r\}$, $D_\sigma = D_{j_1} \circ \ldots \circ D_{j_r}$ and $\psi(Z)$ is defined as follows.

Let S be a section of π defined in the domain of Z, $\theta_1 = [S]^1_x$, and $x = \pi_1(\theta_1)$; then

$$\psi(Z)(\theta_1) = \frac{d}{dt}(\,\varphi_t^{(0)} \circ S \circ \varphi_t^{-1}\,)\Big|_{t=0}(x)\,.$$

If

$$Z = \sum_{i=1}^{5} Z^i\frac{\partial}{\partial x^i}\,,$$

then the lifting $Z^{(\infty)}$ is defined by the formula (see [4, 5])

$$Z^{(\infty)} = \sum_{j=1}^{5} Z^j D_j + \Ә_{\psi(Z)} \,. \tag{21}$$

It follows from this formula that

$$Z^{(k)} = \sum_{j=1}^{5} Z^j D_j^k + \Ә^k_{\psi(Z)} \,, \tag{22}$$

where

$$D_j^k = \frac{\partial}{\partial x^j} + \sum_{0\le|\sigma|\le k} \sum_{i=1}^{4} u^i_{\sigma j} \frac{\partial}{\partial u^i_\sigma}\,, \quad \Ә^k_{\psi(Z)} = \sum_{0\le|\sigma|\le k} \sum_{i=1}^{4} D_\sigma\big(\psi^i(Z)\big) \frac{\partial}{\partial u^i_\sigma}\,.$$

Let f be the generating function of the contact vector field Z (see formula (2)) and $\theta_1 = (x, y, z, p, q, u^i, u^i_x, u^i_y, u^i_z, u^i_p, u^i_q)$. Then the vector $\psi(Z_f)(\theta_1)$ is $(\psi^1, \ldots, \psi^4)$ with

$$\begin{aligned}
\psi^1 &= -u^1_z f - u^1_p f_x - u^1_q f_y + (-pu^1_p - qu^1_q + u^1) f_z \\
&\quad + (u^1_x + pu^1_z) f_p + (u^1_y + qu^1_z) f_q + f_{xx} + 2p f_{xz} + p^2 f_{zz} \\
&\quad + 2u^1 f_{xp} + (u^2 + u^3) f_{xq} + 2pu^1 f_{zp} + p(u^2 + u^3) f_{zq} \\
&\quad + (u^1)^2 f_{pp} + (u^2 + u^3) u^1 f_{pq} + u^2 u^3 f_{qq}, \\
\psi^2 &= -u^2_z f - u^2_p f_x - u^2_q f_y + (-pu^2_p - qu^2_q + u^2) f_z \\
&\quad + (u^2_x + pu^2_z) f_p + (u^2_y + qu^2_z) f_q + f_{xy} + q f_{xz} + p f_{yz} + pq f_{zz} \\
&\quad + u^2 f_{xp} + u^4 f_{xq} + u^1 f_{yp} + u^2 f_{yq} + (qu^1 + pu^2) f_{zp} \\
&\quad + (qu^2 + pu^4) f_{zq} + u^1 u^2 f_{pp} + (u^1 u^4 + (u^2)^2) f_{pq} + u^2 u^4 f_{qq}, \\
\psi^3 &= -u^3_z f - u^3_p f_x - u^3_q f_y + (-pu^3_p - qu^3_q + u^3) f_z \\
&\quad + (u^3_x + pu^3_z) f_p + (u^3_y + qu^3_z) f_q + f_{xy} + q f_{xz} + p f_{yz} + pq f_{zz} \\
&\quad + u^3 f_{xp} + u^4 f_{xq} + u^1 f_{yp} + u^3 f_{yq} + (qu^1 + pu^3) f_{zp} \\
&\quad + (qu^3 + pu^4) f_{zq} + u^1 u^3 f_{pp} + (u^1 u^4 + (u^3)^2) f_{pq} + u^3 u^4 f_{qq}, \\
\psi^4 &= -u^4_z f - u^4_p f_x - u^4_q f_y + (-pu^4_p - qu^4_q + u^4) f_z \\
&\quad + (u^4_x + pu^4_z) f_p + (u^4_y + qu^4_z) f_q + f_{yy} + 2q f_{yz} + q^2 f_{zz} \\
&\quad + (u^2 + u^3) f_{yp} + 2u^4 f_{yq} + q(u^2 + u^3) f_{zp} + 2qu^4 f_{zq} \\
&\quad + u^2 u^3 f_{pp} + (u^2 + u^3) u^4 f_{pq} + (u^4)^2 f_{qq}.
\end{aligned} \tag{23}$$

3.4 Differential invariants

By Γ we denote the pseudogroup of all contact transformations of M. Its action is lifted to $J^k\pi,\ k \ge 0$, as explained above.

A function (vector field, differential form, or any other natural geometric object on $J^k\pi$) is a *k th-order differential invariant* of Γ if for any $\varphi \in \Gamma$ the lifted transformation $\varphi^{(k)}$ preserves this object. In this work these differential invariants are also called *differential invariants (of order k) of Monge–Ampère equations* or simply *differential invariants (of order k).*

Let $\mathcal{E}$ be a Monge–Ampère equation, $S_{\mathcal{E}}$ the section of π identified with $\mathcal{E}$, and I a differential invariant of order k. Then the *value of I on* $\mathcal{E}$ is defined as $(j_k S_{\mathcal{E}})^*(I)$ and denoted by $I_{\mathcal{E}}$. If a contact transformation f transforms $\mathcal{E}$ to $\tilde{\mathcal{E}}$, then, obviously, $f^{(k)}$ transforms $I_{\mathcal{E}}$ to $I_{\tilde{\mathcal{E}}}$, for any kth order invariant I.

Differential invariants that are functions are also called *scalar differential invariants*. By A_k we denote the $\mathbb{R}$-algebra of all scalar differential invariants of order $\leq k$. By identifying A_k with $\pi^*_{l,k}(A_k) \subset A_l$, $\forall k \leq l$, one gets a sequence of inclusions

$$A_0 \subset A_1 \subset \ldots \subset A_k \subset A_{k+1} \subset \ldots$$

The $\mathbb{R}$-algebra $A = \bigcup_{k=0}^{\infty} A_k$ is called the *algebra of scalar differential invariants of Monge–Ampère equations.*

Remark 3.3. It is worth noticing that a differential invariant I is completely determined by its values $I_{\mathcal{E}}$ on concrete equations $\mathcal{E}$. This observation will be used below.

Let Z be a contact vector field in M and I a differential invariant of order k. Then $L_{Z^{(k)}}(I) = 0$, where L stands for the Lie derivative. This means, in particular, that kth order scalar invariants are first integrals of all contact vector fields lifted to $J^k\pi$. Obviously, a scalar differential invariant of order k is constant on any orbit of the action of Γ on $J^k\pi$. Such an orbit consists, generally, of two components, since contact transformations need not be orientation preserving (e.g., the famous Legendre transformation $x' = p$, $y' = q$, $z' = xp + yq - z$, $p' = x$, $q' = y$ is not). In other words, the above-mentioned first integrals of $Z^{(k)}$ are, generally, invariant only with respect to the unit component of Γ and will be called *almost invariant.* Anyway, generic orbits of contact transformations and of contact vector fields have the same dimension:

Proposition 3.4. *(1) $J^k\pi$ is an orbit of the action of Γ iff $k = 0, 1$,*
(2) Codimension of a generic orbit of $J^2\pi$ is equal to 2*.*
(3) Codimension of a generic orbit of $J^3\pi$ is equal to 29*.*

Proof. Let θ_k be a generic point of $J^k\pi$ and Orb_{θ_k} the orbit of the action of Γ on $J^k\pi$ passing through θ_k. Then $\operatorname{codim} \mathrm{Orb}_{\theta_k} = \dim J^k\pi - \dim \mathrm{Orb}_{\theta_k}$. The dimension of Orb_{θ_k} is the dimension of the subspace spanned by all vectors $X^{(k)}(\theta_k)$ which can be calculated with the help of computer algebra using formulas (22) and (23).

Recall that for an arbitrary smooth function ϕ of $k = 1, 2, \ldots$ arguments and arbitrary scalar differential invariants $I_1, \ldots, I_k \in A_r$, the function $\phi(I_1, \ldots, I_k)$ is a scalar differential invariant belonging to A_r, $r = 0, 1, 2, \ldots$. Now the above proposition immediately implies

Corollary 3.5. *(1) The algebra of scalar differential invariants A_2 is generated by 2 functionally independent invariants.*
(2) The algebra of scalar differential invariants A_3 is generated by 29 functionally independent invariants.

Differential invariants constructed below come mainly form natural geometric constructions without saying that these are invariant with respect to the full pseudo-group Γ. Although not impossible, it is quite challenging task to obtain first integrals of $Z^{(k)}$ analytically even for small k.

4 Differential invariants on $J^2\pi$

The next step to be done is explicit construction of differential invariants that generate A_2 as a C^∞-closed algebra.

4.1 Base projectors

Let $\mathcal{D}$ be a distribution on M. Denote by $\mathcal{D}^{(1)}$ the distribution generated by all vector fields X and $[X,Y]$, $\forall\ X,Y \in \mathcal{D}$. Setting $\mathcal{D}^{(0)} = \mathcal{D}$, we define $\mathcal{D}^{(r+1)}$, $r = 0,1,\ldots$, inductively by the formula $\mathcal{D}^{(r+1)} = (\mathcal{D}^{(r)})^{(1)}$.

Lemma 4.1. *For a hyperbolic Monge–Ampère equation $\mathcal{E}$*

$$\dim(\mathcal{D}^1_{\mathcal{E}})^{(1)} = \dim(\mathcal{D}^2_{\mathcal{E}})^{(1)} = 3.$$

Proof. Let $\omega \in \Lambda^1(M)$ and $X,Y \in D(M)$ be such that $\omega(X) = \omega(Y) = 0$. Then, by applying formula $d\omega(X,Y) = L_X(Y \lrcorner\, \omega) - L_Y(X \lrcorner\, \omega) - [X,Y] \lrcorner\, \omega$, one easily finds that

$$\omega([X,Y]) = -d\omega(X,Y).$$

If now $\omega = U_1$ and vector fields $X,Y \in \mathcal{D}^i_{\mathcal{E}}$, $i = 1,2$, are independent, then $dU_1(X,Y) \neq 0$ due to hyperbolicity of $\mathcal{E}$. So, the above formula shows that $U_1([X,Y]) \neq 0$, i.e., that $[X,Y]$ does not belong to the Cartan distribution on M. So, the fields $[X,Y]$, X and Y are linearly independent at every point of M.

Restricting ourselves to the generic case only, we assume from now on that

$$\dim(\mathcal{D}^1_{\mathcal{E}})^{(2)} = \dim(\mathcal{D}^2_{\mathcal{E}})^{(2)} = 5\,. \tag{24}$$

Suppose that vector fields X_1, X_2 generate the distribution $\mathcal{D}^1_{\mathcal{E}}$ and vector fields X_3, X_4 generate the distribution $\mathcal{D}^2_{\mathcal{E}}$. The 3-dimensional generic distributions $\langle X_1, X_2, [X_1,X_2]\rangle$ and $\langle X_3, X_4, [X_3,X_4]\rangle$ intersect along a one-dimensional subdistribution $\mathcal{D}^3_{\mathcal{E}} = \langle X_1, X_2,$

$[X_1, X_2]\rangle \cap \langle X_3, X_4, [X_3, X_4]\rangle$. Hence, equation $\mathcal{E}$ generates a direct sum decomposition [12]

$$T(M) = \mathcal{D}^1_{\mathcal{E}} \oplus \mathcal{D}^2_{\mathcal{E}} \oplus \mathcal{D}^3_{\mathcal{E}}. \tag{25}$$

This decomposition generates six projections

$$\begin{aligned}
\mathcal{P}_i &: T(M) \to \mathcal{D}^i_{\mathcal{E}}\,, \quad i = 1, 2, 3\,,\\
\mathcal{P}^{(1)}_j &: T(M) \to \mathcal{D}^i_{\mathcal{E}} \oplus \mathcal{D}^3_{\mathcal{E}}\,, \quad j = 1, 2\,,\\
\mathcal{P}_{\mathcal{C}} &: T(M) \to \mathcal{C} = \mathcal{D}^1_{\mathcal{E}} \oplus \mathcal{D}^2_{\mathcal{E}}\,.
\end{aligned}$$

These projections may be viewed as vector-valued 1-forms. Namely, let X_5 be a vector field generating $\mathcal{D}^3_{\mathcal{E}}$. Consider the co-frame $\{\omega^1, \ldots, \omega^5\}$ on M dual to the frame $\{X_1, \ldots, X_5\}$, i.e., $\omega_i(X_j) = \delta_{ij}$. Then

$$\begin{aligned}
\mathcal{P}_1 &= \omega^1 \otimes X_1 + \omega^2 \otimes X_2\,,\\
\mathcal{P}_2 &= \omega^3 \otimes X_3 + \omega^4 \otimes X_4\,,\\
\mathcal{P}_3 &= \omega^5 \otimes X_5\,,\\
\mathcal{P}^{(1)}_j &= \mathcal{P}_j + \mathcal{P}_3\,, \quad j = 1, 2\,,\\
\mathcal{P}_{\mathcal{C}} &= \mathcal{P}_1 + \mathcal{P}_2.
\end{aligned} \tag{26}$$

These vector-valued differential 1-forms are, obviously, differential invariants of $\mathcal{E}$ with respect to contact transformations. Moreover, according to proposition 3.2, the original equation $\mathcal{E}$ is completely determined by each of the projectors $\mathcal{P}_1$, $\mathcal{P}_2$.

4.2 Coordinate-wise description of base projectors

In order to find local expressions for the above projectors, consider vector fields $X_1, \ldots, X_4$ given by (18) and use the notation (20), i.e.,

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + u^1\frac{\partial}{\partial p} + u^2\frac{\partial}{\partial q}\,, \ X_2 = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + u^3\frac{\partial}{\partial p} + u^4\frac{\partial}{\partial q}\,,\\
X_3 &= \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} + u^1\frac{\partial}{\partial p} + u^3\frac{\partial}{\partial q}\,, \ X_4 = \frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + u^2\frac{\partial}{\partial p} + u^4\frac{\partial}{\partial q}\,.
\end{aligned} \tag{27}$$

The remaining field X_5 is defined by the relation

$$X_5 = \lambda^1 X_1 + \lambda^2 X_2 + \kappa[X_1, X_2] = \lambda^3 X_3 + \lambda^4 X_4 + \chi[X_3, X_4]\,. \tag{28}$$

A simple computation shows that

$$\lambda^3 = \lambda^1\,, \quad \lambda^4 = \lambda^2\,, \quad \chi = -\kappa \neq 0\,,$$

with

$$\begin{aligned}
\lambda^1 &= \frac{1}{u^2 - u^3}\Big((u^2+u^3)_y + q(u^2+u^3)_z + u^4(u^2+u^3)_q\\
&\qquad - 2(u^4_x + pu^4_z + u_1 u^4_p) - (u^2+u^3)u^4_q + u^3u^2_p + u^2u^3_p\Big),\\
\lambda^2 &= \frac{1}{u^2 - u^3}\Big((u^2+u^3)_x + p(u^2+u^3)_z + u^1(u^2+u^3)_p\\
&\qquad - 2(u^1_y + qu^1_z + u_4 u^1_q) - (u^2+u^3)u^1_p + u^2u^3_q + u^3u^2_q\Big)
\end{aligned} \tag{29}$$

provided that X_5 is normalized by the requirement $\kappa = 1$.

Brackets of vector fields $X_1, \ldots, X_5$ are described by means of the coefficients b^i_{jk}:

$$[X_j, X_k] = \sum_{i=1}^{5} b^i_{jk} X_i \,.$$

Obviously, $b^i_{jk} = -b^i_{kj}$.

4.3 Convenient coordinates on $J^k\pi$

Vector fields X_i, $i = 1, \ldots, 5$ induce vector fields $\mathcal{X}_i$ on the bundle $J^\infty\pi$, uniquely defined by the condition $j^k(S_\mathcal{E})_* X_i = \mathcal{X}_i$ for all sections $S_\mathcal{E}$. Thus, $\mathcal{X}_1 = D_1 + pD_3 + u^1 D_4 + u^2 D_5$, etc., where D_i denote the total derivatives, see Sect. 3.3.3.

Differential invariants of hyperbolic Monge–Ampère equations constructed bellow are described in terms of the quantities $\mathcal{X}_{i_1} \ldots \mathcal{X}_{i_h} b^k_{ij}$. So, we need to know all algebraic relations connecting them, at least for $h = 0, 1$. To find these efficiently it is convenient to use a non-standard local chart in $J^k\pi$.

Lemma 4.2. *Functions*

$$\tilde{u}^j_{i_1\ldots i_h} = \mathcal{X}_{i_1} \ldots \mathcal{X}_{i_h} u^j, \quad i_1 \leq \ldots \leq i_h,\ h \leq k. \tag{30}$$

together with functions x^i, u^j constitute a local chart on $J^k\pi$. Moreover, the standard jet coordinates on $J^k\pi$ are rational functions of these coordinates.

Proof. For $k = 2$ the assertion is verified directly. For $k > 2$ one can express the standard jet coordinates $u^j_{i_1\ldots i_k} = D_{i_1\ldots i_k} u^j$ in terms of coordinates (30) by making use of the following obvious facts. First, fields D_i are linear combinations of fields $\mathcal{X}_i$ with coefficients in $C^\infty(J^2\pi)$. Second, the coefficients $b^j_{i_1 i_2}$ are functions on $J^2\pi$. Third, $\mathcal{X}_{i_2}\mathcal{X}_{i_1} f = -b^j_{i_1 i_2}\mathcal{X}_j f + \mathcal{X}_{i_1}\mathcal{X}_{i_2} f$ for every function $f \in C^\infty(J^k\pi)$, $k \geq 2$.

Omitting explicit expression of quantities b^k_{ij} in terms of coordinates $\tilde{u}^j_{i_1\ldots i_h}$, we only remark that they are essentially simpler than those in terms of the standard jet coordinates $u^j_{i_1\ldots i_h}$. Then it is easy to find the following complete system of functional relations among quantities b^k_{ij}:

$$\begin{aligned}
&b^1_{34} = 0, && b^2_{34} = 0, && \\
&b^3_{12} = 0, && b^4_{12} = 0, && b^5_{12} = 1, \\
&b^3_{13} = -b^1_{13}, && b^4_{13} = -b^2_{13}, && b^5_{13} = 0, \\
&b^3_{23} = -b^1_{23}, && b^4_{23} = -b^2_{23}, && b^5_{23} = 0, \\
&b^3_{14} = -b^1_{14}, && b^4_{14} = -b^2_{14}, && b^5_{14} = 0, \\
&b^3_{24} = -b^1_{24}, && b^4_{24} = -b^2_{24}, && b^5_{24} = 0, \\
&b^3_{34} = -b^1_{12}, && b^4_{34} = -b^2_{12}, && b^5_{34} = -1, \\
&b^5_{15} = -b^2_{14} - b^1_{13}, && b^5_{25} = -b^2_{24} - b^1_{23}, && \\
&b^5_{35} = -b^1_{13} - b^2_{23}, && b^5_{45} = -b^2_{24} - b^1_{14}, && \\
&b^4_{45} = -b^3_{35} + b^2_{25} + b^1_{15}. && &&
\end{aligned} \tag{31}$$

Naturally, these relations reflect basic geometric properties of fields $X_1, \dots, X_5$. For instance, the relation $b^3_{12} = b^4_{12} = 0$ is implied by the fact that $[X_1, X_2]$ belongs to the distribution $(\mathcal{D}^1_{\mathcal{E}})^{(1)}$ generated by X_1, X_2 and X_5, etc.

Henceforth we shall simplify the notation by using X_i for $\mathcal{X}_i$.

4.4 Curvatures

Using formulas (3), (4) and the direct sum decomposition (25), it is easy to compute the curvature forms of projectors $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_1^{(1)}$, $\mathcal{P}_2^{(1)}$, $\mathcal{P}_{\mathcal{C}}$, which are

$$\begin{aligned}
\mathcal{R}_1 &= \omega^1 \wedge \omega^2 \otimes X_5\,, \\
\mathcal{R}_2 &= -\omega^3 \wedge \omega^4 \otimes X_5\,, \\
\mathcal{R}_1^1 &= -(b^3_{15}\omega^1 + b^3_{25}\omega^2) \wedge \omega^5 \otimes X_3 - (b^4_{15}\omega^1 + b^4_{25}\omega^2) \wedge \omega^5 \otimes X_4\,, \\
\mathcal{R}_2^1 &= -(b^1_{35}\omega^3 + b^1_{45}\omega^4) \wedge \omega^5 \otimes X_1 - (b^2_{35}\omega^3 + b^2_{45}\omega^4) \wedge \omega^5 \otimes X_2\,, \\
\mathcal{R} &= \mathcal{R}_1 + \mathcal{R}_2\,,
\end{aligned} \tag{32}$$

respectively. It is clear that these curvature forms are differential invariants of $\mathcal{E}$.

Frölicher–Nijenhuis brackets of base projectors give new invariant vector-valued forms. These, however, turn out to be linear combinations of curvature forms. More exactly, a direct computation, which is omitted, shows that

$$\begin{aligned}
[\![\mathcal{P}_1, \mathcal{P}_2]\!] &= \tfrac{1}{2}(-[\![\mathcal{P}_1, \mathcal{P}_1]\!] - [\![\mathcal{P}_2, \mathcal{P}_2]\!] + [\![\mathcal{P}_3, \mathcal{P}_3]\!])\,, \\
[\![\mathcal{P}_1, \mathcal{P}_3]\!] &= \tfrac{1}{2}(-[\![\mathcal{P}_1, \mathcal{P}_1]\!] + [\![\mathcal{P}_2, \mathcal{P}_2]\!] - [\![\mathcal{P}_3, \mathcal{P}_3]\!])\,, \\
[\![\mathcal{P}_2, \mathcal{P}_3]\!] &= \tfrac{1}{2}([\![\mathcal{P}_1, \mathcal{P}_1]\!] - [\![\mathcal{P}_2, \mathcal{P}_2]\!] - [\![\mathcal{P}_3, \mathcal{P}_3]\!])
\end{aligned}$$

and

$$\begin{aligned}&[\![\mathcal{P}_1,\mathcal{P}_1]\!] = -2(\mathcal{R}_2^1+\mathcal{R}_1), \quad [\![\mathcal{P}_2,\mathcal{P}_2]\!] = -2(\mathcal{R}_1^1+\mathcal{R}_2)\,,\\ &[\![\mathcal{P}_3,\mathcal{P}_3]\!] = -2(\mathcal{R}_1+\mathcal{R}_2).\end{aligned}$$

4.5 Scalar invariants on $J^2\pi$

The following three invariant 5-forms with values in $\mathcal{D}^3_{\mathcal{E}} = \langle X_5\rangle$:

$$\begin{aligned}\tfrac{1}{2}\big(\mathcal{R}_2^1 \lrcorner \mathcal{R}_1\big) \lrcorner \big(\mathcal{R}_2^1 \lrcorner \mathcal{R}_1\big) &= \Lambda_1\,\omega^1\wedge\ldots\wedge\omega^5\otimes X_5\,,\\ \tfrac{1}{2}\big(\mathcal{R}_1^1 \lrcorner \mathcal{R}_2\big) \lrcorner \big(\mathcal{R}_1^1 \lrcorner \mathcal{R}_2\big) &= \Lambda_2\,\omega^1\wedge\ldots\wedge\omega^5\otimes X_5\,,\\ \big(\mathcal{R}_2^1 \lrcorner \mathcal{R}_1\big) \lrcorner \big(\mathcal{R}_1^1 \lrcorner \mathcal{R}_2\big) &= \Lambda_{12}\,\omega^1\wedge\ldots\wedge\omega^5\otimes X_5\,,\end{aligned} \tag{33}$$

with

$$\begin{aligned}\Lambda_1 = b^2_{35}b^1_{45} - b^1_{35}b^2_{45}, \qquad \Lambda_2 = b^4_{15}b^3_{25} - b^3_{15}b^4_{25},\\ \Lambda_{12} = b^3_{15}b^1_{35} + b^4_{15}b^1_{45} + b^3_{25}b^2_{35} + b^4_{25}b^2_{45}\,.\end{aligned} \tag{34}$$

are proportional. Therefore, the corresponding proportionality factors are scalar differential invariants. In particular, such are

$$\begin{aligned}I^1 &= \Lambda_{12}/\Lambda_1\,,\\ I^2 &= \Lambda_{12}/\Lambda_2\,.\end{aligned} \tag{35}$$

Below it will be shown that Λ_1, Λ_2 are nowhere zero.

Theorem 4.3. *The algebra of scalar differential invariants on $J^2\pi$ is generated by the invariants I^1 and I^2.*

Proof. In view of Corollary 3.5, it is sufficient to show that I^1 and I^2 are functionally independent (on $J^2\pi$). But this is straightforward from the complete list of functional relations (31).

Coefficients Λ_σ, $\sigma = 1, 2, 12$, introduced in (34) have a geometrical meaning explained below. Fix a generator $W = fX_5$ in $\mathcal{D}^3_{\mathcal{E}}$ and consider maps

$$\Box_1^W : \mathcal{D}^2 \to \mathcal{D}^1, \qquad \Box_2^W : \mathcal{D}^1 \to \mathcal{D}^2,$$

defined by formulas

$$\Box_1^W(Z_2) = \mathcal{P}_1([Z_2, W]), \qquad \Box_2^W(Z_1) = \mathcal{P}_2([Z_1, W]),$$

with $Z_1 \in \mathcal{D}^1$, $Z_2 \in \mathcal{D}^2$. Since $\mathcal{P}_1(\mathcal{D}^2_{\mathcal{E}}) = \mathcal{P}_2(\mathcal{D}^1_{\mathcal{E}}) = 0$ both $\Box_1^W$ and $\Box_1^W$ are $C^\infty(M)$-linear. This is seen as well from their local expressions

$$\begin{aligned}\Box_1^W &= fb^i_{j5}\,\omega^j\otimes X_i,\ i = 1, 2,\ j = 3, 4,\\ \Box_2^W &= fb^i_{j5}\,\omega^j\otimes X_i,\ i = 3, 4,\ j = 1, 2.\end{aligned}$$

Consider also 2-forms $\rho_i^W : \mathcal{D}^i \times \mathcal{D}^i \to \mathbb{R}$, $i = 1, 2$, defined by

$$\rho_i^W(U_i, V_i)W = \mathcal{R}_i(U_i, V_i), \quad U_i, V_i \in \mathcal{D}^i_{\mathcal{E}}. \tag{36}$$

Then, obviously, $\rho_1^W = (1/f)\omega^1 \wedge \omega^2$, $\rho_2^W = -(1/f)\omega^3 \wedge \omega^4$, so that both are volume forms of $\mathcal{D}^1$ and D^2, respectively. Moreover, we have

$$\begin{aligned} (\Box_1^W)^*(\rho_1^W) &= f^2\Lambda_2\rho_2^W, \\ (\Box_2^W)^*(\rho_2^W) &= f^2\Lambda_1\rho_1^W, \\ \operatorname{tr}(\Box_1^W \circ \Box_2^W) = \operatorname{tr}(\Box_2^W \circ \Box_1^W) &= f^2\Lambda_{12}. \end{aligned} \tag{37}$$

Proposition 4.4. *If $\mathcal{E}$ is generic, then functions Λ_1, Λ_2 are nowhere zero.*

Proof. By genericity condition (24), $\Box_1^W$ and $\Box_2^W$ are surjective, hence Λ_1, Λ_2 are nonzero.

4.5.1

Now consider operators $\nabla_1^W = \Box_1^W \circ \Box_2^W$ and $\nabla_2^W = \Box_2^W \circ \Box_1^W$ acting on $\mathcal{D}^1$ and $\mathcal{D}^2$, respectively. It follows from (37) that

$$\lambda^2 - f^2\Lambda_{12}\lambda + f^4\Lambda_1\Lambda_2 \tag{38}$$

is the characteristic polynomial for each of them. Another peculiarity of the situation is that $\Box_1^W$ send eigenvectors of ∇_2^W to that of ∇_1^W and similarly for $\Box_1^W$.

The discriminant of polynomial (38) is

$$f^4\Lambda_1\Lambda_2(I^1I^2 - 4).$$

Its sign coincides, obviously, with the sign of

$$I^1I^2(I^1I^2 - 4)\ .$$

This proves that generic hyperbolic Monge–Ampère equations are subdivided into three subclasses as follows:

(1) subclass "h": the operator ∇_i^W has two different real eigenfunctions
$\Leftrightarrow \quad I^1I^2(I^1I^2 - 4) > 0$,

(2) subclass "p": the operator ∇_i has a unique real eigenfunction
$\Leftrightarrow \quad I^1I^2(I^1I^2 - 4) = 0$,

(3) subclass "e": the operator ∇_i has no real eigenfunctions
$\Leftrightarrow \quad I^1I^2(I^1I^2 - 4) < 0$.

4.5.2 Some almost invariants

The previous considerations lead to an almost invariant choice of generator $W = fX_5$ in $\mathcal{D}^3_{\mathcal{E}}$. Namely, define functions Λ_i^W, $i = 1, 2$, by relations

$$(\Box_1^W)^*(\rho_1^W) = \Lambda_2^W\rho_2^W, \quad (\Box_2^W)^*(\rho_2^W) = \Lambda_1^W\rho_1^W\ .$$

Obviously, $\Lambda_i^W = f^2\Lambda_i$. This shows that, up to sign, vector fields

$$W_i = \frac{1}{\sqrt{|\Lambda_i^W|}} W, \quad i = 1, 2,$$

do not depend on the choice of W. In particular, $\Lambda_i^{X_5} = \Lambda_i$, so that

$$W_i = \frac{1}{\sqrt{|\Lambda_i|}} X_5, \quad i = 1, 2.$$

By duality, 1-forms

$$\vartheta_i = \sqrt{|\Lambda_i|}\omega_5, \ i = 1, 2,$$

are almost invariant as well.

It is not difficult to construct further almost invariant forms. For instance, the forms

$$\vartheta_{ij} = \mathcal{R}_i \lrcorner \vartheta_j, \ i = 1, 2,$$

are manifestly almost invariant and have the following local expressions:

$$\vartheta_{1j} = \sqrt{|\Lambda_j|}\omega^1 \wedge \omega^2 \quad \vartheta_{2j} = \sqrt{|\Lambda_j|}\omega^3 \wedge \omega^4 \ .$$

The products

$$\rho_j = (-\operatorname{sign}\Lambda_j)\vartheta_{1j} \wedge \vartheta_{2j} = \Lambda_j\, \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4, \qquad j = 1, 2, \tag{39}$$

which are volume forms on the Cartan distribution $\mathcal{D}^1_{\mathcal{E}}, \oplus \mathcal{D}^2_{\mathcal{E}}$, are, obviously, fully invariant. This is a very simple example on how an invariant can be constructed from almost invariants. Forms ρ_j can be described in a manifestly invariant way as follows:

$$\rho_1 = \tfrac{1}{2}\big\langle (\mathcal{R}_2^1 \lrcorner \mathcal{R}_1) \lrcorner (\mathcal{R}_2^1 \lrcorner \mathcal{R}_1)\big\rangle, \quad \rho_2 = \tfrac{1}{2}\big\langle (\mathcal{R}_1^1 \lrcorner \mathcal{R}_2) \lrcorner (\mathcal{R}_1^1 \lrcorner \mathcal{R}_2)\big\rangle$$

where $\langle \cdot \rangle$ stands for the self-contraction. Note that the form

$$\rho^{12} = I^j \rho_j = \Lambda_{12}\, \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 \tag{40}$$

is invariant too.

Similarly, one can construct many other invariant forms. Some of them are :

$$\begin{aligned} \big\langle\, \mathcal{R}_2^1 \lrcorner \mathcal{R}_1 \,\big\rangle &= -(b^1_{35}\omega^3 + b^1_{45}\omega^4) \wedge \omega^2 + (b^2_{35}\omega^3 + b^2_{45}\omega^4) \wedge \omega^1, \\ \big\langle\, \mathcal{R}_1^1 \lrcorner \mathcal{R}_2 \,\big\rangle &= (b^3_{15}\omega^1 + b^3_{25}\omega^2) \wedge \omega^4 - (b^4_{15}\omega^1 + b^4_{25}\omega^2) \wedge \omega^3, \\ \mathcal{R}_1^1 \lrcorner \big\langle\, \mathcal{R}_2 \lrcorner \mathcal{R}_1^1 \,\big\rangle &= 2\Lambda_2\, \omega^1 \wedge \omega^2 \wedge \omega^5, \\ \mathcal{R}_2^1 \lrcorner \big\langle\, \mathcal{R}_2^1 \lrcorner \mathcal{R}_1 \,\big\rangle &= 2\Lambda_1\, \omega^3 \wedge \omega^4 \wedge \omega^5. \end{aligned} \tag{41}$$

Now it is easy to construct almost invariant volume forms :

$$\vartheta_j \wedge \rho_j = |\Lambda_j|^{3/2}\, \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 \wedge \omega^5, \quad j = 1, 2. \tag{42}$$

5 Differential invariants on $J^3\pi$

Since $\omega^k(X_l) = const$, namely, δ_{kl}, we have

$$d\omega_k(X_i, X_j) = -\omega^k([X_i, X_j]) .$$

(see the proof of lemma 4.1). This implies the useful formula

$$d\omega^k = -\sum_{i<j} b^k_{ij}\, \omega^i \wedge \omega^j . \tag{43}$$

5.1 The complete parallelism

First, note that invariant differential 1-forms dI^1 and dI^2 live on $J^3\pi$. This leads us immediately to another set of invariant differential 1-forms on $J^3\pi$:

$$\begin{aligned}
&\Omega^1 = \mathcal{P}_1 \lrcorner\, dI^1 = X_1(I^1)\,\omega^1 + X_2(I^1)\,\omega^2 ,\\
&\Omega^2 = \mathcal{P}_1 \lrcorner\, dI^2 = X_1(I^2)\,\omega^1 + X_2(I^2)\,\omega^2 ,\\
&\Omega^3 = \mathcal{P}_2 \lrcorner\, dI^1 = X_3(I^1)\,\omega^3 + X_4(I^1)\,\omega^4 ,\\
&\Omega^4 = \mathcal{P}_2 \lrcorner\, dI^2 = X_3(I^2)\,\omega^3 + X_4(I^2)\,\omega^4 ,\\
&\Omega^5_1 = \mathcal{P}_3 \lrcorner\, dI^1 = X_5(I^1)\,\omega^5 , \quad \Omega^5_2 = \mathcal{P}_3 \lrcorner\, dI^2 = X_5(I^2)\,\omega^5 .
\end{aligned} \tag{44}$$

Supposing that $\mathcal{E}$ is a generic equation, we henceforth assume that

$$X_5(I^1) \neq 0 , \quad X_5(I^2) \neq 0 , \tag{45}$$

and

$$\Delta_1 = \begin{vmatrix} X_1(I^1) & X_2(I^1) \\ X_1(I^2) & X_2(I^2) \end{vmatrix} \neq 0 , \quad \Delta_2 = \begin{vmatrix} X_3(I^1) & X_4(I^1) \\ X_3(I^2) & X_4(I^2) \end{vmatrix} \neq 0 . \tag{46}$$

This means that two sets of forms $\{\Omega^1, \ldots, \Omega^4, \Omega^5_1\}$ and $\{\Omega^1, \ldots, \Omega^4,\ \Omega^5_2\}$ are invariant coframes on M (we omit the subscript $\mathcal{E}$ according to Remark 3.3). Each of these coframes determines an invariant complete parallelism on M.

The frames $\{Y_1, \ldots,\ Y_4,\ Y^1_5\}$ and $\{Y_1, \ldots, Y_4, Y^2_5\}$ dual to the above constructed coframes are obviously invariant. An explicit description of them is :

$$\begin{aligned}
Y_1 &= \frac{1}{\Delta_1}\big(X_2(I^2)X_1 - X_1(I^2)X_2\big) ,\\
Y_2 &= \frac{1}{\Delta_1}\big(-X_2(I^1)X_1 + X_1(I^1)X_2\big) ,\\
Y_3 &= \frac{1}{\Delta_2}\big(X_4(I^2)X_3 - X_3(I^2)X_4\big) ,\\
Y_4 &= \frac{1}{\Delta_2}\big(-X_4(I^1)X_3 + X_3(I^1)X_4\big) ,\\
Y^1_5 &= \frac{1}{X_5(I^1)}X_5 , \quad Y^2_5 = \frac{1}{X_5(I^2)}X_5 .
\end{aligned} \tag{47}$$

5.2 More scalar invariants on $J^3\pi$

Among numerous invariants constructed previously there are functions, (vector-valued) differential forms, and vector fields. Further invariants can by obtained just by applying various operations of tensor algebra, Frölicher–Nijenhuis brackets, etc., to these objects. Moreover, components of an invariant object with respect to an invariant basis are scalar differential invariants. These simple general tricks are rather efficient and were already used in constructing differential invariants on $J^2\pi$. As for $J^3\pi$ we shall proceed along these lines as well.

The invariant 1-forms Ω_1^5 and Ω_2^5 are proportional. So, the proportionality factor

$$\tilde{I}^3 = \frac{X_5(I^1)}{X_5(I^2)} \tag{48}$$

is a scalar differential invariant on $J^3\pi$.

Consider now invariant 2-forms on $J^3\pi$:

$$\begin{aligned}
\mathcal{R}_1 \lrcorner\, dI^1 &= I^6\Omega^1 \wedge \Omega^2 \,, \\
\mathcal{R}_1 \lrcorner\, dI^2 &= I^7\Omega^1 \wedge \Omega^2 \,, \\
\mathcal{R}_2 \lrcorner\, dI^1 &= I^8\Omega^3 \wedge \Omega^4 \,, \\
\mathcal{R}_2 \lrcorner\, dI^2 &= I^9\Omega^3 \wedge \Omega^4 \,, \\
\mathcal{R}_1^1 \lrcorner\, dI^1 &= I^{10}\Omega^1 \wedge \Omega_1^5 + I^{11}\Omega^2 \wedge \Omega_1^5 \,, \\
\mathcal{R}_1^1 \lrcorner\, dI^2 &= I^{12}\Omega^1 \wedge \Omega_1^5 + I^{13}\Omega^2 \wedge \Omega_1^5 \,, \\
\mathcal{R}_2^1 \lrcorner\, dI^1 &= I^{14}\Omega^3 \wedge \Omega_1^5 + I^{15}\Omega^4 \wedge \Omega_1^5 \,, \\
\mathcal{R}_2^1 \lrcorner\, dI^2 &= I^{16}\Omega^3 \wedge \Omega_1^5 + I^{17}\Omega^4 \wedge \Omega_1^5 \,.
\end{aligned} \tag{49}$$

Their components $I^6, \dots, I^{17}$ with respect to the base $\Omega^1, \dots, \Omega^5$ are further scalar differential invariants on $J^3\pi$. The simplest among them are $I^6 = \Delta_1/X_5(I^1)$ and $I^8 = \Delta_2/X_5(I^1)$.

In the same manner one easily find numerous non-scalar differential invariants on $J^3\pi$. For instance, such are 3-forms $[\![\mathcal{P}_i, \mathcal{R}_j]\!]$ or $[\![\mathcal{P}_i, \mathcal{R}_j^1]\!]$, 4-forms $[\![\mathcal{P}_i, (\mathcal{R}_j^1 \lrcorner\, \mathcal{R}_k^1)]\!]$, 5-forms $[\![\mathcal{P}_i, \mathcal{R}_j^1]\!] \lrcorner\, [\![\mathcal{P}_k, \mathcal{R}_l^1]\!]$, etc.

5.3 Better manageable invariants

From the above said one can see that there are sufficient resources for constructing differential invariants and the main problem becomes to select functionally independent ones in the simplest possible way. From technical point of view this forces us to look for *manageable* invariants, for instance, those that have local expression as simple as possible. In the considered context a help comes from almost invariant objects as it is illustrated below.

In view of (39), (40) and (43), for $\sigma = 1, 2, 12$ we have the invariant 5-forms

$$\begin{aligned}
d\rho_\sigma &= d(\Lambda_\sigma\, \omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4) \\
&= \big(X_5(\Lambda_\sigma) + \Lambda_\sigma B\big)\omega^1 \wedge \omega^2 \wedge \omega^3 \wedge \omega^4 \wedge \omega^5,
\end{aligned} \tag{50}$$

where $B = b^1_{15} + b^2_{25} + b^3_{35} + b^4_{45} = 2(b^1_{15} + b^2_{25}) = 2(b^3_{35} + b^4_{45})$ according to (31).

By comparing these 5-forms with (42) we obtain almost scalar invariants

$$I^j_\sigma = \frac{X_5(\Lambda_\sigma) + \Lambda_\sigma B}{|\Lambda_j|^{3/2}}, \qquad \sigma = 1, 2, 12, \quad j = 1, 2, \tag{51}$$

on $J^3\pi$ which are better manageable in comparison to those constructed in the previous subsection. The squares $(I^j_\sigma)^2$ are, obviously, full scalar invariants. Apart from the obvious relation $(I^1_\sigma/I^2_\sigma)^2 = (I^1/I^2)^3$ they are functionally independent. Some of the earlier constructed invariants can be expressed in terms of I^j_σ's, e.g.,

$$\frac{X_5(I^1)}{X_5(I^2)} = \frac{(I^j_{12} - I^j_1 I^1)I^1}{(I^j_{12} - I^j_2 I^2)I^2}, \quad j = 1, 2.$$

6 The equivalence problem

So far we obtained two independent second-order scalar invariants I^1, I^2 (see (35)) and a number of third-order invariants. Put (see (51))

$$I^3 = (I^1_1)^2, \qquad I^4 = (I^1_2)^2, \qquad I^5 = (I^1_{12})^2,$$

The following statement can be checked by a direct computer-supported calculation in coordinates (30):

Theorem 6.1. *For a generic hyperbolic Monge-Ampére equation $\mathcal{E}$ values $I^j_\mathcal{E}$'s of invariants I^j's, $j = 1, ..., 5$, on $\mathcal{E}$ are functionally independent on the base M.*

Of course, this choice of basic scalar invariants is not unique. For instance, functions $I^1_\mathcal{E}, I^2_\mathcal{E}, \tilde{I}^3_\mathcal{E}, I^6_\mathcal{E}, I^8_\mathcal{E}$ (see (48), (49)) are functionally independent on $\mathcal{E}$ as well. However, this and other reasonable choices are "less manageable" with respect to those made in the above theorem. Unfortunately, this fact is not clearly seen from the above exposition, since we were forced to skip technical details of computations.

According to "the principle of n invariants" [24], any quintuple of functionally independent scalar invariants gives a solution of the equivalence problem for generic hyperbolic Monge–Ampère equations. Theorem 6.1 guarantees existence of such a one, namely, $I^1, \ldots, I^5$.

More exactly, let $\mathcal{E}$ be a generic hyperbolic Monge–Ampère equation considered as an unordered pair of 2-dimensional, skew-orthogonal subdistributions $\mathcal{D}^1_\mathcal{E}$ and $\mathcal{D}^2_\mathcal{E}$ of the Cartan distribution on M, that is $\mathcal{E} = (\langle X_{1,\mathcal{E}}, X_{2,\mathcal{E}}\rangle, \langle X_{3,\mathcal{E}}, X_{4,\mathcal{E}}\rangle)$, where vector fields $X_1, \ldots, X_4$ are defined by (18).

Observe now that the distributions $\mathcal{D}^1_\mathcal{E} = \langle X_{1,\mathcal{E}}, X_{2,\mathcal{E}}\rangle$ and $\mathcal{D}^2_\mathcal{E} = \langle X_{3,\mathcal{E}}, X_{4,\mathcal{E}}\rangle$ are determined uniquely by values on $\mathcal{E}$ of *invariant* bivectors $\mathcal{W}_1 = Y_1 \wedge Y_2$ and $\mathcal{W}_2 = Y_3 \wedge Y_4$ with fields Y_i defined in (47), respectively. Obviously,

$$\mathcal{W}_1 = \frac{1}{\Delta_1} X_1 \wedge X_2, \quad \mathcal{W}_2 = \frac{1}{\Delta_2} X_3 \wedge X_4.$$

Since $I^1_{\mathcal{E}}, \ldots, I^5_{\mathcal{E}}$ are functionally independent, they form an *invariant* chart in M denoted by $I_{\mathcal{E}}$. Observe that $Y_{i+\varepsilon}(I^j) = \delta_{ij}$ for $i, j = 1, 2,\ \varepsilon = 0, 2$. So,

$$Y_{i+\varepsilon} = \delta_{ij}\frac{\partial}{\partial I^j} + Y^{\alpha}_{i+\varepsilon}\frac{\partial}{\partial I^{\alpha}}$$

with $i, j = 1, 2,\ \varepsilon = 0, 2,\ \alpha = 3, 4, 5$. Functions $\nu^{\alpha}_k = Y^{\alpha}_k(I^1, ..., I^5)$ are, obviously, differential invariants. They determine completely vector fields $Y_1, ..., Y_4$ and, so, the distributions $\mathcal{D}^i_{\mathcal{E}},\ i = 1, 2$. Consider now functions $\nu^{\alpha}_{k,\mathcal{E}} = Y^{\alpha}_{k,\mathcal{E}} \circ I^{-1}_{\mathcal{E}}$. They are functions in a certain domain of the arithmetic vector space $\mathbb{R}^5 = \{(z_1, ..., z_5)\}$ and will be called *normal parameters* of the equation $\mathcal{E}$.

Theorem 6.2. *Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be generic hyperbolic Monge–Ampère equations. Then $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are (locally) equivalent iff their normal parameters coincide, i.e., iff $\nu^{\alpha}_{k,\mathcal{E}}(z_1, ..., z_5) \equiv \nu^{\alpha}_{k,\tilde{\mathcal{E}}}(z_1, ..., z_5),\ k = 1, ..., 4,\ \alpha = 3, 4, 5.$*

Proof. Let $I_{\mathcal{E}} = (I^1_{\mathcal{E}}, \ldots, I^5_{\mathcal{E}})$ and $I_{\tilde{\mathcal{E}}} = (I^1_{\tilde{\mathcal{E}}}, \ldots, I^5_{\tilde{\mathcal{E}}})$ be invariant charts for $\mathcal{E}$ and $\tilde{\mathcal{E}}$, respectively.

The "if" part of the theorem is obvious. Now assume that normal parameters of $\mathcal{E}$ and $\tilde{\mathcal{E}}$ coincide. Then the diffeomorphism $f = I^{-1}_{\tilde{\mathcal{E}}} \circ I_{\mathcal{E}}$ is such that $I^i_{\mathcal{E}} = f^*(I^i_{\tilde{\mathcal{E}}})$ and, consequently, $Y^{\alpha}_{k,\mathcal{E}} = f^*(Y^{\alpha}_{k,\tilde{\mathcal{E}}})$. This shows that f sends vector fields $Y_{k,\mathcal{E}}$ to vector fields $Y_{k,\tilde{\mathcal{E}}}$, $k = 1, ..., 4$, and, therefore, $\mathcal{D}^i_{\mathcal{E}}$ to $\mathcal{D}^i_{\tilde{\mathcal{E}}},\ i = 1, 2$. Since $\mathcal{C} = \mathcal{D}^1_{\mathcal{E}} \oplus \mathcal{D}^2_{\mathcal{E}}$ and $\tilde{\mathcal{C}} = \mathcal{D}^1_{\tilde{\mathcal{E}}} \oplus \mathcal{D}^2_{\tilde{\mathcal{E}}}$ the diffeomorphism f is automatically contact.

Remark 6.3. A system of functions f^{α}_k in a domain of $\mathbb{R}^5$ can be realized as the system of normal parameters $\nu^{\alpha}_{k,\mathcal{E}}$ of a hyperbolic Monge-Ampère equation $\mathcal{E}$ iff it is a solution of a system of partial differential equations (see [1, 24]) and the algebra of differential invariants of Monge-Ampère equations is then interpreted to be the smooth function algebra on the infinite prolongation of this system. According to [1, 24]), it is not difficult to describe explicitly this system but the result is rather cumbersome and not very instructive. This is why we do not report it here. More satisfactory results in this direction will be presented in a separate paper. Nevertheless, it is worth mentioning that, in principle, the differential invariants constructed above allow a solution of the *classification problem* for generic hyperbolic Monge-Ampère equations.

There are alternative equivalent formulations of the classification theorem. For instance, one of them is as follows.

Consider the 1-forms $\Omega^1, \ldots, \Omega^5$, defining the complete parallelism on M. In the invariant coordinate system $I^1_{\mathcal{E}}, \ldots, I^5_{\mathcal{E}}$, these forms are described in the terms of functions $\Omega^i_j(I^1_{\mathcal{E}}, \ldots, I^5_{\mathcal{E}})$:

$$\Omega^i = \sum_{j=1}^{5} \Omega^i_j(I^1_{\mathcal{E}}, \ldots, I^5_{\mathcal{E}}) dI^j_{\mathcal{E}}\,, \quad i = 1, \ldots, 5\,.$$

Theorem 6.4. *The (local) equivalence class of a generic equation $\mathcal{E}$ with respect to con-*

tact transformations is uniquely determined by the family of functions $\Omega^i_j(I^1_{\mathcal{E}},\dots,I^5_{\mathcal{E}})$, $i = 1,\dots,5$.

Proof. Let $\tilde{\mathcal{E}}$ be another generic Monge–Ampère equation such that there exists a contact transformation transforming it to $\mathcal{E}$. Then, obviously, the functions $\Omega^i_j(I^1_{\mathcal{E}},\dots,I^5_{\mathcal{E}})$ and $\tilde{\Omega}^i_j(I^1_{\tilde{\mathcal{E}}},\dots,I^5_{\tilde{\mathcal{E}}})$ coincide for all i and j.

Let $\mathcal{E},\tilde{\mathcal{E}}$ be Monge–Ampère equations such that for all i and j the functions $\Omega^i_j(I^1_{\mathcal{E}},\dots,I^5_{\mathcal{E}})$ and $\tilde{\Omega}^i_j(I^1_{\tilde{\mathcal{E}}},\dots,I^5_{\tilde{\mathcal{E}}})$ coincide. Let $I_{\mathcal{E}} = (I^1_{\mathcal{E}},\dots,I^5_{\mathcal{E}})$ and $I_{\tilde{\mathcal{E}}} = (I^1_{\tilde{\mathcal{E}}},\dots,I^5_{\tilde{\mathcal{E}}})$ be invariant coordinate systems in M for $\mathcal{E}$ and $\tilde{\mathcal{E}}$ respectively. Then $I^{-1}_{\tilde{\mathcal{E}}}\circ I_{\mathcal{E}}$ is a locally defined diffeomorphism $M\to M$. This diffeomorphism is a contact transformation because it transforms $\Omega^i = \sum_{j=1}^5 \Omega^i_j(I^1_{\mathcal{E}},\dots,I^5_{\mathcal{E}})dI^j_{\mathcal{E}}$ to $\sum_{j=1}^5 \tilde{\Omega}^i_j(I^1_{\tilde{\mathcal{E}}},\dots,I^5_{\tilde{\mathcal{E}}})dI^j_{\tilde{\mathcal{E}}} = \tilde{\Omega}^i$, $i = 1,\dots,5$, and, in particular, the contact form Ω^5 to the contact form $\tilde{\Omega}^5$. By obvious reasons it also transforms the pair of distributions $(\mathcal{D}^1_{\mathcal{E}},\mathcal{D}^2_{\mathcal{E}})$ to the pair $(\mathcal{D}^1_{\tilde{\mathcal{E}}},\mathcal{D}^2_{\tilde{\mathcal{E}}})$ and hence $\mathcal{E}$ to $\tilde{\mathcal{E}}$. □

7 Examples

Examples discussed in this section aim to illustrate the character and complexity of problems related with actual computations and use of differential invariants. Henceforth invariants I^i are denoted by I_i.

Example 7.1. Consider the equation

$$\tfrac14(z_{xx}z_{yy} - z_{xy}^2) + y^2 z_{xx} - 2xyz_{xy} + x^2 z_{yy} + x^2y^2z^2 = 0.$$

The first two invariants are $I_1 = zn_+/d$, $I_2 = zn_-/d$, where

$$\begin{aligned}
n_\pm &= 2(z + 3y^4 \mp 2)x^2z_x^2 - (z + 12x^2y^2)xyz_xz_y + 2(z + 3x^4 \pm 2)y^2z_y^2\\
&\quad + (z^2 + 8x^2y^2z + 4y^4z \pm 4z \pm 16x^2y^2 \pm 16y^4 - 12)xz_x\\
&\quad + (z^2 + 4x^4z + 8x^2y^2z \mp 4z \mp 16x^4 \mp 16x^2y^2 - 12)yz_y\\
&\quad + 2z^3 + 36x^4y^4z^3 + 6y^4z^2 - 4x^2y^2z^2 + 6x^4z^2\\
&\quad - 8z \mp 16x^4z \pm 16y^4z + 8x^4 + 16x^2y^2 + 8y^4,\\
d &= 4(z^2 + 3y^4z + 4)x^2z_x^2 - 2(z^2 + 12x^2y^2z - 16)xyz_xz_y\\
&\quad + 4(z^2 + 3x^4z + 4)y^2z_y^2 + 2(z^2 + 8x^2y^2z + 4y^4z + 20)xzz_x\\
&\quad + 2(z^2 + 4x^4z + 8x^2y^2z + 20)yzz_y + 4(18x^4y^4z^3 + z^3\\
&\quad + 3x^4z^2 - 2x^2y^2z^2 + 3y^4z^2 + 12z + 4x^4 + 8x^2y^2 + 4y^4)z.
\end{aligned}$$

The invariants I_3, I_4, I_5 are large fractions whose non-reducible numerators are polynomials of order three in z_x, z_y, five in z, and six in x, y. Invariants I_s, $s > 5$, are even more cumbersome.

Computation shows that the jacobian $\partial(I_1, I_2, I_3, I_4, I_5)/\partial(x, y, z, z_x, z_y)$ is nonzero, hence the first five invariants are functionally independent and can be chosen to be local coordinates on $J^1(\tau)$. Although an explicit inversion is rather hopeless, one can still find algorithmically the relations connecting principal invariants $I_1, \dots, I_5$ and higher I_k at least in principle. This kind of procedure is outlined in Example 7.2 below.

Example 7.2. Put $\zeta = z_x + z_y + e$ and consider the family of equations

$$(4z_x z_y + \zeta^2)(z_{xx} z_{yy} - z_{xy}^2) + 4\zeta^2(z_y z_{xx} + z_x z_{yy} + \zeta^2) = 0. \tag{52}$$

depending on parameter e. Assuming that $e \neq 0$, we have

$$\begin{aligned}
I_1 &= 2\frac{(z_x + z_y)^2 + 3e(z_x + z_y) + 4e^2}{5ez_x + ez_y + 4e^2},\\
I_2 &= 2\frac{(z_x + z_y)^2 + 3e(z_x + z_y) + 4e^2}{ez_x + 5ez_y + 4e^2},\\
I_3 &= 2^{3/2}\,\frac{7z_x^2 + 6z_x z_y - z_y^2 + 33ez_x + 5ez_y + 21e^2}{e^{1/2}(5z_x + z_y + 4e)^{3/2}},\\
I_4 &= 2^{3/2}\,\frac{-z_x^2 + 6z_x z_y + 7z_y^2 + 5ez_x + 33ez_y + 21e^2}{e^{1/2}(5z_x + z_y + 4e)^{3/2}},\\
I_5 &= 2^{5/2}\,\frac{(z_x + z_y)^3 + 7e(z_x + z_y)^2 + 17e^2(z_x + z_y) + 21e^3}{e^{3/2}(5z_x + z_y + 4e)^{3/2}}.
\end{aligned}$$

All invariants are independent of x, y, z, reflecting the fact that $x \mapsto x + t_1$, $y \mapsto y + t_2$, $z \mapsto z + t_3$ are symmetries of equation (52). One easily checks that I_1, I_2 are functionally independent, but it is still not straightforward to express z_x, z_y in terms of I_1, I_2 explicitly.

To establish the dependence of I_s, $s > 3$, on I_1, I_2, we observe that for every s there exists a polynomial $P_s(z_x, z_y, I_s)$ such that I_s is a solution of the equation $P_s = 0$. Then what we need is eliminating z_x, z_y from the system

$$\begin{aligned}
&I_1 - 2\frac{(z_x + z_y)^2 + 3e(z_x + z_y) + 4e^2}{5ez_x + ez_y + 4e^2} = 0,\\
&I_2 - 2\frac{(z_x + z_y)^2 + 3e(z_x + z_y) + 4e^2}{ez_x + 5ez_y + 4e^2} = 0,\\
&P_s(z_x, z_y, I_s) = 0.
\end{aligned}$$

To this end, it suffices to compute the Gröbner basis of the last system with respect to an "elimination ordering" of monomials. With the help of the *Groebner* package of *Maple 10* the following quadratic equation for I_3,

$$\begin{aligned}
0 = 4096\, I_2^6 I_3^2 - I_2^3(&729\, I_1^3 I_2^3 - 1971\, I_1^3 I_2^2 + 20493\, I_1^2 I_2^3\\
&+ 3563\, I_1^3 I_2 - 51114\, I_1^2 I_2^2 + 183915\, I_1 I_2^3\\
&+ 3951\, I_1^3 - 52723\, I_1^2 I_2 + 45517\, I_1 I_2^2 + 102191\, I_2^3) I_3\\
+ (&27\, I_1^3 I_2^2 - 81\, I_1^2 I_2^3 - 32\, I_1^3 I_2 + 426\, I_1^2 I_2^2 - 1206\, I_1 I_2^3\\
&- 44\, I_1^3 + 270\, I_1^2 I_2 - 800\, I_1 I_2^2 - 1114\, I_2^3)^2
\end{aligned}$$

can be found rather quickly as well as similar quadratic equations for I_4, I_5. The assumptions of Sect. 5.1 are satisfied as well. In particular, Δ_1, Δ_2 are nonzero since

$$\Delta_1 = \Delta_2 = -128\frac{(z_x + z_y)(3z_x + 3z_y + 8e)(z_x + z_y + e)^4}{e^2(z_x + 5z_y + 4e)^2(5z_x + z_y + 4e)^2} \times \frac{z_x^2 + 2z_xz_y + z_y^2 + 3ez_x + 3ez_y + 4e^2}{z_x^2 + 6z_xz_y + z_y^2 + 2ez_x + 2ez_y + e^2}.$$

This enables us to compute the higher invariants. For instance, I_6 is solution of the quadratic equation

$$\begin{aligned} 0 = &-16I_1^2(27I_1^4I_2 - 27I_1^3I_2^2 + 22I_1^4 - 56I_1^3I_2 - 2I_1^2I_2^2 \\ &\quad + 8I_1^2I_2 - 42I_1^3 + 50I_1I_2^2 + 28I_1^2 + 56I_1I_2 + 28I_2^2)I_6^2 \\ &+ I_1(I_1I_2 - I_1 - I_2)(9I_1I_2 + 7I_1 + 7I_2)(3I_1^3I_2 - 3I_1^2I_2^2 \\ &\quad - 26I_1^3 - 34I_1^2I_2 - 8I_1I_2^2 + 18I_1^2 + 36I_1I_2 + 18I_2^2)I_6 \\ &+ (I_1 + I_2)^2(I_1I_2 - I_1 - I_2)(9I_1I_2 + 7I_1 + 7I_2)(I_1I_2 - 2I_1 - 2I_2)^2. \end{aligned}$$

Although every invariant computed so far depends on e, its expression in terms of I_1, I_2 does not. This suggests the idea that the parameter e is removable. And indeed, after substitution $z \mapsto ez$ equation (52) becomes equivalent to itself with $e = 1$. Thus, the family of equations (52) consists of a continuum of generic members with $e \neq 0$, which are all mutually equivalent, and a single non-generic member with $e = 0$ (in which case $\Lambda_1 = \Lambda_2 = 0$).

Example 7.3. Consider the family of equations

$$\tfrac{1}{4}(z_{xx}z_{yy} - z_{xy}^2) + y^2z_{xx} - 2xyz_{xy} + x^2z_{yy} + ex^2y^2 = 0,$$

depending on a real parameter $e \neq 4$. Then the first five invariants are constants

$$\begin{aligned} I_1 = I_2 &= 2\frac{e + 12}{e - 4}, \\ I_3 = I_4 &= \frac{800}{e - 4}, \\ I_5 &= 3200\frac{(e + 12)^2}{(e - 4)^3}, \end{aligned}$$

while the higher invariants I_s are undefined.

The equation belongs to the subclass "h", or "p", or "e" (see 4.5.1) if $e > -4$ or $e = -4$ or $e < -4$, respectively.

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