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Comparison theorems for noncanonical third order nonlinear differential equations

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Abstract: The aim of our paper is to study oscillatory and asymptotic properties of solutions of nonlinear differential equations of the third order with quasiderivatives. We prove comparison theorems on property A between linear and nonlinear equations. Some integral criteria ensuring property A for nonlinear equations are also given. Our assumptions on the nonlinearity of f are restricted to its behavior only in a neighborhood of zero and a neighborhood of infinity.

Keywords: Comparison theorem, property A, quasiderivative, noncanonical form, nonlinear equation
MSC (2000): 34C10

1 Introduction

Consider the third-order nonlinear differential equations that have quasiderivatives of the form:

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)f(x(t)) = 0, \quad t \geq 0 \tag{N}$$

where

$$r, p, q \in C([0,\infty), \mathbb{R}),\ r(t) > 0,\ p(t) > 0,\ q(t) > 0 \text{ on } [0,\infty), \tag{H1}$$

$$f \in C(\mathbb{R}, \mathbb{R}), \quad f(u)u > 0 \quad \text{for} \quad u \neq 0. \tag{H2}$$

With no restatement of conditions (H1) and (H2), we shall assume their validity throughout this paper.

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For the sake of brevity, we put

$$x^{[0]} = x, \quad x^{[1]} = \frac{1}{r}x', \quad x^{[2]} = \frac{1}{p}\left(\frac{1}{r}x'\right)' = \frac{1}{p}\left(x^{[1]}\right)', \quad x^{[3]} = \frac{1}{q}\left(\frac{1}{p}\left(\frac{1}{r}x'\right)'\right)' = \frac{1}{q}\left(x^{[2]}\right)'.$$

The functions $x^{[i]}$, i=0, 1, 2, 3, we call the *quasiderivatives* of x. In addition to $(H1)$ and $(H2)$, we shall occasionally assume that

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > 0 \tag{H3}$$

and

$$\liminf_{u\to 0} \frac{f(u)}{u} > 0. \tag{H4}$$

By a *solution* of an equation of the form (N), we mean a function $w \in C^1([0,\infty),\mathbb{R})$ such that $w^{[1]}(t)$, $w^{[2]}(t) \in C^1([0,\infty),\mathbb{R})$, satisfying equation (N) for all $t \geq 0$. Any solution of (N) is said to be *proper* if it is defined on the interval $[0,\infty)$ and is nontrivial in any neighborhood of infinity. A proper solution is said to be *oscillatory* if it has a sequence of zeros converging to ∞; otherwise it is said to be *nonoscillatory*. Furthermore, equation (N) is called *oscillatory* if it has at least one nontrivial oscillatory solution; it is called *nonoscillatory* if all of its solutions are nonoscillatory.

The relevance of the study of the asymptotic behavior of solutions is often established by introducing the concept of *equation with property A*. More precisely, equation (N) is said to have *property A* if any proper solution x of (N) is either oscillatory or satisfies the condition

$$|x^{[i]}(t)| \downarrow 0 \text{ as } t \to \infty \quad \text{for } \ i = 0, 1, 2.$$

The notation $u(t) \downarrow 0$ means that function u monotonically decreases to zero as $t \to \infty$.

The special case of equation (N) satisfying (H1)–(H4) is the linear equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}x'(t)\right)'\right)' + q(t)x(t) = 0, \quad t \geq 0 \tag{L}$$

the oscillatory and asymptotic properties of which are studied in [1, 2, 4–6, 10]. The nonlinear case, equation (N), has been thoroughly investigated in [1, 2, 5]. In particular, many papers have been devoted to the study of the oscillatory and asymptotic properties of solutions of differential equations of the n-th order with quasiderivatives. Among the extensive literature on this field, we refer the reader to [7, 8, 11, 12] and to the references contained therein.

As is customary, we shall say that equation (N) $[(L)]$ is in the *canonical form* if $\int^{\infty} r(t)\,dt = \int^{\infty} p(t)\,dt = \infty$. We would like to point out that most of the results of such research (especially the comparison theorems) necessitate the canonical form of the differential equations under investigation (see, e.g., [2, 3, 5, 7–10, 12]).

The aim of this paper is to continue the study of equation (N) $[(L)]$ in the *noncanonical form* (i.e., the case where $\int^{\infty} r(t)\,dt < \infty$ or $\int^{\infty} p(t)\,dt < \infty$ or both integrals converge).

Our research is based on a study of the asymptotic behavior of nonoscillatory solutions of equation (N) as well as on a linearization device. The paper is organized as follows: The second section summarizes some established results for linear equation (L) and some notation that will be useful in our ensuing investigations. In section 3, we prove the comparison theorems on property A between the linear and nonlinear equations. As a result, we obtain sufficient conditions that ensure property A for equation (N). Such results are presented as integral criteria that involve only the functions p, r, q. Our findings expand on some of the results in [2, 5] for the canonical case. Several examples illustrating our main theorems are also provided.

We point out that our assumptions on the nonlinearity of f are restricted to its behavior only in a neighborhood of zero and a neighborhood of infinity. Not only are monotonicity conditions unnecessary but also no assumptions on the behavior of f in $\mathbb{R}$ are required.

2 Preliminary results

In the recent papers [1, 2, 5, 6], others have studied relationships between property A and oscillation as well as the oscillatory and asymptotic properties of solutions of linear equation (L). We recount some of these results insofar as they inform this sequel.

We start with the following oscillation and nonoscillation criteria, results proved in [6]. Consider the notation

$$I(u_i) = \int_0^\infty u_i(t)\,dt, \qquad I(u_i, u_j) = \int_0^\infty u_i(t) \int_0^t u_j(s)\,ds\,dt, \quad i, j = 1, 2$$

$$I(u_i, u_j, u_k) = \int_0^\infty u_i(t) \int_0^t u_j(s) \int_0^s u_k(b)\,db\,ds\,dt, \quad i, j, k = 1, 2, 3,$$

where u_i, $i = 1, 2, 3$, are continuous positive functions on $[0, \infty)$.
For simplicity, we shall sometimes write $u(\infty)$ instead of $lim_{t\to\infty} u(t)$.

Theorem 2.1. *([6], Theorems 8 and 10) Suppose one of conditions*
(i) $I(p) = I(r) = I(q, r) = \infty$,
(ii) $I(q) = I(p) = I(r, p) = \infty$,
(iii) $I(r) = I(q) = I(p, q) = \infty$.
is satisfied. Then equation (L) *is oscillatory.*

Theorem 2.2. *([6], Theorems 5 and 7) Suppose one of the integrals*

$$I(q, r, p), \qquad I(p, q, r), \qquad I(r, p, q)$$

is convergent. Then equation (L) *is nonoscillatory.*

Let $\mathcal{N}(N)$ and $\mathcal{N}(L)$ denote the sets of all proper nonoscillatory solutions of (N) and (L), respectively. The sets $\mathcal{N}(N)$ and $\mathcal{N}(L)$ can be divided into the following four classes in the same way as in [1, 2, 5]:

$$\mathcal{N}_0 = \{x \in \mathcal{N}(N)\, [x \in \mathcal{N}(L)],\, \exists T_x\, :\, x(t)x^{[1]}(t) < 0,\, x(t)x^{[2]}(t) > 0\, \text{for}\, t \geq T_x\}$$

$$\mathcal{N}_1 = \{x \in \mathcal{N}(N)\, [x \in \mathcal{N}(L)],\, \exists T_x\, :\, x(t)x^{[1]}(t) > 0,\, x(t)x^{[2]}(t) < 0\, \text{for}\, t \geq T_x\}$$

$$\mathcal{N}_2 = \{x \in \mathcal{N}(N)\, [x \in \mathcal{N}(L)],\, \exists T_x\, :\, x(t)x^{[1]}(t) > 0,\, x(t)x^{[2]}(t) > 0\, \text{for}\, t \geq T_x\}$$

$$\mathcal{N}_3 = \{x \in \mathcal{N}(N)\, [x \in \mathcal{N}(L)],\, \exists T_x\, :\, x(t)x^{[1]}(t) < 0,\, x(t)x^{[2]}(t) < 0\, \text{for}\, t \geq T_x\}$$

If $x \in \mathcal{N}_0$, then its quasiderivatives satisfy the inequality $x^{[i]}(t)x^{[i+1]}(t) < 0$ for $i = 0, 1, 2$, for all sufficiently large t. Using the terminology in [1, 2, 4, 5], we call it a *Kneser solution*.

The asymptotic properties of Kneser solutions of equation (L) are given by the following lemma.

Lemma 2.3. *([5], Lemma 4)*
(i) If there exists $x \in \mathcal{N}_0$ such that $\lim_{t\to\infty} x(t) \neq 0$, then $I(q,p,r) < \infty$.
(ii) If there exists $x \in \mathcal{N}_0$ such that $\lim_{t\to\infty} x^{[1]}(t) \neq 0$, then $I(r,q,p) < \infty$.
(iii) If there exists $x \in \mathcal{N}_0$ such that $\lim_{t\to\infty} x^{[2]}(t) \neq 0$, then $I(p,r,q) < \infty$.

Finally, we introduce the result that connects property A to oscillation and to the integral behavior of functions p, r, q.

Theorem 2.4. *([1], Theorem 2.2) The following assertions are equivalent:*
(i) (L) has property A.
(ii) (L) is oscillatory and $I(q,p,r) = I(r,q,p) = I(p,r,q) = \infty$.

3 Main results

We begin our consideration with a comparison theorem.

Theorem 3.1. *Assume* $(H3)$, $(H4)$, $I(r) < \infty$ *and* $I(p) = I(r,p) = I(q) = \infty$*. If equation* (L) *has property A, then equation* (N) *has property A.*

Proof. Let x be a proper nonoscillatory solution of (N). We know that any proper nonoscillatory solution x of equation (N) belongs to one of four classes. More specifically, $x \in \mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$. We assume that there exists $T \geq 0$ such that $x(t) > 0$ for all $t \geq T$. Now, suppose that (N) does not have property A. There are, then, four possibilities: *I.* $x \in \mathcal{N}_1$, *II.* $x \in \mathcal{N}_2$, *III.* $x \in \mathcal{N}_0$ such that $\lim_{t\to\infty} x^{[i]}(t) \neq 0$ for some $i \in \{0,1,2\}$, *IV.* $x \in \mathcal{N}_3$.

Case I. Since x is a positive nonoscillatory solution of (N) in the class $\mathcal{N}_1$, there exists $T_1 \geq T$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) < 0$ for all $t \geq T_1$. Moreover, since x is a positive increasing function, either *i)* $\lim_{t\to\infty} x(t) = \alpha < \infty$ or *ii)* $\lim_{t\to\infty} x(t) = \infty$.

In case (i), the continuity of the function $f(u)$ on the interval $[x(T_2), \alpha]$ (where $T_2 \geq T_1$) ensures the existence of a positive constant K such that

$$\frac{f(x(t))}{x(t)} \geq K \qquad \text{for all } t \geq T_2. \tag{1}$$

In case (ii), the corresponding inequality (1) holds for some positive constant K_1 for all $t \geq T_3 \geq T_1$ insofar as $(H3)$ holds. Now we see that there exists a positive number K_2 and $T_4 \geq T_1$ such that

$$\frac{f(x(t))}{x(t)} \geq K_2 \qquad \text{for all } t \geq T_4\,. \tag{2}$$

As a positive decreasing function, $x^{[1]}(t)$ is bounded. By integrating equation (N) twice in the interval $[T_4, t]$, we see that

$$x^{[1]}(t) = x^{[1]}(T_4) + x^{[2]}(T_4)\int_{T_4}^{t} p(s)\,ds - \int_{T_4}^{t} p(s)\int_{T_4}^{s} q(u)f(x(u))\,du\,ds$$

or

$$x^{[1]}(t) < x^{[1]}(T_4) - \int_{T_4}^{t} p(s)\int_{T_4}^{s} q(u)f(x(u))\,du\,ds\,.$$

Using this expression with (2), we obtain

$$x^{[1]}(T_4) - x^{[1]}(t) > \int_{T_4}^{t} p(s)\int_{T_4}^{s} q(u)f(x(u))\,du\,ds \geq$$

$$\geq K_2\int_{T_4}^{t} p(s)\int_{T_4}^{s} q(u)x(u)\,du\,ds \geq K_2 x(T_4)\int_{T_4}^{t} p(s)\int_{T_4}^{s} q(u)\,du\,ds\,.$$

When $t \to \infty$, we have that $I(p,q) < \infty$. Taken together with $I(r) < \infty$, we have that $I(r,p,q) < \infty$. However, inasmuch as equation (L) has property A, equation (L) is oscillatory by Theorem 2.4. Now, Theorem 2.2 yields $I(r,p,q) = \infty$, a contradiction.

Case II. Inasmuch as x is a positive nonoscillatory solution of (N) in the class $\mathcal{N}_2$, there exists $T_1 \geq T$ such that $x(t) > 0$, $x^{[1]}(t) > 0$, $x^{[2]}(t) > 0$ for all $t \geq T_1$. Since $\left(x^{[2]}(t)\right)' = -q(t)f(x(t)) < 0$ for all $t \geq T_1$, $x^{[2]}(t)$ is a positive decreasing function. Hence, $0 \leq x^{[2]}(\infty) < \infty$. Just as in case I, from the positive increasing nature of the function x, we establish the validity of (2). Integrating equation (N) in $[T_4, \infty)$, we obtain

$$x^{[2]}(T_4) - x^{[2]}(\infty) = \int_{T_4}^{\infty} q(t)f(x(t))\,dt\,.$$

Given that $0 \leq x^{[2]}(\infty) < \infty$ and (2), there exists a positive constant c such that

$$c = \int_{T_4}^{\infty} q(t)f(x(t))\,dt \geq K_2\int_{T_4}^{\infty} q(t)x(t)\,dt \geq K_2 x(T_4)\int_{T_4}^{\infty} q(t)\,dt\,,$$

contradicting $I(q) = \infty$.

Case III. Let $x \in \mathcal{N}_0$ such that $\lim_{t\to\infty} x^{[i]}(t) \neq 0$ for some $i \in \{0,1,2\}$. Consider the linearized equation

$$\left(\frac{1}{p(t)}\left(\frac{1}{r(t)}w'(t)\right)'\right)' + q(t)F(t)w(t) = 0\,, \tag{$\mathrm{L_F}$}$$

where $F(t) = \dfrac{f(x(t))}{x(t)}$. For its nonoscillatory solution $w \equiv x$, equation (L_F) has a Kneser solution such that $\lim_{t\to\infty} w^{[i]}(t) \neq 0$ for some $i \in \{0,1,2\}$. By Lemma 2.3, at least one of integrals $I(qF,p,r)$, $I(r,qF,p)$, or $I(p,r,qF)$ is convergent. For positive decreasing x, either *i)* $\lim_{t\to\infty} x(t) = \beta > 0$ or *ii)* $\lim_{t\to\infty} x(t) = 0$.
In case (i), the continuity of the function $f(u)$ on the interval $[\beta, x(T)]$ (where $T \geq 0$) ensures the existence of a positive constant M such that

$$F(t) = \frac{f(x(t))}{x(t)} \geq M \qquad \text{for all } t \text{ sufficiently large.} \tag{3}$$

In case (ii), based on $(H4)$, the inequality of the form (3) holds for some positive constant M_1 for all t sufficiently large. We see, then, that there exists a positive number M_2 such that

$$\frac{f(x(t))}{x(t)} \geq M_2 \qquad \text{for all } t \text{ sufficiently large}\,. \tag{4}$$

Now, from the inequality (4), we get

$$M_2 I(q,p,r) \leq I(qF,p,r),\ \ M_2 I(r,q,p) \leq I(r,qF,p),\ \ M_2 I(p,r,q) \leq I(p,r,qF)$$

and, so, at least one of integrals $I(q,p,r)$, $I(r,q,p)$, or $I(p,r,q)$ is convergent. However, since equation (L) has property A, it follows from Theorem 2.4 that all of these integrals are divergent, a contradictory result.

Case IV. Let $x \in \mathcal{N}_3$. Then there exists $T_1 \geq T$ such that $x(t) > 0$, $x^{[1]}(t) < 0$, $x^{[2]}(t) < 0$ for all $t \geq T_1$. As a positive decreasing function, x is bounded. We have $-x^{[2]}(t) \geq -x^{[2]}(T_1) > 0$ for all $t \geq T_1$ because $x^{[2]}$ is a negative decreasing function. Integrating this inequality twice in $[T_1, t]$, we obtain

$$x(t) \leq x(T_1) + x^{[1]}(T_1)\int_{T_1}^{t} r(s)\,ds + x^{[2]}(T_1)\int_{T_1}^{t} r(s)\int_{T_1}^{s} p(u)\,du\,ds$$

or

$$x(t) < x(T_1) + x^{[2]}(T_1)\int_{T_1}^{t} r(s)\int_{T_1}^{s} p(u)\,du\,ds\,.$$

When $t \to \infty$, we get a contradiction because the function x is positive for all $t \geq T_1$.

The case $x(t) < 0$ for all $t \geq T^*$ may be treated similarly. Thus, we have proved that any proper solution x of equation (N) is either oscillatory or belongs to the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} x^{[i]}(t) = 0$ for all $i \in \{0,1,2\}$. This completes the proof. □

Theorem 3.1, together with integral criteria that ensure property A for equation (L), implies the following result.

Corollary 3.2. *Assume (H3) and (H4) and assume that one of the following conditions is satisfied:*
(i) $I(r) < \infty$, $I(p) = I(r,p) = I(q) = I(r,q) = \infty$,
(ii) $I(r,q) < \infty$, $I(p) = I(r,p) = I(q) = \infty$ *and*

$$\int_0^\infty p(t)\left(\int_t^\infty r(s)\,ds\right)\left(\int_t^\infty q(s)\int_s^\infty r(a)\,da\,ds\right)dt = \infty.$$

Then Equation (N) has property A.

Proof. From Theorems 4 and 5 in [4], it follows that equation (L) has property A. Now we get the assertion from Theorem 3.1. □

The following example illustrates the statement of Theorem 3.1 .

Example 3.3. Consider the differential equation given by

$$\left(\frac{1}{t+1}\left((t+1)^3 x'(t)\right)'\right)' + (t+1)^2\left(x^3(t)+x(t)\right) = 0\,, \quad t \geq 0\,. \tag{5}$$

This equation has the form (N) where $f(u) = u^3 + u$, $r(t) = 1/(t+1)^3$, $p(t) = t+1$ and $q(t) = (t+1)^2$. Here, $I(r) < \infty$ and, so, the equation under consideration is in the non-canonical form. It is easy to verify that the assumptions of Theorem 3.1 are satisfied. Moreover, the corresponding linear equation has property A (see Theorem 4, part (iii) in [4]). Thus, by Theorem 3.1, the nonlinear equation (5) has property A.

Theorem 3.4. *Assume (H3), (H4),* $I(p) < \infty$, *and* $I(r) = I(p,q) = I(q) = \infty$. *If equation (L) has property A, then equation (N) has property A.*

Proof. Let x be a proper nonoscillatory solution of (N). Then $x \in \mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ and we assume that there exists $T \geq 0$ such that $x(t) > 0$ for all $t \geq T$. Now, suppose that (N) does not have property A. There are, then, four possibilities: *I.* $x \in \mathcal{N}_1$, *II.* $x \in \mathcal{N}_2$, *III.* $x \in \mathcal{N}_0$ such that $\lim_{t\to\infty} x^{[i]}(t) \neq 0$ for some $i \in \{0,1,2\}$, *IV.* $x \in \mathcal{N}_3$.

Given that $I(r) = \infty$ implies $I(r,p) = \infty$, cases II, III, and IV yield contradictions in the same way as in the proof of Theorem 3.1. In case I, proceeding as in the proof of Theorem 3.1, we get $I(p,q) < \infty$, a contradiction.

The case $x(t) < 0$ for all $t \geq T^*$ may be treated similarly. This completes the proof. □

From Theorem 3.4, we obtain the following:

Corollary 3.5. *Assume (H3) and (H4) and assume that one of the following conditions is satisfied:*
(i) $I(p) < \infty$, $I(r) = I(p,q) = I(q) = I(p,r) = \infty$,

(ii) $I(p,r) < \infty$, $I(r) = I(p,q) = I(q) = \infty$ *and*

$$\int_0^\infty q(t) \left(\int_t^\infty p(s)\, ds \right) \left(\int_t^\infty r(s) \int_s^\infty p(a)\, da\, ds \right) dt = \infty.$$

Then equation (N) *has property A.*

Proof. From Theorems 4 and 5 in [4], it follows that equation (L) has property A. Now we get the assertion from Theorem 3.4. □

Remark 3.6. Note that the resulting integral criteria extend to equation (N) from other criteria that are stated for equation (L) (see Theorems 4 and 5 in [4]) and for equation (N) in the canonical form (see Corollary 4 in [5]).

The following example illustrates the statement of Theorem 3.4 .

Example 3.7. Consider the differential equation

$$\left((t+1)^3 \left(\frac{1}{t+1} x'(t) \right)' \right)' + (t+1)^3 \left(x^3(t) + x(t) \right) = 0 \,, \quad t \geq 0 \,. \tag{6}$$

This equation has the form (N) where $f(u) = u^3 + u$, $r(t) = t+1$, $p(t) = 1/(t+1)^3$ and $q(t) = (t+1)^3$. Here, $I(p) < \infty$ and, so, the equation under consideration is in the non-canonical form. It is easy to verify that the assumptions of Theorem 3.4 are satisfied. Moreover, the corresponding linear equation has property A (see Theorem 4, part (i) in [4]). Thus, by Theorem 3.4, the nonlinear equation (6) has property A.

Remark 3.8. Our comparison results (Theorems 3.1 and 3.4) extend other results that have been proved for the differential equations in the canonical form (see, e.g., Theorem 3 in [2] and Theorem 4 in [5]).

We obtain a somewhat weaker result than the one above if the integrals $I(p)$ and $I(r)$ are both convergent. The following theorem holds:

Theorem 3.9. *Assume* $(H3)$, $(H4)$, $I(p) < \infty$ *and* $I(r) < \infty$. *If equation* (L) *has property A, then equation* (N) *has property A or any solution* x *of equation* (N) *from the class* $\mathcal{N}_3$ *tends to zero as* $t \to \infty$.

Proof. Let x be a proper nonoscillatory solution of (N). It follows that $x \in \mathcal{N}_0 \cup \mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$. Moreover, we assume that there exists $T \geq 0$ such that $x(t) > 0$ for all $t \geq T$. Now, suppose that (N) does not have property A and that there exists a solution x of equation (N) from the class $\mathcal{N}_3$ that tends to a positive constant as t tends to infinity. Then, there are four possibilities: *I.* $x \in \mathcal{N}_1$, *II.* $x \in \mathcal{N}_2$, *III.* $x \in \mathcal{N}_0$ such that $\lim_{t\to\infty} x^{[i]}(t) \neq 0$ for some $i \in \{0,1,2\}$, *IV.* $x \in \mathcal{N}_3$ such that $\lim_{t\to\infty} x(t) = c > 0$.

In each of cases I and III, we get a contradiction in the same way as in the proof of Theorem 3.1.

Case II. Proceeding as in the proof of Theorem 3.1, we get $I(q) < \infty$. Taken together with $I(r) < \infty$ and $I(p) < \infty$, we have that $I(q,p,r) < \infty$. However, since equation (L) has property A, it follows from Theorem 2.4 that $I(q,p,r) = \infty$, a contradiction.

Case IV. Let $x \in \mathcal{N}_3$ such that $\lim_{t\to\infty} x(t) = c > 0$. Then, there exists $T_1 \geq T$ such that $x(t) > 0$, $x^{[1]}(t) < 0, x^{[2]}(t) < 0$ for all $t \geq T_1$. Integrating equation (N) three times in the interval $[T_1, t]$ we obtain

$$x(t) = x(T_1) + x^{[1]}(T_1)\int_{T_1}^{t} r(s)\,ds + x^{[2]}(T_1)\int_{T_1}^{t} r(s)\int_{T_1}^{s} p(k)\,dk\,ds - \\ - \int_{T_1}^{t} r(s)\int_{T_1}^{s} p(k)\int_{T_1}^{k} q(a)f(x(a))\,da\,dk\,ds$$

or

$$x(t) < x(T_1) - \int_{T_1}^{t} r(s)\int_{T_1}^{s} p(k)\int_{T_1}^{k} q(a)f(x(a))\,da\,dk\,ds \qquad \text{for all } t \geq T_1\,.$$

Inasmuch as x is a positive decreasing function such that $\lim_{t\to\infty} x(t) = c > 0$, we have that $0 < c \leq x(t) \leq x(T_2) < \infty$ for all $t \geq T_2$ where $T_2 \geq T_1$. The continuity of the function $f(u)/u$ on the interval $[c, x(T_2)]$ ensures the existence of a positive constant K_1 such that $f(x(t)) \geq K_1 x(t)$ for all $t \geq T_2$. Thus, we have that

$$x(t) \leq x(T_1) - K_1\int_{T_2}^{t} r(s)\int_{T_2}^{s} p(k)\int_{T_2}^{k} q(a)x(a)\,da\,dk\,ds \qquad \text{for all } t \geq T_2\,.$$

Based on the positive decreasing nature of x, we obtain

$$x(t) \leq x(T_1) - K_1 x(t)\int_{T_2}^{t} r(s)\int_{T_2}^{s} p(k)\int_{T_2}^{k} q(a)\,da\,dk\,ds \qquad \text{for all } t \geq T_2$$

or

$$x(t)\left(1 + K_1\int_{T_2}^{t} r(s)\int_{T_2}^{s} p(k)\int_{T_2}^{k} q(a)\,da\,dk\,ds\right) \leq x(T_1) \qquad \text{for all } t \geq T_2.$$

When $t \to \infty$, $I(r,p,q) < \infty$. However, by Theorem 2.4, equation (L) is oscillatory because equation (L) has property A. Now, according to Theorem 2.2, $I(r,p,q) = \infty$, a contradictory result.

The case $x(t) < 0$ for all $t \geq T^*$ may be treated similarly. We have just proved that any proper solution x of equation (N) either is oscillatory or belongs to the class $\mathcal{N}_0$ such that $\lim_{t\to\infty} x^{[i]}(t) = 0$ for all $i \in \{0,1,2\}$ or belongs to the class $\mathcal{N}_3$ such that $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

Remark 3.10. Let us remark that, so far, we have not found any results that are comparable with our Theorem 3.9.

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