Central European Journal of Mathematics
DOI: 10.2478/s11533-006-0045-2
Research article
CEJM 5(1) 2007 1–18

A family of regular vertex operator algebras with two generators

Dražen Adamović*

Department of Mathematics,
University of Zagreb,
10 000 Zagreb, Croatia

Received 7 September 2006; accepted 20 November 2006

Abstract: For every $m \in \mathbb{C} \setminus \{0, -2\}$ and every nonnegative integer k we define the vertex operator (super)algebra $D_{m,k}$ having two generators and rank $\frac{3m}{m+2}$. If m is a positive integer then $D_{m,k}$ can be realized as a subalgebra of a lattice vertex algebra. In this case, we prove that $D_{m,k}$ is a regular vertex operator (super)algebra and find the number of inequivalent irreducible modules.

Keywords: Vertex operator algebras, vertex operator superalgebras, rationality, regularity, lattice vertex operator algebras
MSC (2000): 17B69

1 Introduction

In the theory of vertex operator (super)algebras, the classification and construction of rational vertex operator (super)algebras are important problems. These problems are connected with the classification of rational conformal field theories in physics. The rationality of certain familiar vertex operator (super)algebras was proved in papers [1–3, 7, 8, 14, 20, 25].

It is natural to consider rational vertex operator (super)algebras of certain rank. In particular, in the rank one case for every positive integer k we have the well-known rational vertex operator (super)algebra F_k associated to the lattice $\sqrt{k}\mathbb{Z}$. These vertex operator (super)algebras are generated by two generators.

In the present paper we will be concentrated on vertex operator (super)algebras of rank

* E-mail: adamovic@math.hr

$c_m = \frac{3m}{m+2}$, $m \in \mathbb{C} \setminus \{0, -2\}$. This rank has the vertex operator algebra $L(m, 0)$ associated to the irreducible vacuum $\hat{sl_2}$–module of level m and the vertex operator superalgebra L_{c_m} associated to the vacuum module for the $N = 2$ superconformal algebra with central charge c_m (cf. [2, 3, 11, 15–17]). In the case $m = 1$ these vertex operator (super)algebras are included into the family F_k, $k \in \mathbb{N}$, since $L(1, 0) \cong F_2$ and $L_{c_1} \cong F_3$. The main purpose of this article is to include $L(m, 0)$ and L_{c_m} into the family $D_{m,k}$, $k \in \mathbb{Z}_{\geq 0}$, of rational vertex operator (super)algebras of rank c_m for arbitrary positive integer m.

In fact, for every $m \in \mathbb{C} \setminus \{0, -2\}$ we define the vertex operator (super)algebra $D_{m,k}$ as a subalgebra of the vertex operator (super)algebra $L(m, 0) \otimes F_k$ (cf. Section 4). In the special case $k = 1$, $D_{m,1}$ is in the $N = 2$ vertex operator superalgebra L_{c_m} constructed by using the Kazama-Suzuki mapping (cf. [15, 19]). We also have that $D_{m,0} \cong L(m, 0)$ and $D_{1,k} \cong F_{k+2}$. Moreover, we shall demonstrate that $D_{m,k}$ has many properties similar to those of affine and $N = 2$ superconformal vertex algebras.

When m is not a nonnegative integer, then $D_{m,k}$ has infinitely many irreducible representations. Thus, it is not rational (cf. Section 4). In order to construct new examples of rational vertex operator (super)algebras we shall consider the case when m is a positive integer. Then $D_{m,k}$ can be embedded into a lattice vertex algebra (cf. Section 5). In fact, we shall prove that

$$D_{m,k} \otimes F_{-k} \cong L(m, 0) \otimes F_{-\frac{k}{2}(mk+2)} \quad (k \text{ even}), \tag{1}$$

$$D_{m,k} \otimes F_{-k} \cong L(m, 0) \otimes F_{-2k(mk+2)} \oplus L(m, m) \otimes MF_{-2k(mk+2)} \quad (k \text{ odd}). \tag{2}$$

These relations completely determine the structure of $D_{m,k} \otimes F_{-k}$ as a weak $L(m, 0)$–module.

In [9] the notion of a regular vertex operator algebra was introduced, i. e. rational vertex operator algebra with the property that every weak module is completely reducible. The relations (1) and (2), together with the regularity results from [9] and [21] imply that $D_{m,k}$ is a simple regular vertex operator algebra if k is even, and a simple regular vertex operator superalgebra if k is odd. It was shown in [5] that regularity is equivalent to rationality and C_2–cofiniteness. Therefore, vertex operator (super)algebras $D_{m,k}$ are also rational and C_2–cofinite.

Let us here discuss the case $k = 2n$, where n is a positive integer. The relation (1) suggests that one can study the dual pair $(D_{m,2n}, F_{-2n})$ directly inside $L(m, 0) \otimes F_{-2n(nm+1)}$. This approach requires many deep results on the structure of the vertex operator algebra $L(m, 0)$ and deserves to be investigated independently. Instead of this approach, we realize the vertex algebra $D_{m,2n} \otimes F_{-2n}$ inside a larger lattice vertex algebra. Then the formulas for the generators are much simpler (cf. Section 6). The similar analysis can be done when k is odd (cf. Section 7). This approach was also used in [3] for studying the fusion rules for the $N = 2$ vertex operator superalgebra $D_{m,1}$.

Our results show that for every $m \in \mathbb{N}$, there exists an infinite family of rational vertex operator algebras of rank c_m. We believe that these algebras will have an important role in the classification of rational vertex operator algebras of this rank. As an example, in this paper we shall consider in detail the vertex operator (super)algebras of rank $c_4 = 2$.

Then our vertex operator (super)algebras $D_{m,k}$ admit nice realizations. In Section 8 we show that $D_{4,k}$ is a $\mathbb{Z}_2$–orbifold model of a lattice vertex operator superalgebra under an automorphism of order two.

This paper is a slightly modified version of the preprint math.QA/0111055.

2 Preliminaries

In this section we recall the definition of vertex operator superalgebras their modules (cf. [12, 13, 18, 20]). We also recall the basic properties of regular vertex operator superalgebras.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be any $\mathbb{Z}_2$–graded vector space. Then any element $u \in V_{\bar{0}}$ (resp. $u \in V_{\bar{1}}$) is said to be even (resp. odd). We define $|u| = \bar{0}$ if u is even and $|u| = \bar{1}$ if u is odd. Elements in $V_{\bar{0}}$ or $V_{\bar{1}}$ are called homogeneous. Whenever $|u|$ is written, it is understood that u is homogeneous.

Definition 2.1. A vertex superalgebra is a triple $(V, Y, \mathbf{1})$ where $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a $\mathbb{Z}_2$–graded vector space, $\mathbf{1} \in V_{\bar{0}}$ is a specified element called the vacuum of V, and Y is a linear map

$$\begin{aligned} Y(\cdot, z):\ & V \to (\mathrm{End}\ V)[[z, z^{-1}]]; \\ & a \mapsto Y(a,z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1} \in (\mathrm{End}\ V)[[z, z^{-1}]] \end{aligned}$$

satisfying the following conditions for $a, b \in V$:

(V1) $|a_n b| = |a| + |b|$.

(V2) $a_n b = 0$ for n sufficiently large.

(V3) $[D, Y(a,z)] = Y(D(a), z) = \frac{d}{dz} Y(a,z)$,
where $D \in \mathrm{End}\ V$ is defined by $D(a) = a_{-2}\mathbf{1}$.

(V4) $Y(\mathbf{1}, z) = I_V$ (the identity operator on V).

(V5) $Y(a,z)\mathbf{1} \in (\mathrm{End}\ V)[[z]]$ and $\lim_{z \to 0} Y(a,z)\mathbf{1} = a$.

(V6) The following Jacobi identity holds

$$\begin{aligned} & z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(a, z_1)Y(b, z_2) - (-1)^{|a||b|} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(b, z_2)Y(a, z_1) \\ = & z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(a, z_0)b, z_2). \end{aligned}$$

A vertex superalgebra V is called a vertex operator superalgebra if there is a special element $\omega \in V_{\bar{0}}$ (called the *Virasoro element*) whose vertex operator we write in the form $Y(\omega, z) = \sum_{n \in \mathbb{Z}} \omega_n z^{-n-1} = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$, such that

(V7) $[L(m), L(n)] = (m-n)L(m+n) + \delta_{m+n,0} \frac{m^3 - m}{12} c, \quad c = \mathrm{rank}\ V \in \mathbb{C}$.

(V8) $L(-1) = D$.

(V9) $V = \oplus_{n \in \frac{1}{2}\mathbb{Z}} V(n)$ is a $\frac{1}{2}\mathbb{Z}$–graded so that $V_{\bar{0}} = \oplus_{n \in \mathbb{Z}} V(n)$, $V_{\bar{1}} = \oplus_{n \in \frac{1}{2} + \mathbb{Z}} V(n)$, $L(0) \mid_{V(n)} = n I_V \mid_{V(n)}$, $\dim V(n) < \infty$, and $V(n) = 0$ for n sufficiently small.

We shall sometimes refer to the vertex operator superalgebra V as quadruple $(V, Y, \mathbf{1}, \omega)$.

Remark 2.2. If in the definition of vertex (operator) superalgebra the odd subspace $V_{\bar{1}} = 0$ we get the usual definition of vertex (operator) algebra.

We will say that the vertex operator superalgebra is generated by the set S if

$$V = \mathrm{span}_{\mathbb{C}}\{u^1_{n_1} \cdots u^r_{n_r}\mathbf{1} |\ u^1, \ldots, u^r \in S,\ n_1, \ldots, n_r \in \mathbb{Z},\ r \in \mathbb{Z}_{\geq 0}\}.$$

A subspace $I \subset V$ is called an ideal in the vertex operator superalgebra V if $a_n I \subset I$ for every $a \in V$ and $n \in \mathbb{Z}$. A vertex operator superalgebra V is called simple if it does not contain any proper non-zero ideal.

There is a canonical automorphism σ_V of the vertex operator superalgebra V such that $\sigma_V | V_{\bar{0}} = 1$ and $\sigma_V | V_{\bar{1}} = -1$.

Definition 2.3. Let V be a vertex operator superalgebra. A weak V–module is a pair (M, Y_M), where $M = M_{\bar{0}} \oplus M_{\bar{1}}$ is a $\mathbb{Z}_2$–graded vector space, and $Y_M(\cdot, z)$ is a linear map

$$Y_M : V \to \mathrm{End}(M)[[z, z^{-1}]],\ a \mapsto Y_M(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

satisfying the following conditions for $a, b \in V$ and $v \in M$:

(M1) $|a_n v| = |a| + |v|$ for any $a \in V$.

(M2) $Y_M(\mathbf{1}, z) = I_M$.

(M3) $a_n v = 0$ for n sufficiently large.

(M4) The following Jacobi identity holds

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_M(a, z_1) Y_M(b, z_2) - (-1)^{|a||b|} z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y_M(b, z_2) Y_M(a, z_1)$$
$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y_M(Y(a, z_0)b, z_2).$$

A weak V–module (M, Y_M) is called a V–module if

(M5) $M = \coprod_{n \in \mathbb{C}} M(n)$;

(M7) $L(0)u = nu,\ u \in M(n)$; $\dim M(n) < \infty$;

(M8) $M(n) = 0$ for n sufficiently small.

We recall the definition of regular vertex operator algebra introduced by C. Dong, H. Li and G. Mason in [9].

Definition 2.4. The vertex operator superalgebra V is called regular if every weak V–module is a direct sum of irreducible modules.

If vertex operator superalgebra V is regular, then V is also a rational vertex operator superalgebra, meaning that V has only finitely many irreducible modules and that every V–module is completely reducible.

A vertex operator superalgebra V is called C_2–cofinite if the subspace $C_2(V) = \mathrm{span}_{\mathbb{C}}\{u_{-2}v \mid u, v \in V\}$ has finite codimension in V. This condition is important in the representation theory of vertex operator superalgebras.

Proposition 2.5. *([5, 23]) The vertex operator superalgebra V is regular if and only if V is rational and C_2–cofinite.*

Remark 2.6. A regularity result for the affine, Virasoro and lattice vertex operator algebras was obtained in [9]. Regularity of vertex operator superalgebras associated to minimal models for the Neveu-Schwarz and $N = 2$ superconformal algebra was proved in [3, 4].

3 Lattice and affine vertex algebras

In this section, we shall recall the lattice construction of vertex superalgebras from [8, 18].

Let L be a lattice. Set $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ and extend the $\mathbb{Z}$-form $\langle \cdot, \cdot \rangle$ on L to $\mathfrak{h}$. Let $\hat{\mathfrak{h}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{h} \oplus \mathbb{C}c$ be the affinization of $\mathfrak{h}$. We also use the notation $h(n) = t^n \otimes h$ for $h \in \mathfrak{h}, n \in \mathbb{Z}$.

Set $\hat{\mathfrak{h}}^+ = t\mathbb{C}[t] \otimes \mathfrak{h}$; $\hat{\mathfrak{h}}^- = t^{-1}\mathbb{C}[t^{-1}] \otimes \mathfrak{h}$. Then $\hat{\mathfrak{h}}^+$ and $\hat{\mathfrak{h}}^-$ are abelian subalgebras of $\hat{\mathfrak{h}}$. Let $U(\hat{\mathfrak{h}}^-) = S(\hat{\mathfrak{h}}^-)$ be the universal enveloping algebra of $\hat{\mathfrak{h}}^-$. Let $\lambda \in \mathfrak{h}$. Consider the induced $\hat{\mathfrak{h}}$-module

$$M(1, \lambda) = U(\hat{\mathfrak{h}}) \otimes_{U(\mathbb{C}[t] \otimes \mathfrak{h} \oplus \mathbb{C}c)} \mathbb{C} \simeq S(\hat{\mathfrak{h}}^-) \ \ (\text{linearly}),$$

where $t\mathbb{C}[t] \otimes \mathfrak{h}$ acts trivially on $\mathbb{C}$, $t^0 \otimes \mathfrak{h}$ acting as $\langle h, \lambda \rangle$ for $h \in \mathfrak{h}$ and c acts on $\mathbb{C}$ as multiplication by 1. We shall write $M(1)$ for $M(1, 0)$. For $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$ write $h(n) = t^n \otimes h$. Set $h(z) = \sum_{n \in \mathbb{Z}} h(n) z^{-n-1}$.

Then $M(1)$ is a vertex operator algebra which is generated by the fields $h(z)$, $h \in \mathfrak{h}$, and $M(1, \lambda)$, for $\lambda \in \mathfrak{h}$, are irreducible modules for $M(1)$.

Let $\hat{L}$ be the canonical central extension of L by the cyclic group $\langle \pm 1 \rangle$:

$$1 \ \to \langle \pm 1 \rangle \ \to \hat{L} \ \bar{\to} L \ \to 1 \tag{3}$$

with the commutator map $c(\alpha, \beta) = (-1)^{\langle \alpha, \beta \rangle + \langle \alpha, \alpha \rangle \langle \beta, \beta \rangle}$ for $\alpha, \beta \in L$. Let $e : L \to \hat{L}$ be a section such that $e_0 = 1$ and $\epsilon : L \times L \to \langle \pm 1 \rangle$ be the corresponding 2-cocycle. Then $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = (-1)^{\langle \alpha, \beta \rangle + \langle \alpha, \alpha \rangle \langle \beta, \beta \rangle}$,

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta + \gamma) \tag{4}$$

and $e_\alpha e_\beta = \epsilon(\alpha, \beta) e_{\alpha+\beta}$ for $\alpha, \beta, \gamma \in L$. Form the induced $\hat{L}$-module

$$\mathbb{C}\{L\} = \mathbb{C}[\hat{L}] \otimes_{\langle \pm 1 \rangle} \mathbb{C} \simeq \mathbb{C}[L] \ \ (\text{linearly}),$$

where $\mathbb{C}[\cdot]$ denotes the group algebra and -1 acts on $\mathbb{C}$ as multiplication by -1. For $a \in \hat{L}$, write $\iota(a)$ for $a \otimes 1$ in $\mathbb{C}\{L\}$. Then the action of $\hat{L}$ on $\mathbb{C}\{L\}$ is given by: $a \cdot \iota(b) = \iota(ab)$ and $(-1) \cdot \iota(b) = -\iota(b)$ for $a, b \in \hat{L}$.

Furthermore we define an action of $\mathfrak{h}$ on $\mathbb{C}\{L\}$ by: $h \cdot \iota(a) = \langle h, \bar{a} \rangle \iota(a)$ for $h \in \mathfrak{h}, a \in \hat{L}$. Define $z^h \cdot \iota(a) = z^{\langle h, \bar{a} \rangle} \iota(a)$.

The untwisted space associated with L is defined to be

$$V_L = \mathbb{C}\{L\} \otimes_{\mathbb{C}} M(1) \simeq \mathbb{C}[L] \otimes S(\hat{\mathfrak{h}}^-) \ \text{(linearly)}.$$

Then $\hat{L}, \hat{\mathfrak{h}}, z^h$ $(h \in \mathfrak{h})$ act naturally on V_L by acting on either $\mathbb{C}\{L\}$ or $M(1)$ as indicated above. Define $\mathbf{1} = \iota(e_0) \in V_L$. We use a normal ordering procedure, indicated by open colons, which signify that in the enclosed expression, all creation operators $h(n)$ $(n < 0)$, $a \in \hat{L}$ are to be placed to the left of all annihilation operators $h(n), z^h$ $(h \in \mathfrak{h}, n \geq 0)$. For $a \in \hat{L}$, set

$$Y(\iota(a), z) =: e^{\int (\bar{a}(z) - \bar{a}(0) z^{-1})} a z^{\bar{a}} : .$$

Let $a \in \hat{L}$; $h_1, \cdots, h_k \in \mathfrak{h}; n_1, \cdots, n_k \in \mathbb{Z}$ $(n_i > 0)$. Set

$$v = \iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k) \in V_L.$$

Define vertex operator $Y(v, z)$ with

$$: \left(\frac{1}{(n_1 - 1)!} (\frac{d}{dz})^{n_1 - 1} h_1(z) \right) \cdots \left(\frac{1}{(n_k - 1)!} (\frac{d}{dz})^{n_k - 1} h_k(z) \right) Y(\iota(a), z) : . \qquad (5)$$

This gives us a well-defined linear map

$$\begin{aligned} Y(\cdot, z) : \ & V_L \to (\mathrm{End} V_L)[[z, z^{-1}]] \\ & v \mapsto Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}, \ (v_n \in \mathrm{End} V_L). \end{aligned}$$

Let $\{ h_i \mid i = 1, \cdots, d\}$ be an orthonormal basis of $\mathfrak{h}$ and set

$$\omega = \frac{1}{2} \sum_{i=1}^{d} h_i(-1) h_i(-1) \in V_L.$$

Then $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$ gives rise to a representation of the Virasoro algebra on V_L with the central charged d and

$$\begin{aligned} & L(0) \left(\iota(a) \otimes h_1(-n_1) \cdots h_n(-n_k) \right) \\ = & \left(\frac{1}{2} \langle \bar{a}, \bar{a} \rangle + n_1 + \cdots + n_k \right) \left(\iota(a) \otimes h_1(-n_1) \cdots h_k(-n_k) \right). \end{aligned} \qquad (6)$$

The following theorem was proved in [8] and [18].

Theorem 3.1.

(i) The structure $(V_L, Y, \mathbf{1})$ *is a vertex (super)algebra.*

(ii) Assume that L *is a positive definite lattice. Then the structure* $(V, Y, \mathbf{1}, \omega)$ *is a vertex operator (super)algebra.*

Define the Schur polynomials $p_r(x_1, x_2, \cdots)$ in variables $x_1, x_2, \cdots$ by the following equation:

$$\exp\left(\sum_{n=1}^{\infty}\frac{x_n}{n}y^n\right) = \sum_{r=0}^{\infty} p_r(x_1, x_2, \cdots)y^r. \tag{7}$$

For any monomial $x_1^{n_1}x_2^{n_2}\cdots x_r^{n_r}$ we have an element $h(-1)^{n_1}h(-2)^{n_2}\cdots h(-r)^{n_r}\mathbf{1}$ in both $M(1)$ and V_L for $h \in \mathfrak{h}$. Then for any polynomial $f(x_1, x_2, \cdots)$, $f(h(-1), h(-2), \cdots)\mathbf{1}$ is a well-defined element in $M(1)$ and V_L. In particular, $p_r(h(-1), h(-2), \cdots)\mathbf{1}$ for $r \in \mathbb{N}$ are elements of $M(1)$ and V_L.

Suppose $a, b \in \hat{L}$ such that $\bar{a} = \alpha, \bar{b} = \beta$. Then

$$\begin{aligned} Y(\iota(a), z)\iota(b) &= z^{\langle\alpha,\beta\rangle}\exp\left(\sum_{n=1}^{\infty}\frac{\alpha(-n)}{n}z^n\right)\iota(ab) \\ &= \sum_{r=0}^{\infty} p_r(\alpha(-1), \alpha(-2), \cdots)\iota(ab)z^{r+\langle\alpha,\beta\rangle}. \end{aligned} \tag{8}$$

Thus

$$\iota(a)_i\iota(b) = 0 \quad \text{for } i \geq -\langle\alpha,\beta\rangle. \tag{9}$$

Especially, if $\langle\alpha,\beta\rangle \geq 0$, we have $\iota(a)_i\iota(b) = 0$ for $i \geq 0$, and if $\langle\alpha,\beta\rangle = -n < 0$, we get

$$\iota(a)_{i-1}\iota(b) = p_{n-i}(\alpha(-1), \alpha(-2), \cdots)\iota(ab) \quad \text{for } i \in \{0, \ldots, n\}. \tag{10}$$

Let $n \in \mathbb{Z}$, $n \neq 0$, and $\langle\beta,\beta\rangle = n$. Define

$$L_n = \mathbb{Z}\beta, \quad F_n = V_{L_n}.$$

Then F_n is a simple vertex algebra if n is even, and a simple vertex superalgebra if n is odd. For $i \in \mathbb{Z}$, let $\bar{i} = i + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$. We define $F_n^{\bar{i}} = V_{\mathbb{Z}\beta + \frac{i}{n}\beta}$. Clearly $F_n = F_n^{\bar{0}}$. It is well-known (cf. [7, 8, 26]) that the set $\{F_n^{\bar{i}}\}_{i=0,\ldots,|n|-1}$ provides all irreducible F_n–modules. In particular, F_n has $|n|$ inequivalent irreducible modules.

If $n = 2k$ is even, we define $\tilde{L}_{2k} = \frac{\beta}{2} + \mathbb{Z}\beta$, and $MF_{2k} = V_{\tilde{L}_{2k}} = F_{2k}^{\bar{k}}$. Then F_{2k} is a vertex algebra, and MF_{2k} is a F_{2k}–module.

We shall also need the following result from [9].

Proposition 3.2. *[9] Assume that $n \in \mathbb{Z}$, $n \neq 0$. Then the vertex (super)algebra F_n is regular, i.e., any (weak) F_n–module is completely reducible.*

Let $\mathfrak{g}$ be the Lie algebra sl_2 with generators e, f, h and relations $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the corresponding affine Lie algebra of type $A_1^{(1)}$. As usual we write $x(n)$ for $x \otimes t^n$ where $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Let Λ_0, Λ_1 denote the fundamental weights for $\hat{\mathfrak{g}}$. For any complex numbers m, j, let $L(m, j) =$

$L((m-j)\Lambda_0 + j\Lambda_1)$ be the irreducible highest weight $\hat{sl}_2$–module with the highest weight $(m-j)\Lambda_0 + j\Lambda_1$. Then $L(m,0)$ has a natural structure of a simple vertex operator algebra. Let $\mathbf{1}_m$ denote the vacuum vector in $L(m,0)$.

If m is a positive integer then $L(m,0)$ is a regular vertex operator algebra, and the set $\{L(m,j)\}_{j=0,\dots,m}$ provides all inequivalent irreducible $L(m,0)$–modules.

We shall now recall the lattice construction of the vertex operator algebra $L(m,0)$. Define the following lattice

$$A_{1,m} = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_m$$
$$\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j},$$

for every $i,j \in \{1,\dots,m\}$. Define also $\tilde{A}_{1,m} = \frac{\alpha_1+\cdots+\alpha_m}{2} + A_{1,m}$. We have:

Lemma 3.3. *[8] The vectors $\mathcal{E} = \iota(e_{\alpha_1}) + \cdots + \iota(e_{\alpha_m})$, $\mathcal{F} = \iota(e_{-\alpha_1}) + \cdots + \iota(e_{-\alpha_m})$, generate a subalgebra of $V_{A_{1,m}}$ isomorphic to $L(m,0)$. Moreover, $L(m,m)$ is a $L(m,0)$ submodule of $V_{\tilde{A}_{1,m}}$.*

4 The definition of $D_{m,k}$

In this section we give the definition of the vertex operator (super)algebra $D_{m,k}$. Let the vertex (super)algebras $L(m,0)$ and F_k be defined as in Section 3.

Definition 4.1. Let $m \in \mathbb{C} \setminus \{0,-2\}$, and let k be a nonnegative integer. Let $D_{m,k}$ be the vertex subalgebra of the vertex operator (super)algebra $L(m,0) \otimes F_k$ generated by the vectors:

$$\bar{X} = e(-1)\mathbf{1}_m \otimes \iota(e_\beta) \quad \text{and} \quad \bar{Y} = f(-1)\mathbf{1}_m \otimes \iota(e_{-\beta}).$$

Let $\mathbf{1}_{m,k} = \mathbf{1}_m \otimes \mathbf{1} \in D_{m,k} \subset L(m,0) \otimes F_k$. Define the following elements of $D_{m,k}$:

$$\bar{H} = \bar{X}_k \bar{Y} = h(-1)\mathbf{1}_m \otimes \mathbf{1} + m\mathbf{1}_m \otimes \beta(-1)\mathbf{1},$$
$$\omega_{m,k} = \frac{1}{2(m+2)} \left(\bar{X}_{k-1}\bar{Y} + \bar{Y}_{k-1}\bar{X} + \frac{1-k}{mk+2}\bar{H}_{-1}^2 \mathbf{1}_{m,k} \right).$$

Assume that $mk+2 \neq 0$. Then the components of the field

$$Y(\omega_{m,k}, z) = \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$$

give rise a representation of the Virasoro algebra of central charge $c_m = \frac{3m}{m+2}$. We shall now investigate the conformal structure on $D_{m,k}$ defined by the Virasoro element $\omega_{m,k}$. For $n \geq 0$ one has

$$L(n)\bar{X} = \delta_{n,0}(1+\frac{k}{2})\bar{X} \quad \text{and} \quad L(n)\bar{Y} = \delta_{n,0}(1+\frac{k}{2})\bar{Y}. \tag{11}$$

Therefore the generators $\bar{X}$ and $\bar{Y}$ of $D_{m,k}$ are primary vectors of conformal weight $1+\frac{k}{2}$ for the Virasoro algebra. Moreover, the operator $L(0)$ defines on $D_{m,k}$ a $\mathbb{Z}_{\geq 0}$–gradation if k is even and a $\frac{1}{2}\mathbb{Z}_{\geq 0}$–gradation if k is odd.

Assume first that k is even. Then $D_{m,k}$ is a subalgebra of the vertex algebra $L(m,0)\otimes F_k$. Since the operator $L(0)$ defines on $D_{m,k}$ a $\mathbb{Z}_{\geq 0}$–gradation we have that $D_{m,k}$ is a vertex operator algebra.

If k is odd, then $\iota(e_\beta)$ and $\iota(e_{-\beta})$ are odd elements in F_k, which implies that $\bar{X}$ and $\bar{Y}$ are also odd elements in the vertex superalgebra $L(m,0)\otimes F_k$. Therefore $D_{m,k}$ carries the structure of a vertex operator superalgebra which is generated by the odd elements $\bar{X}$ and $\bar{Y}$ of half-integer conformal weight.

In this way we get the following theorem.

Theorem 4.2. *Let $m \in \mathbb{C}\setminus\{0,-2\}$, and let k be a nonnegative integer. Assume that $mk+2\neq 0$. Then $D_{m,k}$ is a vertex operator algebra if k is even and a vertex operator superalgebra if k is odd. The Virasoro element is $\omega_{m,k}$, the vacuum vector is $\mathbf{1}_{m,k}$ and the rank is c_m.*

Let $k=0$. Then $D_{m,0}$ is isomorphic to the $\hat{sl_2}$ vertex operator algebra $L(m,0)$. Note also that the vector

$$\omega_{m,0}=\frac{1}{2(m+2)}\left(\bar{X}_{-1}\bar{Y}+\bar{Y}_{-1}\bar{X}+\frac{1}{2}\bar{H}_{-1}^2\mathbf{1}_{m,0}\right)$$

coincides with the Virasoro element in $L(m,0)$ constructed using the Sugawara construction.

For $k=1$, $D_{m,1}$ is in fact the vertex operator superalgebra associated to the vacuum representation of the $N=2$ superconformal algebra constructed using the Kazama-Suzuki mapping (cf. [15, 19]). The Virasoro element in $D_{m,1}$ is

$$\omega_{m,1}=\frac{1}{2(m+2)}(\bar{X}_0\bar{Y}+\bar{Y}_0\bar{X}).$$

Its representation theory was studied in [2, 3, 11] . It was proved in [3] that if m is a positive integer, then $D_{m,1}$ is a regular vertex operator superalgebra and that the vertex superalgebra $D_{m,1}\otimes F_{-1}$ is a simple current extension of the vertex algebra $L(m,0)\otimes F_{-2(m+2)}$. When m is not a nonnegative integer then $D_{m,1}$ is not rational. In Theorem 4.4 we will generalize this fact for every positive integer k.

The definition of $D_{m,k}$ implies that for every weak $L(m,0)$–module M, $M\otimes F_k$ is a weak module for $D_{m,k}$. Thus, the representation theory of $D_{m,k}$ is closely related to the representation theory of the vertex operator algebra $L(m,0)$. The case when m is a nonnegative integer will be studied in following sections. When $m\neq -2$ and m is not an admissible rational number, then every highest weight $\hat{sl_2}$–module of level m is a module for the vertex operator algebra $L(m,0)$. This easily gives that $D_{m,k}$ is not rational. In the case when m is an admissible rational number, by using the similar arguments to that of

[2], and by using the representation theory of the vertex operator algebra $L(m,0)$ in this case (cf. [6]) one can construct infinitely many inequivalent irreducible $D_{m,k}$–modules. In order to be more precise, we shall state the following lemma.

Lemma 4.3. *Assume that m is not a nonnegative integer and $m \neq -2$, $mk+2 \neq 0$. Let $k \geq 1$. Then for every $t \in \mathbb{C}$ there is an ordinary $D_{m,k}$–module N_t such that $N_t = \oplus_{n \in \frac{1}{2}\mathbb{Z}_{\geq 0}} N_t(n)$, and the top level $N_t(0)$ satisfies*

$$N_t(0) = \mathbb{C}w, \quad L(n)w = t\delta_{n,0}w \quad \textit{for } n \geq 0.$$

Proof. The proof will use a similar consideration to that in [2], Section 6.

Assume that m is not a positive integer and $t \in \mathbb{C}$. The results from [6] gives that for every $q \in \mathbb{C}$ there is a $\mathbb{Z}_{\geq 0}$–graded $L(m,0)$–module $M_q = \oplus_{n \in \mathbb{Z}_{\geq 0}} M_q(n)$ and a weight vector $v_q \in M_q(0)$ such that

$$\Omega(0)|M_q(0) \equiv \frac{(m+2)m}{2}\mathrm{Id}, \quad h(0)v_q = qv_q,$$

where $\Omega(0) = e(0)f(0) + f(0)e(0) + \frac{1}{2}h(0)^2$ is the Casimir element acting on the sl_2–module $M_q(0)$. Then $M_q \otimes F_k$ is a weak $D_{m,k}$–module. Choose $q \in \mathbb{C}$ such that

$$\frac{m}{4} - \frac{k}{4(mk+2)}q^2 = t.$$

Let N_t be the $D_{m,k}$–submodule of $M_q \otimes F_k$ generated by the vector $w = v_q \otimes \mathbf{1}$. Then for $n \geq 0$ we have that

$$L(n)w = \delta_{n,0}(\frac{m}{4} - \frac{k}{4(mk+2)}q^2)w = \delta_{n,0}tw.$$

Now it is easy to see that N_t is an ordinary $\frac{1}{2}\mathbb{Z}_{\geq 0}$–graded $D_{m,k}$–module with the top level $N_t(0) = \mathbb{C}w$ and that $L(0)|N_t(0) \equiv t\mathrm{Id}$. Thus, the lemma holds. □

In fact, Lemma 4.3 gives that there is uncountably many inequivalent irreducible $D_{m,k}$–modules. Thus, we conclude that the following theorem holds.

Theorem 4.4. *Let k be a nonnegative integer. Assume that m is not a nonnegative integer and that $m \neq -2$, $mk+2 \neq 0$. Then for every positive integer k, the vertex operator (super)algebra $D_{m,k}$ is not rational.*

Remark 4.5. In what follows we will prove that if m is a positive integer, then $D_{m,k}$ is rational. In fact, we will establish more general complete reducibility theorem, which will imply that $D_{m,k}$ is regular in the sense of [9].

5 The lattice construction of $D_{m,k}$ for $m \in \mathbb{N}$

In this section we give the lattice construction of the vertex operator (super)algebra $D_{m,k}$. This construction is a generalization of the lattice constructions of the vertex operator algebra $L(m, 0)$ (cf. [8] and our Lemma 3.3) and of the N=2 vertex operator superalgebra L_{c_m} (cf. [3]).

Let $m \in \mathbb{N}$ and $k \in \mathbb{Z}_{\geq 0}$. Define the lattice

$$\begin{aligned} &\Gamma_{m,k} = \mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_m, \\ &\langle \gamma_i, \gamma_j \rangle = 2\delta_{i,j} + k \end{aligned}$$

for every $i, j \in \{1, \ldots, m\}$.

Then $V_{\Gamma_{m,k}}$ is a vertex operator algebra if k is even and a vertex operator superalgebra if k is odd.

Proposition 5.1. *Let $m \in \mathbb{N}$ and $k \in \mathbb{Z}_{\geq 0}$. The vertex operator (super)algebra $D_{m,k}$ is isomorphic to the subalgebra of the vertex operator (super)algebra $V_{\Gamma_{m,k}}$ generated by the vectors*

$$\begin{aligned} &\bar{X} = \iota(e_{\gamma_1}) + \cdots + \iota(e_{\gamma_m}), \\ &\bar{Y} = \iota(e_{-\gamma_1}) + \cdots + \iota(e_{-\gamma_m}). \end{aligned}$$

Set $\bar{H} = \bar{X}_k \bar{Y}$. Then the Virasoro element in $D_{m,k}$ is given by

$$\begin{aligned} \bar{\omega}_{m,k} &= \frac{1}{2(m+2)} \left(\bar{X}_{k-1}\bar{Y} + \bar{Y}_{k-1}\bar{X} + \frac{1-k}{mk+2} \bar{H}^2_{-1}\mathbf{1} \right) \\ &= \frac{1}{2(m+2)} \sum_{i=1}^{m} \gamma_i(-1)^2 \mathbf{1} + \frac{1}{m+2} \sum_{i \neq j} \iota(e_{\gamma_i - \gamma_j}) + \\ &+ \frac{1-k}{2(m+2)(mk+2)} \left(\sum_{i=1}^{m} \gamma_i(-1) \right)^2 \mathbf{1}. \end{aligned}$$

Proof. Define the lattice Γ_1 by

$$\begin{aligned} &\Gamma_1 = \mathbb{Z}\alpha_1 + \cdots + \mathbb{Z}\alpha_m + \mathbb{Z}\beta, \\ &\langle \alpha_i, \alpha_j \rangle = 2\delta_{i,j}, \quad \langle \alpha_i, \beta \rangle = 0, \quad \langle \beta, \beta \rangle = k. \end{aligned}$$

For $i = 1, \ldots, m$ set $\gamma_i = \alpha_i + \beta$. It is clear that the lattice $\Gamma_{m,k}$ can be identified with the sublattice $\mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_m$ of the lattice Γ_1. In the same way $V_{\Gamma_{m,k}}$ can treated as a subalgebra of the vertex operator (super)algebra V_{Γ_1}. Lemma 3.3 implies that $\mathcal{E} = \iota(e_{\alpha_1}) + \cdots + \iota(e_{\alpha_m})$, $\mathcal{F} = \iota(e_{-\alpha_1}) + \cdots + \iota(e_{-\alpha_m})$, generate a subalgebra of V_{Γ_1} isomorphic to $L(m, 0)$, and the elements $\iota(e_\beta)$, $\iota(e_{-\beta})$ generate a subalgebra isomorphic to F_k. Since

$$\bar{X} = \mathcal{E}_{-1}\iota(e_\beta) \quad \text{and} \quad \bar{Y} = \mathcal{F}_{-1}\iota(e_{-\beta}),$$

we conclude that the vertex subalgebra generated by the elements $\bar{X}, \bar{Y} \in V_{\Gamma_{m,k}} \subset V_{\Gamma_1}$ is isomorphic to the vertex operator (super)algebra $D_{m,k}$. This concludes the proof of the theorem. □

The previous result implies that we can identify the generators of $D_{m,k}$ in $L(m,0)\otimes F_k$ with the generators of $D_{m,k}$ in $V_{\Gamma_{m,k}}$. We shall also prove an interesting proposition which identifies some regular subalgebras of $D_{m,k}$.

Proposition 5.2. *For every positive integer n we have that*

$$\iota(e_{n(\gamma_1+\cdots+\gamma_m)}),\ \ \iota(e_{-n(\gamma_1+\cdots+\gamma_m)}) \in D_{m,k}.$$

In particular, $D_{m,k}$ has a vertex subalgebra isomorphic to $F_{n^2m(mk+2)}$.

Proof. Using relations (9) and (10), it is easy to prove that:

$$\begin{aligned}&\left(\bar{X}_{-(nm-1)k-2n+1}\cdots\bar{X}_{-(n-1)mk-2n+1}\right)\cdots\left(\bar{X}_{-(2m-1)k-3}\cdots\bar{X}_{-mk-3}\right)\cdot\\&\cdot\left(\bar{X}_{-(m-1)k-1}\cdots\bar{X}_{-k-1}\bar{X}_{-1}\mathbf{1}\right)=C\iota(e_{n(\gamma_1+\cdots+\gamma_m)})\end{aligned}$$

for some nontrivial constant C. Thus $\iota(e_{n(\gamma_1+\cdots+\gamma_m)})\in D_{m,k}$. Similarly we prove that $\iota(e_{-n(\gamma_1+\cdots+\gamma_m)})\in D_{m,k}$. The second assertion of the proposition follows from the fact that the vectors $\iota(e_{\pm n(\gamma_1+\cdots+\gamma_m)})$ generate a subalgebra of $V_{\Gamma_{m,k}}$ isomorphic to $F_{n^2m(mk+2)}$. □

6 Regularity of the vertex operator algebra $D_{m,2n}$

In this section we study the vertex algebra $L(m,0)\otimes F_{-2n(mn+1)}$ where m,n are positive integers. We know that $L(m,0)\otimes F_{-2n(mn+1)}$ is a simple regular vertex algebra. Its irreducible modules are:

$$L(m,r)\otimes F^{\bar{s}}_{-2n(mn+1)},\quad r\in\{1,\ldots,m\},\ \bar{s}\in\frac{\mathbb{Z}}{-2n(mn+1)\mathbb{Z}}.$$

The fusion rules can be calculated easily from the fusion rules for $L(m,0)$ and $F_{-2n(mn+1)}$.

Our main goal is to show that the vertex operator algebra $D_{m,2n}$ is isomorphic to a subalgebra of $L(m,0)\otimes F_{-2n(mn+1)}$. In order to do this, we shall first give the lattice construction of the vertex algebra $L(m,0)\otimes F_{-2n(mn+1)}$.

Define the following lattice:

$$\begin{aligned}&L=\mathbb{Z}\alpha_1+\cdots+\mathbb{Z}\alpha_m+\mathbb{Z}\beta,\\&\langle\alpha_i,\alpha_j\rangle=2\delta_{i,j},\quad\langle\alpha_i,\beta\rangle=0,\quad\langle\beta,\beta\rangle=-2n(mn+1)\end{aligned}$$

for every $i,j\in\{1,\ldots,m\}$.

We shall now give another description of the lattice L.

For $i=1,\ldots,m$, we define

$$\begin{aligned}&\delta=n\alpha_1+\cdots+n\alpha_m+\beta,\\&\gamma_i=\alpha_i+\delta.\end{aligned}$$

Since

$$\alpha_i=\gamma_i-\delta,\ \beta=(nm+1)\delta-n(\gamma_1+\cdots+\gamma_m),$$

we have that

$$L = \mathbb{Z}\gamma_1 + \cdots + \mathbb{Z}\gamma_m + \mathbb{Z}\delta,$$
$$\langle \gamma_i, \gamma_j \rangle = 2\delta_{i,j} + 2n, \quad \langle \gamma_i, \delta \rangle = 0, \quad \langle \delta, \delta \rangle = -2n$$

for every $i, j \in \{1, \dots, m\}$.

In fact, we have proved that

$$L \cong \Gamma_{m,2n} + L_{-2n} \cong A_{1,m} + L_{-2n(mn+1)}, \tag{12}$$

which implies that

$$V_L \cong V_{\Gamma_{m,2n}} \otimes F_{-2n} \cong V_{A_{1,m}} \otimes F_{-2n(mn+1)}. \tag{13}$$

Define the following vectors in the vertex algebra V_L:

$$\mathcal{E} = \iota(e_{\alpha_1}) + \cdots + \iota(e_{\alpha_m});$$
$$\mathcal{F} = \iota(e_{-\alpha_1}) + \cdots + \iota(e_{-\alpha_m}).$$

These vectors generate a subalgebra of V_L isomorphic to $L(m, 0)$.

As in Section 5 we define:

$$\bar{X} = \iota(e_{\gamma_1}) + \cdots + \iota(e_{\gamma_m});$$
$$\bar{Y} = \iota(e_{-\gamma_1}) + \cdots + \iota(e_{-\gamma_m}).$$

Clearly $\bar{X}, \bar{Y}$ generate a subalgebra isomorphic to $D_{m,2n}$. In fact, the definition of elements $\mathcal{E}$, $\mathcal{F}$, $\bar{X}$, $\bar{Y}$ together with relations (12) and (13) imply the following lemma.

Lemma 6.1.

(1) Let V be the subalgebra of V_L generated by the vectors

$$\mathcal{E}, \mathcal{F}, \iota(e_\beta), \iota(e_{-\beta}).$$

Then $V \cong L(m, 0) \otimes F_{-2n(mn+1)}$.

(2) Let W be the subalgebra of V_L generated by the vectors

$$\bar{X}, \bar{Y}, \iota(e_\delta), \iota(e_{-\delta}).$$

Then $W \cong D_{m,2n} \otimes F_{-2n}$.

Now using standard calculations in lattice vertex algebras one easily gets the following important lemma.

Lemma 6.2. *In the vertex algebra V_L the following relations hold:*

(1) $\bar{X} = (\mathcal{E}_{-2n-1}\iota(e_{n(\alpha_1+\cdots+\alpha_m)}))_{-1}\iota(e_\beta)$;

(2) $\bar{Y} = (\mathcal{F}_{-2n-1}\iota(e_{-n(\alpha_1+\cdots+\alpha_m)}))_{-1}\iota(e_{-\beta})$;

(3) $\iota(e_{\delta}) = \iota(e_{n(\alpha_1+\cdots+\alpha_m)})_{-1}\iota(e_{\beta})$;
(4) $\iota(e_{-\delta}) = \iota(e_{-n(\alpha_1+\cdots+\alpha_m)})_{-1}\iota(e_{-\beta})$;
(5) $\mathcal{E} = \bar{X}_{-1}\iota(e_{-\delta})$;
(6) $\mathcal{F} = \bar{Y}_{-1}\iota(e_{\delta})$;
(7) $\iota(e_{\beta}) = \iota(e_{(nm+1)\delta})_{-1}\iota(e_{-n(\gamma_1+\cdots+\gamma_m)})$;
(8) $\iota(e_{-\beta}) = \iota(e_{-(nm+1)\delta})_{-1}\iota(e_{n(\gamma_1+\cdots+\gamma_m)})$.

Theorem 6.3. *The vertex subalgebras V and W coincide. In particular, we have the following isomorphism of vertex algebras:*

$$L(m,0) \otimes F_{-2n(mn+1)} \cong D_{m,2n} \otimes F_{-2n}. \tag{14}$$

Proof. Using the same arguments as in the proof of Proposition 5.2 we get

$$\iota(e_{\pm n(\alpha_1+\cdots+\alpha_m)}) \in V, \quad \iota(e_{\pm n(\gamma_1+\cdots+\gamma_m)}) \in W.$$

Then the relations (1) - (4) in Lemma 6.2 implies that $\bar{X}, \bar{Y}, \iota(e_{\pm\delta}) \in V$. Thus $W \subset V$. Similarly, the relations (5) - (8) in Lemma 6.2 gives that $V \subset W$. Hence, $V = W$. Then Lemma 6.1 implies that $L(m,0) \otimes F_{-2n(mn+1)} \cong D_{m,2n} \otimes F_{-2n}$. □

The next result follows from [9, 10] and [12].

Proposition 6.4. *Let V be a vertex operator (super) algebra and $s \in \mathbb{Z}$, $s \neq 0$. Then we have:*
(1) $V \otimes F_s$ is a simple vertex superalgebra if and only if V is a simple vertex operator (super)algebra.
(2) $V \otimes F_s$ is a regular vertex superalgebra if and only if V is a regular vertex operator (super)algebra.

Theorem 6.5. *Let $m, m_1, \ldots, m_r$ be positive integers and let $k, k_1, \ldots, k_r$ be positive even integers.*
(1) The vertex operator algebra $D_{m,k}$ is simple and regular. In particular, $D_{m,k}$ is rational and C_2–cofinite.
(2) The vertex operator algebra $D_{m_1,k_1} \otimes \cdots \otimes D_{m_r,k_r}$ is simple and regular.

Proof. Since $L(m,0)$ and $F_{-2n(nm+1)}$ are simple regular vertex algebras, Proposition 6.4 implies that $L(m,0) \otimes F_{-2n(nm+1)}$ is also simple and regular. Since

$$L(m,0) \otimes F_{-2n(nm+1)} \cong D_{m,2n} \otimes F_{-2n},$$

using again Proposition 6.4 we get that the vertex operator algebra $D_{m,2n}$ is simple and regular. This gives (1). The proof of (2) is now standard (cf. [9]). □

Since $L(m,0)$ has $(m+1)$ inequivalent irreducible modules, and for every $k \in \mathbb{Z}$, $k \neq 0$, F_k has $|k|$ inequivalent irreducible modules, we get:

Corollary 6.6. *The vertex operator algebra $D_{m,2n}$ has exactly $(m+1)(nm+1)$ inequivalent irreducible representations.*

7 Regularity of the vertex operator superalgebra $D_{m,k}$ for k odd

In this section, we shall consider the case when k is an odd natural number. When $k = 1$, then $D_{m,1}$ is the vertex operator superalgebra associated to the unitary vacuum representation for the $N = 2$ superconformal algebra. This case was studied in [3].

First we see that the following relation between lattices holds:

$$\Gamma_{m,k} + L_{-k} \cong (A_{1,m} + L_{-2k(mk+2)}) \cup (\tilde{A}_{1,m} + \tilde{L}_{-2k(mk+2)}), \tag{15}$$

which implies the following isomorphism of vertex algebras:

$$V_{\Gamma_{m,k}} \otimes F_{-k} \cong (V_{A_{1,m}} \otimes F_{-2k(mk+2)}) \oplus (V_{\tilde{A}_{1,m}} \otimes MF_{-2k(mk+2)}). \tag{16}$$

Using (15), (16) and a completely analogous proof to that of Theorem 7.1 in [3], we get the following result.

Theorem 7.1. *We have the following isomorphism of vertex superalgebras:*

$$D_{m,k} \otimes F_{-k} \cong L(m,0) \otimes F_{-2k(km+2)} \oplus L(m,m) \otimes MF_{-2k(km+2)}.$$

In other words, the vertex superalgebra $D_{m,k} \otimes F_{-k}$ is a simple current extension of the vertex algebra $L(m,0) \otimes F_{-2k(km+2)}$.

By using Proposition 6.4, Theorem 7.1 and the fact that a simple current extension of a simple regular vertex algebra is a simple regular vertex (super)algebra (cf. [21]) we get the following theorem.

Theorem 7.2. *Let $m, m_1, \ldots, m_r$ be positive integers and let $k, k_1, \ldots, k_r$ be positive odd integers.*

(1) The vertex operator superalgebra $D_{m,k}$ is simple and regular. In particular, $D_{m,k}$ is rational and C_2–cofinite.

(2) The vertex operator superalgebra $D_{m_1,k_1} \otimes \cdots \otimes D_{m_r,k_r}$ is simple and regular.

We also have:

Corollary 7.3. *The vertex operator superalgebra $D_{m,k}$ has exactly $\frac{(m+1)(km+2)}{2}$ inequivalent irreducible representations.*

Proof. The results from [21] imply that the extended vertex superalgebra

$$L(m,0) \otimes F_{-2k(km+2)} \oplus L(m,m) \otimes MF_{-2k(km+2)}$$

has exactly $\frac{1}{2}(m+1)k(km+2)$ inequivalent irreducible representations (see also [3, 22]). Since the vertex superalgebra F_{-n} has n inequivalent irreducible representations, we conclude that $D_{m,k}$ has to have $\frac{(m+1)(km+2)}{2}$ inequivalent irreducible representations. □

8 Realization of the vertex operator algebra $D_{4,k}$

The lattice construction of $D_{m,k}$ in Section 5 is based on a very general lattice realization of the vertex operator algebra $L(m,0)$. Since in some special cases $L(m,0)$ admits other realizations, one can apply them in the theory of our vertex operator algebras $D_{m,k}$. As an example, in this section we shall consider the case $m=4$. We will show that the vertex operator (super)algebra $D_{4,k}$ is the fixed point subalgebra of an automorphism g of the lattice vertex operator (super)algebra V_{P_k}. Our construction generalizes the fact that the vertex operator algebra $L(4,0)$ can be constructed as a subalgebra of the lattice vertex operator algebra V_{A_2}.

For every $k \in \mathbb{Z}_{\geq 0}$, we define the following lattice

$$P_k = \mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2, \quad \langle \gamma_1, \gamma_1 \rangle = \langle \gamma_2, \gamma_2 \rangle = k+2, \quad \langle \gamma_1, \gamma_2 \rangle = k-1.$$

Then V_{P_k} is a vertex operator algebra if k is even and a vertex operator superalgebra if k is odd.

Set $P = P_k + L_{-k}$, where $L_{-k} = \mathbb{Z}\delta$ and

$$\langle \delta, \gamma_1 \rangle = \langle \delta, \gamma_2 \rangle = 0, \quad \langle \delta, \delta \rangle = -k.$$

Define

$$\alpha_1 = \gamma_1 + \delta, \quad \alpha_2 = \gamma_2 + \delta, \quad \beta = k(\gamma_1 + \gamma_2) + (2k+1)\delta.$$

It is easy to see that $P = A_2 + \mathbb{Z}\beta$, where $A_2 = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ is the root lattice of type A_2. Since $\langle \beta, \beta \rangle = -k(2k+1)$ we get that the following relation between lattices holds:

$$P_k + L_{-k} \cong A_2 + L_{-k(2k+1)}.$$

Therefore, we have the following isomorphism of vertex (super)algebras:

$$V_P \cong V_{P_k} \otimes F_{-k} \cong V_{A_2} \otimes F_{-k(2k+1)}. \tag{17}$$

Let g be the automorphism V_P which is uniquely determined by

$$g(\iota(e_{\pm\gamma_1})) = \iota(e_{\pm\gamma_2}), \quad g(\iota(e_{\pm\gamma_2})) = \iota(e_{\pm\gamma_1}), \quad g(\iota(e_{\pm\delta})) = \iota(e_{\pm\delta}).$$

g is the automorphism of order two of the vertex (super)algebra V_P and it is lifted from the automorphism $\gamma_1 \mapsto \gamma_2$, $\gamma_2 \mapsto \gamma_1$, $\delta \mapsto \delta$ of the lattice P. The definition of g implies that

$$g(\iota(e_{\pm\alpha_1})) = \iota(e_{\pm\alpha_2}), \quad g(\iota(e_{\pm\alpha_2})) = \iota(e_{\pm\alpha_1}), \quad g(\iota(e_{\pm\beta})) = (-1)^k \iota(e_{\pm\beta}).$$

Let W be one of the subalgebras V_{P_k}, V_{A_2} or $V_{\mathbb{Z}\beta}$. Then W is g–invariant and $W = W^0 \oplus W^1$, where

$$W^0 = \{w \in W \mid gw = w\}, \quad W^1 = \{w \in W \mid gw = -w\}.$$

We have the following isomorphism of vertex algebras

$$V_{P_k}^0 \otimes F_{-k} \cong \begin{cases} V_{A_2}^0 \otimes F_{-k(2k+1)} & \text{if } k \text{ is even} \\ V_{A_2}^0 \otimes F_{-4k(2k+1)} \oplus V_{A_2}^1 \otimes MF_{-4k(2k+1)} & \text{if } k \text{ is odd} \end{cases}.$$

Next we recall the important fact (see Note 7.3.2 of [24]) that

$$V_{A_2}^0 \cong L(4,0), \quad V_{A_2}^1 \cong L(4,4). \tag{18}$$

Combining (18), Theorem 6.3 and Theorem 7.1 we get that

$$V_{P_k}^0 \otimes F_{-k} \cong D_{4,k} \otimes F_{-k}.$$

This implies that $D_{4,k} \cong V_{P_k}^0$. In this way we have proved the following result.

Theorem 8.1. *We have:*

$$D_{4,k} \cong V_{P_k}^0.$$

Under this isomorphism, the generators of $D_{4,k}$ are mapped to

$$\bar{X} \mapsto \sqrt{2}(\iota(e_{\gamma_1}) + \iota(e_{\gamma_2})), \quad \bar{Y} \mapsto \sqrt{2}(\iota(e_{-\gamma_1}) + \iota(e_{-\gamma_2})).$$

Acknowledgment

We would like to thank the referee for his valuable comments.

References

[1] D. Adamović: "Rationality of Neveu-Schwarz vertex operator superalgebras", *Int. Math. Res. Not.*, Vol. 17, (1997), pp. 865–874

[2] D. Adamović: "Representations of the $N = 2$ superconformal vertex algebra", *Int. Math. Res. Not.*, Vol. 2, (1999), pp. 61–79

[3] D. Adamović: "Vertex algebra approach to fusion rules for $N = 2$ superconformal minimal models", *J. Algebra*, Vol. 239, (2001), pp. 549–572

[4] D. Adamović: *Regularity of certain vertex operator superalgebras*, Contemp. Math., Vol. 343, Amer. Math. Soc., Providence, 2004, pp. 1-16.

[5] T. Abe, G. Buhl and C. Dong: "Rationality, regularity and C_2–cofiniteness", *Trans. Amer. Math. Soc.*, Vol. 356, (2004), pp. 3391–3402.

[6] D. Adamović and A. Milas: "Vertex operator algebras associated to the modular invariant representations for $A_1^{(1)}$", *Math. Res. Lett.*, Vol. 2, (1995), pp. 563–575

[7] C. Dong: "Vertex algebras associated with even lattices", *J. Algebra*, Vol. 160, (1993), pp. 245–265.

[8] C. Dong and J. Lepowsky: *Generalized vertex algebras and relative vertex operators*, Birkhäuser, Boston, 1993.

[9] C. Dong, H. Li and G. Mason: "Regularity of rational vertex operator algebras", *Adv. Math.*, Vol. 132, (1997), pp. 148–166

[10] C. Dong, G. Mason and Y. Zhu: "Discrete series of the Virasoro algebra and the Moonshine module", *Proc. Sympos. Math. Amer. Math. Soc.*, Vol. 56(2), (1994), pp. 295–316

[11] W. Eholzer and M.R. Gaberdiel: "Unitarity of rational $N = 2$ superconformal theories", *Comm. Math. Phys.*, Vol. 186, (1997), pp. 61–85.

[12] I.B. Frenkel, Y.-Z. Huang and J. Lepowsky: "On axiomatic approaches to vertex operator algebras and modules", *Memoirs Am. Math. Soc.*, Vol. 104, 1993.

[13] I. B. Frenkel, J. Lepowsky and A. Meurman: *Vertex Operator Algebras and the Monster*, Pure Appl. Math., Vol. 134, Academic Press, New York, 1988.

[14] I.B. Frenkel and Y. Zhu: "Vertex operator algebras associated to representations of affine and Virasoro algebras", *Duke Math. J.*, Vol. 66, (1992), pp. 123–168.

[15] B.L. Feigin, A.M. Semikhatov and I.Yu. Tipunin: "Equivalence between chain categories of representations of affine $sl(2)$ and $N = 2$ superconformal algebras", *J. Math. Phys.*, Vol. 39, (1998), pp. 3865–3905

[16] B.L. Feigin, A.M. Semikhatov and I.Yu. Tipunin: "A semi-infinite construction of unitary N=2 modules", *Theor. Math. Phys.*, Vol. 126(1), (2001), pp. 1–47.

[17] Y.-Z Huang and A. Milas: "Intertwining operator superalgebras and vertex tensor categories for superconformal algebras", *II. Trans. Amer. Math. Soc.*, Vol. 354, (2002), pp. 363–385.

[18] V.G. Kac: *Vertex Algebras for Beginners*, University Lecture Series, Vol. 10, 2nd ed., AMS, 1998.

[19] Y. Kazama and H. Suzuki: "New N=2 superconformal field theories and superstring compactifications", *Nuclear Phys. B*, Vol. 321, (1989), pp. 232–268.

[20] H. Li: "Local systems of vertex operators, vertex superalgebras and modules", *J. Pure Appl. Algebra*, Vol. 109, (1996), pp. 143–195.

[21] H. Li: "Extension of Vertex Operator Algebras by a Self-Dual Simple Module", *J. Algebra*, Vol. 187, (1997), pp. 236–267.

[22] H. Li: "Certain extensions of vertex operator algebras of affine type", *Comm. Math. Phys.*, Vol. 217, (2001), pp. 653–696.

[23] H. Li: "Some finiteness properties of regular vertex operator algebras", *J. Algebra*, Vol. 212, (1999), pp. 495–514.

[24] M. Wakimoto: *Lectures on infinite-dimensional Lie algebra, algebra*, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[25] W. Wang: "Rationality of Virasoro Vertex operator algebras", *Internat. Math. Res. Notices*, Vol 71(1), (1993), PP. 197–211.

[26] Xu Xiaoping: *Introduction to vertex operator superalgebras and their modules*, Mathematics and Its Applications, Vol. 456, Kluwer Academic Publishers, 1998.