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The signed k-domination number of directed graphs

Research Article

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Abstract: Let $k \geq 1$ be an integer, and let $D = (V, A)$ be a finite simple digraph, for which $d_D^-(v) \geq k-1$ for all $v \in V$. A function $f : V \to \{-1, 1\}$ is called a signed k-dominating function (SkDF) if $f(N^-[v]) \geq k$ for each vertex $v \in V$. The weight $w(f)$ of f is defined by $\sum_{v \in V} f(v)$. The signed k-domination number for a digraph D is $\gamma_{kS}(D) = \min\{w(f) \mid f$ is an SkDF of $D\}$. In this paper, we initiate the study of signed k-domination in digraphs. In particular, we present some sharp lower bounds for $\gamma_{kS}(D)$ in terms of the order, the maximum and minimum outdegree and indegree, and the chromatic number. Some of our results are extensions of well-known lower bounds of the classical signed domination numbers of graphs and digraphs.

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1. Introduction

In this paper, D is a finite simple digraph with vertex set $V(D) = V$ and arc set $A(G) = A$. A digraph without directed cycles of length 2 is an *oriented graph*. The *order* $n(D) = n$ of a digraph D is the number of its vertices, and the number of its arcs is the *size* $m(D) = m$. We write $d_D^+(v) = d^+(v)$ for the outdegree of a vertex v and $d_D^-(v) = d^-(v)$ for its indegree. The *minimum* and *maximum indegree* and *minimum* and *maximum outdegree* of D are denoted by $\delta^-(D) = \delta^-$, $\Delta^-(D) = \Delta^-$, $\delta^+(D) = \delta^+$ and $\Delta^+(D) = \Delta^+$, respectively. If uv is an arc of D, then we also write $u \to v$, and we say

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that v is an *out-neighbor* of u and u is an *in-neighbor* of v. For every vertex $v \in V$, let $N_D^-(v) = N^-(v)$ be the set consisting of all vertices of D from which arcs go into v, and let $N_D^-[v] = N^-[v] = N^-(v) \cup \{v\}$. If $X \subseteq V(D)$, then $D[X]$ is the subdigraph induced by X. If $X \subseteq V(D)$ and $v \in V(D)$, then $E(X, v)$ is the set of arcs from X to v. For a real-valued function $f : V(D) \to \mathbb{R}$, the weight of f is $w(f) = \sum_{v \in V} f(v)$, and for $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$, so $w(f) = f(V)$. Consult [6] for the notation and terminology which are not defined here.

Let $k \geq 1$ be an integer and let D be a digraph such that $\delta^-(D) \geq k - 1$. A *signed k-dominating function* (abbreviated SkDF) of D is a function $f : V \to \{-1, 1\}$ such that $f(N^-[v]) \geq k$ for every $v \in V$. The *signed k-domination number* for a digraph D is

$$\gamma_{kS}(D) = \min \left\{ w(f) \mid f \text{ is an S}k\text{DF of } D \right\}.$$

As the assumption $\delta^-(D) \geq k - 1$ is clearly necessary, we always assume that when we discuss $\gamma_{kS}(D)$, all digraphs involved satisfy $\delta^-(D) \geq k - 1$ and thus $n(D) \geq k$. A $\gamma_{kS}(D)$-function is an SkDF of D of weight $\gamma_{kS}(D)$. For any SkDF f of D we define $P = \{v \in V \mid f(v) = 1\}$ and $M = \{v \in V \mid f(v) = -1\}$. When $k = 1$, the signed k-domination number $\gamma_{kS}(D)$ is the usual *signed domination number* $\gamma_S(D)$, which was introduced by Zelinka in [7] and has been studied by several authors (see for example [3]).

The concept of the signed k-domination number $\gamma_{kS}(G)$ of undirected graphs G was introduced by Wang [5]. The special case $k = 1$ was defined and investigated in [1]. In this article, we present some sharp lower bounds on the signed k-domination number of digraphs. We make use of the following results and observations.

Theorem A ([4]).

For any graph G,

$$\chi(G) \leq 1 + \max \left\{ \delta(H) \mid H \text{ is a subgraph of } G \right\}.$$

Theorem B ([3]).

Let D be a digraph of order $n \geq 2$ and let r be a nonnegative integer such that $\delta^+(D) \geq r$. Then

$$\gamma_S(D) \geq 2\left(\chi(G) + r + 1 - \Delta(G)\right) - n,$$

where G is the underlying graph of D.

Observation 1.1.

For any digraph D of order $n \geq 2$, $\gamma_{kS}(D) \equiv n \pmod 2$.

Proof. Let f be a $\gamma_{kS}(D)$-function. Since $n = |P| + |M|$ and $\gamma_{kS}(D) = |P| - |M|$, we obtain $n - \gamma_{kS}(D) = 2|M|$ and this implies the desired result. □

Observation 1.2.

Let u be a vertex of indegree at most k in D. If f is an SkDF on D, then f assigns 1 to each vertex of $N_D^-[u]$.

Proof. Since $f(N_D^-[u]) \geq k$ and $|N_D^-[u]| \leq k + 1$, the result follows. □

Observation 1.3.

If $k \geq 2$ is an integer and D a digraph with $\delta^-(D) \geq k - 1$, then

(i) $\gamma_{kS}(D) \geq \gamma_{(k-1)S}(D)$, *and*

(ii) if $k \geq 3$, *then* $\gamma_{kS}(D) \geq \gamma_{(k-2)S}(D) + 2$.

Proof. (i) Since every signed k-dominating function of D is also a signed $(k-1)$-dominating function of D, inequality (i) is proved.

(ii) Let f be a $\gamma_{kS}(D)$-function with $f(v) = 1$ for $v \in P$ and $f(v) = -1$ for $v \in M$. We choose an arbitrary vertex $w \in P$ and define $P' = P \setminus \{w\}$ and $M' = M \cup \{w\}$. In addition, we define $g : V(D) \to \{-1, 1\}$ by $g(v) = 1$ for $v \in P'$ and $g(v) = -1$ for $v \in M'$. Now it is a simple matter to verify that g is a signed $(k-2)$-dominating function of D of weight

$$w(g) = |P'| - |M'| = |P| - |M| - 2 = \gamma_{kS}(D) - 2.$$

This implies that $\gamma_{(k-2)S}(D) \leq w(g) = \gamma_{kS}(D) - 2$, and the proof is complete. □

The *associated digraph* $D(G)$ of a graph G is the digraph obtained when each edge e of G is replaced by two oppositely oriented arcs with the same ends as e. We denote the associated digraph $D(K_n)$ of the complete graph K_n of order n by K_n^*.

Let $D = K_n^*$ and let k be an integer with $1 \leq k \leq n$. It is straighforward to show that $\gamma_{kS}(D) = k$ when $n + k$ is even, and $\gamma_{kS}(D) = k + 1$ when $n + k$ is odd. It follows that, if $k \geq 3$ and $n + k$ is even, then $\gamma_{kS}(D) = k$, $\gamma_{(k-1)S}(D) = k$ and $\gamma_{(k-2)S}(D) = k - 2$, and therefore we have equality in Observation 1.3, (i) and (ii). This example demonstrates that Observation 1.3 is sharp.

Observation 1.4.

Let D be a digraph of order n. Then $\gamma_{kS}(D) = n$ if and only if $k - 1 \leq \delta^-(D) \leq k$, and for each $v \in V(D)$ there exists a vertex $u \in N^+[v]$ with indegree at most k.

Proof. If $k - 1 \leq \delta^-(D) \leq k$ and for each $v \in V(D)$ there exists a vertex $u \in N^+[v]$ with indegree at most k, then trivially $\gamma_{ks}(D) = n$.

Conversely, assume that $\gamma_{kS}(D) = n$. By assumption $k - 1 \leq \delta^-(D)$. Suppose to the contrary that $\delta^-(D) > k$ or there exists a vertex $v \in V(D)$ such that $d^-(u) \geq k + 1$ for each $u \in N^+[v]$. If $\delta^-(D) > k$, define $f : V(D) \to \{-1, 1\}$ by $f(v) = -1$ for some fixed v and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Obviously, f is a signed k-dominating function of D with weight less than n, which is a contradiction. Thus $k - 1 \leq \delta^-(D) \leq k$. Now let $v \in V(D)$ and $d^-(u) \geq k + 1$ for each $u \in N^+[v]$. Define $f : V(D) \to \{-1, 1\}$ by $f(v) = -1$ and $f(x) = 1$ for $x \in V(D) \setminus \{v\}$. Again, f is a signed k-dominating function of D, which is a contradiction. This completes the proof. □

Corollary 1.5.

If D is a digraph of order n such that $\Delta^-(D) \leq k$, then $\gamma_{kS}(D) = n$.

A *tournament* is a digraph in which for every pair u, v of vertices, either $u \to v$ or $v \to u$, but not both. Next we determine the exact value of the signed k-domination number for particular types of tournaments. Let n be an odd positive integer. We have $n = 2r + 1$, where r is a positive integer. We define the circulant tournament $\mathrm{CT}(n)$ with n vertices as follows. The vertex set of $\mathrm{CT}(n)$ is $V(\mathrm{CT}(n)) = \{u_0, u_1, \ldots, u_{n-1}\}$. For each i, the arcs are going from u_i to the vertices $u_{i+1}, u_{i+2}, \ldots, u_{i+r}$, where the indices are taken modulo n.

Proposition 1.6.

Let $n = 2r + 1$ where r is a positive integer and let $1 \leq k \leq r + 1$ be an integer. Then

$$\gamma_{kS}(\mathrm{CT}(n)) = \begin{cases} 2k + 1 & \text{if } r \equiv k + 1 \pmod 2, \\ 2k + 3 & \text{if } r \equiv k \pmod 2. \end{cases}$$

Proof. If $n = 3$, then obviously $\gamma_{kS}(\mathrm{CT}(n)) = n$. If $k = r$ or $k = r + 1$ then by Observation 1.4, $\gamma_{kS}(\mathrm{CT}(n)) = n$. Thus we assume that $n \geq 5$ and $k \leq r - 1$. Let f be a $\gamma_{kS}(\mathrm{CT}(n))$-function. If $f(x) = 1$ for each $x \in V(\mathrm{CT}(n))$, then $w(f) = n \geq 2k + 1$. We may assume, without loss of generality, that $f(u_0) = -1$. Consider the sets $N^-[u_0]$ and $N^-[u_r]$.

Since f is an SkDF on CT(n), we have $f(N^{-}[u_0]) \geq k$, $f(N^{-}[u_r]) \geq k$ if $r \equiv k+1 \pmod 2$ and $f(N^{-}[u_0]) \geq k+1$, $f(N^{-}[u_r]) \geq k+1$ when $r \equiv k \pmod 2$. Therefore

$$\omega(f) = f(V(\mathrm{CT}(n))) = f(N^{-}[u_0]) + f(N^{-}[u_r]) - f(u_0) \geq \begin{cases} 2k+1 & \text{if } r \equiv k+1 \pmod 2, \\ 2k+3 & \text{if } r \equiv k \pmod 2. \end{cases}$$

This implies that

$$\gamma_{kS}(\mathrm{CT}(n)) \geq \begin{cases} 2k+1 & \text{if } r \equiv k+1 \pmod 2, \\ 2k+3 & \text{if } r \equiv k \pmod 2. \end{cases}$$

Suppose now that $s = \left\lfloor \frac{r-k-1}{2} \right\rfloor$, $V^{-} = \{u_1, \ldots, u_s, u_{r+1}, \ldots, u_{r+s}\}$ and $V^{+} = V(\mathrm{CT}(n)) - V^{-}$. Define $f : V(\mathrm{CT}(n)) \to \{-1, 1\}$ by $f(u_0) = -1$, $f(v) = 1$ if $v \in V^{+}$ and $f(v) = -1$ when $v \in V^{-}$. For any vertex $v \in V(\mathrm{CT}(n))$ we have $|N^{-}[v]| = r+1$ and $|N^{-}[v] \cap V^{-}| \leq s+1$. Therefore $f(N^{-}[v]) = r - 2s - 1 \geq k$ and so f is an SkDF on CT(n). Now we have

$$\gamma_{kS}(\mathrm{CT}(n)) \leq \omega(f) = \begin{cases} 2k+1 & \text{if } r \equiv k+1 \pmod 2, \\ 2k+3 & \text{if } r \equiv k \pmod 2. \end{cases}$$

This completes the proof. □

2. Lower bounds on signed k-domination numbers of digraphs

In this section we present some sharp lower bounds for $\gamma_{kS}(D)$ in terms of the order, the maximum and minimum outdegree and indegree, and the chromatic number of D. Recall that the complement of a graph G is denoted as $\overline{G}$.

Theorem 2.1.
Let $k \geq 1$ be an integer, and let D be a digraph of order $n \geq k+1$ with $\delta^{-}(D) \geq k-1$. Then

$$\gamma_{kS}(D) \geq 2(k+1) - n,$$

with equality if and only if D is $H \vee \overline{K_{n-k-1}}$, where H is a digraph of order $k+1$ with $\delta^{-}(H) \geq k-1$ such that $u \to v$ for each $u \in V(H)$ and each $v \in V(\overline{K_{n-k-1}})$. Also, if $d_H^{-}(w) = k - i$ with $i = 0, 1$ for a vertex $w \in V(H)$, then there are at most i arcs from $V(\overline{K_{n-k-1}})$ to w.

Proof. Let f be an SkDF of D. If f assigns 1 to each vertex, then the statement is true, since $n \geq k+1$. Now assume that there exists a vertex $v \in V$ with $f(v) = -1$. Then f assigns 1 to at least $k+1$ vertices in $N_D^{-}(v)$ and so $|M| \leq n-k-1$. Thus

$$\gamma_{kS}(D) = |P| - |M| \geq k+1-(n-k-1) = 2(k+1) - n,$$

as desired.

Let H be a digraph of order $k+1$ with $\delta^{-}(H) \geq k-1$ such that $u \to v$ for each $u \in V(H)$ and each $v \in V(\overline{K_{n-k-1}})$. If D is $H \vee \overline{K_{n-k-1}}$, and if for every vertex $w \in V(H)$ with $d_H^{-}(w) = k-i$ for $i = 0, 1$, there are at most i arcs from $V(\overline{K_{n-k-1}})$ to w, then we define $f : V(D) \to \{-1, 1\}$ by $f(v) = 1$ if $v \in V(H)$ and $f(v) = -1$ if $v \in V(\overline{K_{n-k-1}})$. It is straightforward to verify that f is an SkDF of D with $w(f) = 2(k+1) - n$ and hence $\gamma_{kS}(D) = 2(k+1) - n$.

Now let D be a digraph such that $\gamma_{kS}(D) = 2(k+1) - n$. Let f be an SkDF of D. Then $|P| = k+1$ and $|M| = n-k-1$. Define H by $D[P]$. Since $f(N_D^{-}[x]) \geq k$ for every vertex x, we deduce that $\delta^{-}(H) \geq k-1$, $u \to v$ for each $u \in V(H)$ and each $v \in M$ and M is an independent set. In addition, we observe that for every vertex $w \in V(H)$ with $d_H^{-}(w) = k$, an arbitrary arc from M to w is admissible. This completes the proof. □

For oriented graphs we now present a sharper lower bound on the signed k-domination number when $k \geq 2$.

Theorem 2.2.
Let $k \geq 2$ be an integer, and let D be an oriented graph of order n with $\delta^-(D) \geq k-1$. Then

$$\gamma_{kS}(D) \geq 2\,(2k-1) - n,$$

with equality if and only if D consists of an arbitrary $(k-1)$-regular tournament T_{2k-1} and a set W of $n-(2k-1)$ further vertices, such that each $w \in W$ has at least $k+1$ in-neighbors in T_{2k-1} and there is no arc from W to T_{2k-1}. Also, if a vertex $w \in W$ has $k+1 \leq t \leq 2k-1$ in-neighbors in T_{2k-1}, then w has at most $t-k-1$ in-neighbors in W.

Proof. Let f be an SkDF of D. Each vertex $v \in P$ has at least $k-1$ in-neighbors in P. This implies that

$$\frac{|P|(|P|-1)}{2} \geq |A(D[P])| \geq (k-1)|P|,$$

and thus $|P| \geq 2k-1$. Therefore $|M| \leq n-2k+1$, and we obtain the desired bound as follows,

$$\gamma_{kS}(D) = |P| - |M| \geq 2k-1-(n-2k+1) = 2\,(2k-1) - n.$$

Assume that D consists of an arbitrary $(k-1)$-regular tournament T_{2k-1} and a set W of $n-(2k-1)$ further vertices, such that each $w \in W$ has at least $k+1$ in-neighbors in T_{2k-1}, and there is no arc from W to T_{2k-1}. Also, assume that if a vertex $w \in W$ has $k+1 \leq t \leq 2k-1$ in-neighbors in T_{2k-1}, then w has at most $t-k-1$ in-neighbors in W. We define $f: V(D) \to \{-1, 1\}$ by $f(v) = 1$ if $v \in V(T_{2k-1})$ and $f(v) = -1$ if $v \in W$. It is easy to see that f is an SkDF of D with $w(f) = 2(2k-1) - n$, and so $\gamma_{kS}(D) = 2(2k-1) - n$.

Now let D be a digraph such that $\gamma_{kS}(D) = 2(2k-1) - n$. If f is an SkDF of D, then $|P| = 2k-1$ and $|M| = n-2k+1$. Define T by $D[P]$. Since $f(N_D^-[x]) \geq k$ for every vertex x, we deduce that $\delta^-(T) \geq k-1$, and therefore it follows that

$$(k-1)(2k-1) \leq \sum_{v \in V(T)} \delta^-(T) \leq \sum_{v \in V(T)} d_T^-(v) = |A(T)| \leq \frac{n(T)\,(n(T)-1)}{2} = (k-1)(2k-1).$$

Hence we have equality in this inequality chain, and thus T is a tournament such that $d_T^-(x) = k-1$ for each $x \in V(T)$. Because $d_T^+(x) + d_T^-(x) = n(T) - 1 = 2k-2$, for each $x \in V(T)$, we conclude that T is a $(k-1)$-regular tournament. Now it follows that there is no arc from M to P, every vertex in M has at least $k+1$ in-neighbors in P, and if a vertex $w \in M$ has $k+1 \leq t \leq 2k-1$ in-neighbors in P, then w has at most $t-k-1$ in-neighbors in M. $\square$

Theorem 2.3.
Let $k \geq 1$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq k-1$. Then

$$\gamma_{kS}(D) \geq n\,\frac{2\left\lceil \frac{\delta^-(D)+k+1}{2} \right\rceil - 1 - \Delta^+(D)}{\Delta^+(D)+1}.$$

Proof. Let f be a $\gamma_{kS}(D)$-function, and let s be the number of arcs from the set P to the set M. The condition $f(N^-[x]) \geq k$ implies that $|E(P,x)| \geq |E(M,x)| + k - 1$ for $x \in P$, and $|E(P,x)| \geq |E(M,x)| + k + 1$ for $x \in M$. Thus we obtain

$$\delta^-(D) \leq d^-(x) = |E(P,x)| + |E(M,x)| \leq 2\,|E(P,x)| - k - 1,$$

and so $|E(P,x)| \geq \left\lceil \frac{\delta^-(D)+k+1}{2} \right\rceil$ for each vertex $x \in M$. Hence we deduce that

$$s = \sum_{x \in M} |E(P,x)| \geq \sum_{x \in M} \left\lceil \frac{\delta^-(D)+k+1}{2} \right\rceil = |M| \left\lceil \frac{\delta^-(D)+k+1}{2} \right\rceil. \tag{1}$$

Since $|E(P,x)| \geq \left\lceil \frac{\delta^-(D)+k-1}{2} \right\rceil$ for $x \in P$, it follows that $|E(D[P])| = \sum_{y \in P} |E(P,y)| \geq |P| \left\lceil \frac{\delta^-(D)+k-1}{2} \right\rceil$, and so we conclude that

$$s = \sum_{y \in P} d^+(y) - |E(D[P])| \leq \sum_{y \in P} d^+(y) - |P| \left\lceil \frac{\delta^-(D)+k-1}{2} \right\rceil \leq |P|\Delta^+(D) - |P| \left\lceil \frac{\delta^-(D)+k-1}{2} \right\rceil. \tag{2}$$

Inequalities (1) and (2) imply that

$$|M| \leq \frac{|P|\Delta^+(D) - |P| \left\lceil \frac{\delta^-(D)+k-1}{2} \right\rceil}{\left\lceil \frac{\delta^-(D)+k+1}{2} \right\rceil}.$$

Since $\gamma_{kS}(D) = |P| - |M|$ and $n = |P| + |M|$, the last inequality leads to

$$\gamma_{kS}(D) \geq |P| - \frac{|P|\Delta^+(D) - |P| \left\lceil \frac{\delta^-(D)+k-1}{2} \right\rceil}{\left\lceil \frac{\delta^-(D)+k+1}{2} \right\rceil} = \frac{n + \gamma_{kS}(D)}{2} \cdot \frac{2\left\lceil \frac{\delta^-(D)+k+1}{2} \right\rceil - 1 - \Delta^+(D)}{\left\lceil \frac{\delta^-(D)+k+1}{2} \right\rceil},$$

and this yields the desired result. □

To see the sharpness of the last result, let $D = K_n^*$. If $k = n$ or $k = n-1$, then Theorem 2.3 leads to $\gamma_{kS}(D) \geq n$, and thus $\gamma_{kS}(D) = n$.

If $D(G)$ is the associate digraph of a graph G, then $N^-_{D(G)}(v) = N_G(v)$ for each $v \in V(G) = V(D(G))$. Thus the following useful observation is valid.

Observation 2.4.

If $D(G)$ is the associate digraph of a graph G, then $\gamma_{kS}(D(G)) = \gamma_{kS}(G)$.

There are many interesting applications of Observation 2.4, such as the following two results.

Corollary 2.5.

Let $k \geq 1$ be an integer, and let G be a graph of order n with $\delta(G) \geq k-1$. Then

$$\gamma_{kS}(G) \geq n\, \frac{2\left\lceil \frac{\delta(G)+k+1}{2} \right\rceil - 1 - \Delta(G)}{\Delta(G)+1}.$$

Proof. Since $\delta(G) = \delta^-(D(G))$, $\Delta(G) = \Delta^+(D(G))$ and $n = n(D(G))$, it follows from Theorem 2.3 and Observation 2.4 that

$$\gamma_{kS}(G) = \gamma_{kS}(D(G)) \geq \frac{2\left\lceil \frac{\delta^-(D(G))+k+1}{2} \right\rceil - 1 - \Delta^+(D(G))}{\Delta^+(D(G))+1}\, n = \frac{2\left\lceil \frac{\delta(G)+k+1}{2} \right\rceil - 1 - \Delta(G)}{\Delta(G)+1}\, n. \quad \square$$

Corollary 2.6 ([5]).

Let $k \geq 1$ be an integer, and let G be an r-regular graph of order n with $r \geq k-1$. Then $\gamma_{kS}(G) \geq \frac{kn}{r+1}$ if $k+r+1$ is even, and $\gamma_{kS}(G) \geq \frac{(k+1)n}{r+1}$ if $k+r+1$ is odd.

The special case $k = 1$ in Corollary 2.6 can be found in [1] and [2]. Counting the arcs from M to P, we next prove an analogue to Theorem 2.3.

Theorem 2.7.
Let $k \geq 1$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq k-1$. Then

$$\gamma_{kS}(D) \geq n\,\frac{\delta^+(D)+1-2\left\lfloor\frac{\Delta^-(D)-k+1}{2}\right\rfloor}{\delta^+(D)+1}.$$

Proof. Let f be a $\gamma_{kS}(D)$-function, and let s be the number of arcs from M to P. If $x \in P$, then

$$\Delta^-(D) \geq d^-(x) = |E(P,x)| + |E(M,x)| \geq 2\,|E(M,x)| + k - 1,$$

and thus $|E(M,x)| \leq \left\lfloor\frac{\Delta^-(D)-k+1}{2}\right\rfloor$ for each $x \in P$. Hence we deduce that

$$s = \sum_{x\in P} |E(M,x)| \leq |P| \left\lfloor \frac{\Delta^-(D)-k+1}{2} \right\rfloor. \tag{3}$$

If $x \in M$, then

$$\Delta^-(D) \geq d^-(x) = |E(P,x)| + |E(M,x)| \geq 2\,|E(M,x)| + k + 1,$$

and thus $|E(M,x)| \leq \left\lfloor\frac{\Delta^-(D)-k-1}{2}\right\rfloor$ for each $x \in M$. It follows that

$$s = \sum_{y\in M} d^+(y) - |E(D[M])| \geq |M|\,\delta^+(D) - |M| \left\lfloor \frac{\Delta^-(D)-k-1}{2} \right\rfloor. \tag{4}$$

Inequalities (3) and (4) imply that

$$|P| \geq \frac{|M|\,\delta^+(D) - |M|\left\lfloor\frac{\Delta^-(D)-k-1}{2}\right\rfloor}{\left\lfloor\frac{\Delta^-(D)-k+1}{2}\right\rfloor}.$$

Since $\gamma_{kS}(D) = |P| - |M|$ and $n = |P| + |M|$, the last inequality leads to

$$\gamma_{kS}(D) \geq \frac{|M|\delta^+(D) - |M|\left\lfloor\frac{\Delta^-(D)-k-1}{2}\right\rfloor}{\left\lfloor\frac{\Delta^-(D)-k+1}{2}\right\rfloor} - |M| = \frac{n-\gamma_{kS}(D)}{2}\cdot\frac{\delta^+(D) - 2\left\lfloor\frac{\Delta^-(D)-k+1}{2}\right\rfloor + 1}{\left\lfloor\frac{\Delta^-(D)-k+1}{2}\right\rfloor},$$

and this yields the desired result immediately. □

Using Observation 2.4 and Theorem 2.7, we obtain an analogue to Corollary 2.5, and this also leads to Corollary 2.6.

Theorem 2.8.
Let $k \geq 1$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq k-1$. Then

$$\gamma_{kS}(D) \geq \frac{\delta^+(D)+2k-\Delta^+(D)}{\delta^+(D)+\Delta^+(D)+2}\,n.$$

Proof. If f is a $\gamma_{kS}(D)$-function, then

$$\begin{aligned} nk &= \sum_{x\in V} k \leq \sum_{x\in V} f(N^-[x]) = \sum_{x\in V}(d^+(x)+1)f(x) = \sum_{x\in P}(d^+(x)+1) - \sum_{x\in M}(d^+(x)+1) \\ &\leq |P|(\Delta^+(D)+1) - |M|(\delta^+(D)+1) = |P|\big(\Delta^+(D)+\delta^+(D)+2\big) - n\,(\delta^+(D)+1). \end{aligned}$$

This implies that

$$|P| \geq \frac{n\,(\delta^+(D)+k+1)}{\delta^+(D)+\Delta^+(D)+2},$$

and hence we obtain the desired bound as follows,

$$\gamma_{kS}(D) = |P|-|M| = 2\,|P|-n \geq \frac{2n\,(\delta^+(D)+k+1)}{\delta^+(D)+\Delta^+(D)+2} - n = \frac{\delta^+(D)+2k-\Delta^+(D)}{\delta^+(D)+\Delta^+(D)+2}\,n. \qquad \square$$

Using Observation 2.4, we obtain the following analogue for graphs.

Corollary 2.9.

If $k \geq 1$ is an integer, and G is a graph of order n with $\delta(G) \geq k-1$, then

$$\gamma_{kS}(G) \geq \frac{\delta(G)+2k-\Delta(G)}{\delta(G)+\Delta(G)+2}\,n.$$

We note that Corollary 2.9 immediately implies a 1999 result by Zhang, Xu, Li and Liu [8] for the case $k = 1$.

Theorem 2.10.

Let $k \geq 1$ be an integer, and let D be a digraph of order n with $\delta^-(D) \geq k-1$. Then

$$\gamma_{kS}(D) \geq \frac{kn+|A(D)|-n\Delta^+(D)}{\Delta^+(D)+1}.$$

Proof. If f is a $\gamma_{kS}(D)$-function, then

$$\begin{aligned} nk &\leq \sum_{x\in V} f(N^-[x]) = \sum_{x\in V}(d^+(x)+1)f(x) = \sum_{x\in P}(d^+(x)+1) - \sum_{x\in M}(d^+(x)+1) \\ &= 2\sum_{x\in P}(d^+(x)+1) - \sum_{x\in V}(d^+(x)+1) \leq 2|P|(\Delta^+(D)+1) - |A(D)| - n. \end{aligned}$$

This implies that

$$|P| \geq \frac{(k+1)n+|A(D)|}{2\Delta^+(D)+2},$$

and hence we obtain the desired bound as follows,

$$\gamma_{kS}(D) = |P|-|M| = 2|P|-n \geq \frac{(k+1)\,n+|A(D)|}{\Delta^+(D)+1} - n = \frac{kn+|A(D)|-n\Delta^+(D)}{\Delta^+(D)+1}. \qquad \square$$

Theorem 2.11.

Let $r \geq k \geq 1$ be integers, and let D be a digraph of order $n \geq 2$ such that $\delta^-(D) \geq k-1$ and $\delta^+(D) \geq r$. Then

$$\gamma_{kS}(D) \geq 2\left(\chi(G)+k+r-\Delta(G)\right) - n,$$

where G is the underlying graph of D.

Proof. By Theorem B, we may assume that $k \geq 2$. If $\Delta^-(D) \leq k$, then $d^+(x) = d^-(x) = k$ for each $x \in V(D)$, and by Observation 1.4, $\gamma_{kS}(D) = n$. The result follows. If $\Delta^-(D) \geq k+1$, then $\Delta(G) \geq 2k+1$.

Let $\alpha = \frac{\Delta(G)-r-k-1}{2}$. We claim that $r \leq \Delta(G) - k - 1$. Suppose to the contrary that $r \geq \Delta(G) - k$. Since $d^+(x) + d^-(x) \leq \Delta(G)$, by the assumption we have $d^-(x) \leq k$ for each $x \in V(D)$. Thus

$$n\,(\Delta(G) - k) \leq \sum_{x \in V(D)} d^+(x) = \sum_{x \in V(D)} d^-(x) \leq nk,$$

which implies that $\Delta(G) \leq 2k$. This is a contradiction, and therefore $\alpha \geq 0$. For each $x \in M$, $|E(P,x)| \geq |E(M,x)| + k + 1$, and so

$$\Delta(G) \geq \deg(x) = |E(P,x)| + |E(M,x)| + d^+(x) \geq r + 2|E(M,x)| + k + 1,$$

which implies $|E(M,x)| \leq \alpha$. Let $H = D[M]$ be the subdigraph induced by M and let $H' = G[M]$ be the underlying graph of H.

Suppose H_1 is an induced subgraph of H. Then $d^-_{H_1}(x) \leq |E(M,x)| \leq \alpha$ for each $x \in V(H_1)$, and hence $\sum_{x \in V(H_1)} d^+_{H_1}(x) = \sum_{x \in V(H_1)} d^-_{H_1}(x) \leq \alpha\,|V(H_1)|$. Therefore there exists a vertex $x \in V(H_1)$ such that $d^+_{H_1}(x) \leq \alpha$. This implies that $\delta(H_1') \leq 2\alpha$, where H_1' is the underlying graph of H_1. By Theorem A,

$$\begin{aligned}\chi(H') &\leq 1 + \max\left\{\delta(H'') \mid H'' \text{ is a subgraph of } H'\right\} \\ &= 1 + \max\left\{\delta(H_1') \mid H_1' \text{ is an induced subgraph of } H'\right\} \leq 1 + 2\alpha.\end{aligned}$$

Since $2|P| - n = \gamma_{kS}(D)$, it follows that

$$\chi(G) \leq \chi(G[P]) + \chi(G[M]) \leq |P| + 1 + 2\alpha = 1 + 2\alpha + \frac{n + \gamma_{kS}(D)}{2}.$$

Thus

$$\gamma_{kS}(D) \geq 2\left(\chi(G) + k + r - \Delta(G)\right) - n,$$

as required. □

Theorem 2.12.

Let D be a digraph of order n and size m, and let G be the underlying graph of D. Then $\gamma_{kS}(D) \geq nk - m$. Furthermore, the bound is sharp.

Proof. Let f be a $\gamma_{kS}(D)$-function and t be the number of arcs from the set P to the set M. Then

$$t = \sum_{x \in M} |E(P,x)| \geq \sum_{x \in M} (k+1) = |M|(k+1) = (n - |P|)(k+1). \tag{5}$$

On the other hand, $m \geq t + |E(P,P)|$. Since $|E(P,P)| \geq |P|(k-1)$, we have $m \geq (n-|P|)(k+1) + |P|(k-1)$ and thus $|P| \geq \frac{n(k+1)-m}{2}$. Since $\gamma_{kS}(D) = 2|P| - n$, the result follows.

To prove the sharpness, note that if $k = n$, the bound is sharp for K_n^*. Thus we may assume $k \leq n-1$. Suppose that C is a directed Hamiltonian cycle in K_{k+1}^* and consider the digraph $D = \left(K_{k+1}^* - E(C)\right) \vee \overline{K_{n-k-1}}$ where the edges are oriented from $V\left(K_{k+1}^* - E(C)\right)$ to $V(\overline{K_{n-k-1}})$. Define $f : V(D) \to \{-1, 1\}$ by $f(v) = 1$ if $v \in V\left(K_{k+1}^* - E(C)\right)$ and $f(v) = -1$ if $v \in V(\overline{K_{n-k-1}})$. Obviously, f is an SkDF of D and $w(f) = nk - m$ where m is the size of D. Hence, $\gamma_{kS}(D) = nk - m$. This completes the proof. □

Theorem 2.13.

Let D be a digraph of order n with outdegree sequence $d_1^+ \geq d_2^+ \geq \cdots \geq d_n^+$ and let s be the smallest positive integer for which $\sum_{i=1}^{s} d_i^+ - \sum_{i=s+1}^{n} d_i^+ \geq (k+1)\,n - 2s$. Then $\gamma_{kS}(D) \geq 2s - n$. Furthermore, this bound is sharp.

Proof. Let f be a $\gamma_{kS}(D)$-function and $p = |P|$. We have

$$kn \le \sum_{x\in V} f(N_D^-[x]) = \sum_{x\in V}(d^+(x)+1)f(x) = \sum_{x\in P}(d^+(x)+1) - \sum_{x\in M}(d^+(x)+1)$$
$$\le |P| - |M| + \left(\sum_{x\in P} d^+(x) - \sum_{x\in M} d^+(x)\right) \le 2p - n + \left(\sum_{i=1}^{p} d_i^+ - \sum_{i=p+1}^{n} d_i^+\right).$$

Thus $(k+1)n - 2p \le \sum_{i=1}^{p} d_i^+ - \sum_{i=p+1}^{n} d_i^+$. By the assumption on s, we must have $p \ge s$. This implies that $\gamma_{kS}(D) = 2p - n \ge 2s - n$.

In order to show that the bound is sharp, suppose that C is a directed Hamiltonian cycle in K_{k+1}^* and consider the digraph $D = \left(K_{k+1}^* - E(C)\right) \vee \overline{K_{n-k-1}}$ where the edges are oriented from $V\left(K_{k+1}^* - E(C)\right)$ to $V(\overline{K_{n-k-1}})$. By Theorem 2.12, $\gamma_{ks}(D) = 2(k+1) - n$. Since the outdegree sequence of D is

$$\overbrace{n-2,\ldots,n-2}^{k+1},\overbrace{0\ldots,0}^{n-k-1}$$

and $(n-2)(k+1) = n\,(k+1) - 2\,(k+1)$, it follows that $k+1$ is the smallest positive integer such that $\sum_{i=1}^{s} d_i^+ - \sum_{i=s+1}^{n} d_i^+ \ge n\,(k+1) - 2\,(k+1)$, and so $\gamma_{kS}(D) \ge 2\,(k+1) - n$. This completes the proof. □

The special case $k = 1$ in Theorems 2.12 and 2.13 was recently proved by Karami, Khodkar and Sheikholeslami in [3].

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