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On the extent of star countable spaces

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Abstract: For a topological property $\mathcal{P}$, we say that a space X is *star* $\mathcal{P}$ if for every open cover $\mathcal{U}$ of the space X there exists $Y \subset X$ such that $\mathrm{St}(Y,\mathcal{U}) = X$ and Y has $\mathcal{P}$. We consider star countable and star Lindelöf spaces establishing, among other things, that there exists first countable pseudocompact spaces which are not star Lindelöf. We also describe some classes of spaces in which star countability is equivalent to countable extent and show that a star countable space with a dense σ-compact subspace can have arbitrary extent. It is proved that for any ω_1-monolithic compact space X, if $C_p(X)$ is star countable then it is Lindelöf.

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Introduction

Given a topological property $\mathcal{P}$, a space X is called *star* $\mathcal{P}$ if for an arbitrary open cover $\mathcal{U}$ of the space X, there exists a set $Y \subset X$ such that $\mathrm{St}(Y,\mathcal{U}) = X$ and Y has the property $\mathcal{P}$. The classes of star $\mathcal{P}$ spaces were first defined (under another name) by Ikenaga in his paper [10] where he studied the cases of star countable, star Lindelöf and star

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σ-compact spaces. Star $\mathcal{P}$ properties were also introduced and studied systematically in the survey of Matveev [12]. The last three authors of this paper published in [14] some results on star $\mathcal{P}$ spaces for compactness-like properties $\mathcal{P}$.

One of the motivations to study star $\mathcal{P}$ properties is a folklore fact that every space is star discrete (and hence star metrizable); in fact, for any cover $\mathcal{U}$ of a space X, there exists a closed discrete set $D \subset X$ such that $\mathrm{St}\,(D,\mathcal{U}) = X$. To see it, choose inductively a point $x_\alpha \notin \mathrm{St}\,(\{x_\beta : \beta < \alpha\}, \mathcal{U})$. If μ is the first ordinal for which this choice is impossible, then $D = \{x_\alpha : \alpha < \mu\}$ is a closed discrete subset of X and $\mathrm{St}\,(D,\mathcal{U}) = X$.

For some concrete classes $\mathcal{P}$, the star $\mathcal{P}$ properties were studied in the papers [4, 5, 8–11] with very individual terminology in each one. In particular, star countable spaces were called "star Lindelöf", "spaces of countable weak extent", "ω-star" and "*Lindelöf". We hope that the paper [14] will be a basis for standardizing the terminology; the term "star $\mathcal{P}$" is logically simple and defines all star covering properties.

In this paper we consider star countable and star Lindelöf spaces continuing the research done in [1]. We solve two questions from [1] showing, in particular, that a dense pseudocompact subspace of a Tychonoff cube need not be star Lindelöf and there exist pseudocompact spaces with a point-countable base which are not star Lindelöf.

Matveev proved in [13] that a star countable space can have arbitrarily large extent. In the same paper he cites the referee's question whether a realcompact star countable space can have arbitrarily big extent. A very natural task, therefore, is to determine whether $\mathbb{R}^\kappa$ is star countable for any cardinal κ; observe that this is true for all $\kappa \leq \mathfrak{c}$ because in this case $\mathbb{R}^\kappa$ is separable; besides, $\mathbb{R}^\kappa$ is star Lindelöf because it has a dense σ-compact subspace. We prove that $\mathbb{R}^\kappa$ is not star countable for any $\kappa \geq 2^{\mathfrak{c}^+}$.

Matveev's example in [13] of a star countable space with a large extent is pseudocompact and it is not dense in any product of "nice" spaces; we complement this example showing that, for any infinite cardinal κ, there exists a dense star countable subspace X of the Cantor cube $\mathbb{D}^\kappa$ such that $X = L \cup D$, where L is a dense σ-compact subspace of X and D is a closed discrete set of cardinality κ. We also study star countable and star Lindelöf P-spaces showing that in the presence of normality, all these classes coincide with the class of spaces of countable extent.

One of our sources of inspiration was Arkhangel'skii's problem cited by Bonanzinga and Matveev in [4, Question 2.2.4]; Arkhangel'skii asked whether for every compact space X, star countability of $C_p(X)$ is equivalent to its Lindelöf property. We prove that this is true for any ω_1-monolithic compact space X.

Notation and terminology

If nothing is said about the axioms of separation of a space X, then X is assumed to be Hausdorff. Given a space X, the family $\tau(X)$ is its topology; if $x \in X$ then $\tau(x, X) = \{U \in \tau(X) : x \in U\}$. Suppose that $\mathcal{A}$ is a family of subsets of X; then $\mathrm{St}\,(Y, \mathcal{A}) = \bigcup \{A \in \mathcal{A} : Y \cap A \neq \emptyset\}$ for any $Y \subset X$. We denote by $\mathbb{R}$ the real line with its natural topology and $\mathbb{I} = [0, 1] \subset \mathbb{R}$. Let $\mathbb{Q}$ be the set of rationals; we will also need the doubleton $\mathbb{D} = \{0, 1\}$ with the discrete topology as well as the set $\mathbb{N} = \omega \setminus \{0\}$.

Our set-theoretic notation is standard; in particular, any ordinal is identified with the set of its predecessors. For any set A we let $[A]^{<\omega} = \{B \subset A : B \text{ is finite}\}$, $[A]^\omega = \{B \subset A : B \text{ is countably infinite}\}$ and $[A]^{\leq\omega} = \{B \subset A : B \text{ is countable}\}$. A family $\mathcal{A}$ of subsets of a set X is said to T_0-*separate the points* of X if, for any distinct $x, y \in X$ there exists $A \in \mathcal{A}$ such that $|A \cap \{x, y\}| = 1$.

If X is a space and $\mathcal{U}$ is an open cover of X then a set $Y \subset X$ is a *kernel* of $\mathcal{U}$ if $\mathrm{St}\,(Y, \mathcal{U}) = X$. Suppose that $\mathcal{P}$ is a topological property; following [14] and [1], say that a space X is *star $\mathcal{P}$* if any open cover $\mathcal{U}$ of the space X has a kernel Y with the property $\mathcal{P}$. For an infinite cardinal κ, a space X is called κ-*monolithic* if $\mathrm{nw}\,(\overline{A}) \leq \kappa$ for any set $A \subset X$ with $|A| \leq \kappa$.

For any space X the *extent* of X (denoted as $\mathrm{ext}\,(X)$) is the supremum of cardinalities of closed discrete subsets of X. A space X is *metalindelöf* if every open cover of X has a point-countable open refinement. We say that X is a *P-space* if every G_δ-subset of X is open in X. A regular space X is a *Moore space* or, equivalently, *a developable space* if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that the family $\{\mathrm{St}\,(x, \mathcal{U}_n) : n \in \omega\}$ is a local base at x for each $x \in X$.

The rest of our terminology is standard and can be found in [7].

1. Star countability and star Lindelöf property vs countable extent

It is well-known and easy to prove that any space of countable extent is star countable. We will show that, in quite a few classes of spaces, star countability and sometimes even star Lindelöfness is equivalent to having countable extent.

Definition 1.1.
If X is a space and $A \subset X$, say that a family $\mathcal{U}$ is an *open expansion of A* if $\mathcal{U} = \{U_a : a \in A\}$ and $U_a \in \tau(a, X)$ for any $a \in A$.

Definition 1.2.
Given an infinite cardinal κ, call a space X *weakly κ-metalindelöf* if for any closed discrete set $D \subset X$ with $|D| = \kappa$ we can find a set $D' \subset D$ such that $|D'| = \kappa$ and D' has a point-countable open expansion.

Definition 1.3.
If κ is an infinite cardinal, we will say that a space X is *weakly κ-collectionwise Hausdorff* if for any closed discrete set $D \subset X$ with $|D| = \kappa$ there exists a set $D' \subset D$ such that $|D'| = \kappa$ and D' has a disjoint open expansion.

The respective definitions imply that any weakly κ-collectionwise Hausdorff space is weakly κ-metalindelöf.

Lemma 1.4.
Suppose that a space X has an uncountable closed discrete subspace D such that some uncountable set $E \subset D$ has a point-countable open expansion. Then X is not star countable.

Proof.   Assume that X is star countable and take a point-countable open expansion $\mathcal{V} = \{V_d : d \in E\}$ of the uncountable set E such that $V_d \cap E = \{d\}$ for any $d \in E$. The family $\mathcal{V}' = \mathcal{V} \cup \{X \setminus E\}$ is an open cover of the space X, so there is a countable set $A \subset X$ such that $\mathrm{St}\,(A, \mathcal{V}') = X$. Since $\mathcal{V}'$ is point-countable, we can find a countable set $B \subset E$ such that, for any $d \in E$, if $V_d \cap A \neq \emptyset$ then $d \in B$. Take any $d \in E \setminus B$ and observe that $X = \mathrm{St}\,(A, \mathcal{V}') \not\ni d$ which is a contradiction. $\square$

Corollary 1.5.
For any weakly ω_1-metalindelöf space X the following conditions are equivalent:

(a) *the space X is star countable;*

(b) $\mathrm{ext}\,(X) \leq \omega$.

Corollary 1.6.
If a space X is weakly ω_1-collectionwise Hausdorff, then X is star countable if and only if $\mathrm{ext}\,(X) \leq \omega$.

Corollary 1.7.
A regular P-space X is star countable if and only if $\mathrm{ext}\,(X) \leq \omega$.

Proof.   It is easy to see that every regular P-space is weakly ω_1-collectionwise Hausdorff, so Corollary 1.6 applies. $\square$

Theorem 1.8.
For a normal P-space X the following conditions are equivalent:

(a) X *is star countable;*

(b) X *is star Lindelöf;*

(c) *every discrete family of non-empty open subsets of X is countable;*

(d) $\mathrm{ext}\,(X) \leq \omega$.

Proof. We have (d)⇒(a)⇒(b)⇒(c) [1, Theorems 2.1 and 2.7], so it suffices to prove that (c)⇒(d). Assume that every discrete family of non-empty open subsets of X is countable and there exists a closed discrete set $D \subset X$ with $|D| = \omega_1$. Using the P-property of X it can be proved in a standard way that there exists a disjoint open expansion $\{U_d : d \in D\}$ of the set D. By normality of X, we can find an open set $G \subset X$ such that $D \subset G \subset \overline{G} \subset U = \bigcup \{U_d : d \in D\}$. If we let $V_d = G \cap U_d$ for each $d \in D$, then it is routine to prove that the family $\{V_d : d \in D\}$ is a discrete expansion of D, which is a contradiction. □

The following fact is well known for Lindelöf P-spaces.

Proposition 1.9.
Suppose that X is a regular P-space such that every discrete family of non-empty open subsets of X is countable and $\psi(x, X) \leq \omega_1$ for some $x \in X$. Then $\chi(x, X) \leq \omega_1$.

Proof. Fix a decreasing ω_1-sequence $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of clopen subsets of X such that $\{x\} = \bigcap \mathcal{U}$. To see that $\mathcal{U}$ is a local base at x, fix any clopen set $U \in \tau(x, X)$ and observe that, in the space $Y = X \setminus U$, every discrete family of non-empty open subsets is countable because Y is clopen in the space X.

Let $V_\alpha = Y \setminus U_\alpha$ for every $\alpha < \omega_1$; then the family $\mathcal{V} = \{V_\alpha : \alpha < \omega_1\}$ is a clopen cover of Y. By the P-property of Y, the set $W_\alpha = V_\alpha \setminus \bigcup \{V_\beta : \beta < \alpha\}$ is clopen in Y for each $\alpha < \omega_1$ and it is straightforward that $\mathcal{W} = \{W_\alpha : \alpha < \omega_1\}$ is a disjoint clopen refinement of $\mathcal{V}$.

This refinement is a discrete family of open subsets of Y so only countably many elements of $\mathcal{W}$ are non-empty. Since the family $\mathcal{V}$ is increasing, we must have $Y \setminus U_\alpha = V_\alpha = Y$, i.e., $U_\alpha \subset U$ for some $\alpha < \omega_1$. □

Proposition 1.10.
Suppose that X is a regular P-space such that every discrete family of non-empty open subsets of X is countable and $l(X) \leq \omega_1$. Then X is Lindelöf.

Proof. Take any open cover $\mathcal{U}$ of the space X. We can assume, without loss of generality, that $|\mathcal{U}| \leq \omega_1$ and all elements of $\mathcal{U}$ are clopen in X. Choose an enumeration $\{U_\alpha : \alpha < \omega_1\}$ of the family $\mathcal{U}$ and let $V_\alpha = U_\alpha \setminus \bigcup \{U_\beta : \beta < \alpha\}$ for every $\alpha < \omega_1$. It is clear that $\mathcal{V} = \{V_\alpha : \alpha < \omega_1\}$ is a disjoint clopen partition of X, so it is a discrete family and hence the collection $\mathcal{V}' = \{V \in \mathcal{V} : V \neq \emptyset\}$ is a countable open refinement of the cover $\mathcal{U}$. □

Proposition 1.11.
If G is a P-group such that every discrete family of non-empty open subsets of G is countable and $\psi(G) \leq \omega_1$, then $w(G) \leq \omega_1$ and G is Lindelöf.

Proof. It follows from Proposition 1.9 that $\chi(G) \leq \omega_1$; fix a base $\{U_\alpha : \alpha < \omega_1\}$ at the identity e of the group G. It is standard from the P-property of G that there exists a clopen subgroup $H_\alpha \subset G$ such that $H_\alpha \subset U_\alpha$ for each $\alpha < \omega_1$. For every $\alpha < \omega_1$ the family $\mathcal{B}_\alpha = \{x \cdot H_\alpha : x \in G\}$ is a clopen partition of G so it is a discrete family of non-empty open subsets of G and hence $|\mathcal{B}_\alpha| \leq \omega$.

The family $\mathcal{B} = \bigcup \{\mathcal{B}_\alpha : \alpha < \omega_1\}$ is easily seen to be a base in G; this implies that $l(G) \leq w(G) \leq \omega_1$ and hence G is Lindelöf by Proposition 1.10. □

Theorem 1.12.
Suppose that κ is an infinite cardinal with $\kappa^\omega = \kappa$ and X is a regular star Lindelöf P-space such that $w(X) = \kappa$. Then X has no closed discrete subset of cardinality κ; in particular, if κ is a successor cardinal then $\operatorname{ext}(X) < \kappa$.

Proof. Assume that there exists a closed discrete subset $D \subset X$ with $|D| = \kappa$. It is easy to find a clopen base $\mathcal{B}$ in X such that $|\mathcal{B}| = \kappa$ and $|U \cap D| \leq 1$ for any $U \in \mathcal{B}$. Let $\{\mathcal{V}_\alpha : \alpha < \kappa\}$ be an enumeration of all countable subfamilies of $\mathcal{B}$. Observe that $\bigcup \mathcal{V}_0$ is a closed subset of X which contains at most countably many points of D so we can take a point $d_0 \in D \setminus \bigcup \mathcal{V}_0$.

Proceeding by induction, suppose that $\alpha < \kappa$ and we have chosen a point $d_\beta \in D$ for any $\beta < \alpha$ in such a way that $\beta \neq \beta'$ implies $d_\beta \neq d_{\beta'}$ and $d_\beta \in D \setminus \bigcup \mathcal{V}_\beta$ for each $\beta < \alpha$. Since the set $P = \{d_\beta : \beta < \alpha\} \cup \bigcup \mathcal{V}_\alpha$ can contain at most $|\alpha| \cdot \omega < \kappa$ points of D, we can pick a point $d_\alpha \in D \setminus P$. Therefore we can construct a faithfully indexed set $D' = \{d_\alpha : \alpha < \kappa\} \subset D$ such that $d_\alpha \in D \setminus \bigcup \mathcal{V}_\alpha$ for every $\alpha < \kappa$. In particular, we can fix a set $W_\alpha \in \tau(d_\alpha, X)$ such that $W_\alpha \cap D' = \{d_\alpha\}$ and $W_\alpha \cap \bigcup \mathcal{V}_\alpha = \emptyset$ for any $\alpha < \kappa$.

The family $\mathcal{U} = \{X \setminus D'\} \cup \{W_\alpha : \alpha < \kappa\}$ is an open cover of X. Therefore there exists a Lindelöf subspace $L \subset X$ such that $\mathrm{St}(L, \mathcal{U}) = X$. Since the family $\mathcal{B}$ covers L, we can find a countable subfamily $\mathcal{V} \subset \mathcal{B}$ such that $L \subset \bigcup \mathcal{V}$. There exists $\alpha < \kappa$ such that $\mathcal{V} = \mathcal{V}_\alpha$, so it follows from $L \subset \bigcup \mathcal{V}_\alpha$ and $W_\alpha \cap \bigcup \mathcal{V}_\alpha = \emptyset$ that $W_\alpha \cap L = \emptyset$, and hence $d_\alpha \notin \mathrm{St}(L, \mathcal{U})$ which is a contradiction. $\square$

Proposition 1.13.
Suppose that X is a regular star Lindelöf metalindelöf space such that $t(X) \leq \kappa$ and $\psi(X) \leq \kappa$. Then $l(X) \leq 2^\kappa$.

Proof. Let $\mathcal{U}$ be an open cover of X; we can assume that $\mathcal{U}$ is point-countable. There exists a Lindelöf subspace $A \subset X$ such that $\mathrm{St}(A, \mathcal{U}) = X$. It follows from the inequality $|A| \leq 2^{l(A) \cdot \psi(A) \cdot t(A)}$ that $|A| \leq 2^{\omega \cdot \kappa \cdot \kappa} = 2^\kappa$. The cover $\mathcal{U}$ being point-countable, the family $\mathcal{U}' = \{U \in \mathcal{U} : U \cap A \neq \emptyset\}$ has cardinality at most $|A| \cdot \omega \leq 2^\kappa$ so $\mathcal{U}'$ is a subcover of $\mathcal{U}$ of cardinality $\leq 2^\kappa$. $\square$

Corollary 1.14.
If a regular star Lindelöf space X has a point-countable base then we have the inequality $l(X) \leq \mathfrak{c}$.

Proof. It is clear that X is metalindelöf and $t(X) \cdot \psi(X) \leq \chi(X) \leq \omega$, so we can apply Proposition 1.13. $\square$

It was asked in [1, Question 1] whether every pseudocompact first countable space must be star Lindelöf. The following result shows that even existence of a point-countable base in a pseudocompact space does not guarantee its star Lindelöf property. The same result also answers [1, Question 3].

Theorem 1.15.
There exists a Tychonoff pseudocompact space X with a point-countable base which is not star Lindelöf.

Proof. By a result of Shakhmatov [15, Theorem 1] there exists a Tychonoff pseudocompact space X such that X has a point-countable base and $\mathrm{ext}(X) \geq \mathfrak{c}^+$. If X is star Lindelöf then Corollary 2.14 shows that we have the inequalities $\mathrm{ext}(X) \leq l(X) \leq \mathfrak{c}$, which is a contradiction. $\square$

Proposition 1.16.
Suppose that X is a star countable ω-monolithic space.

(a) If $|X|^\omega = |X|$, then X has no closed discrete subspace of cardinality $|X|$; in particular, if $|X|$ is a successor cardinal then $\mathrm{ext}(X) < |X|$.

(b) If $|X| = \omega_1$ then $\mathrm{ext}(X) \leq \omega$.

Proof. (a) Assume the contrary; let $\kappa = |X|$ and fix a closed discrete set $D \subset X$ with $|D| = \kappa$. Choose an enumeration $\{C_\alpha : \alpha < \kappa\}$ of all countably infinite subsets of X. By ω-monolithity of X, for any countable set $A \subset X$ we have the inequalities $|\overline{A} \cap D| \leq \omega < \kappa$, so we can construct by induction a faithfully indexed set $D' = \{d_\alpha : \alpha < \kappa\} \subset D$ such that $d_\alpha \notin \overline{C_\alpha}$ for all ordinals $\alpha < \kappa$. Pick a set $U_\alpha \in \tau(d_\alpha, X)$ such that $U_\alpha \cap C_\alpha = \emptyset$ and $U_\alpha \cap D = \{d_\alpha\}$ for each $\alpha < \kappa$. It is straightforward that the cover $\mathcal{U} = \{X \setminus D'\} \cup \{U_\alpha : \alpha < \kappa\}$ witnesses that the space X is not star countable, a contradiction.

(b) Suppose that there exists a closed discrete set $D \subset X$ with $|D| = \omega_1$. Choose an enumeration $\{x_\alpha : \alpha < \omega_1\}$ of the set X and let $X_\alpha = \{x_\beta : \beta < \alpha\}$ for each $\alpha < \omega_1$. Applying the ω-monolithity of X we can construct by induction a set $D' = \{d_\alpha : \alpha < \omega_1\} \subset D$ and an open expansion $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\}$ of the set D' such that $\alpha \neq \beta$ implies $d_\alpha \neq d_\beta$, while $U_\alpha \cap D = \{d_\alpha\}$ and $U_\alpha \cap X_\alpha = \emptyset$ for every $\alpha < \omega_1$. It is immediate that $\mathcal{U}$ is point-countable, so we can apply Lemma 1.4 to see that X is not star countable, which is a contradiction. $\qquad\square$

Assume that a space X contains a dense hereditarily Lindelöf subspace L. If L is separable then X is separable and hence star countable. Since hereditarily Lindelöf spaces are often separable, it is interesting to describe the situations in which existence of a dense L-space in X implies that X is star countable.

Example 1.17.
There exists a Hausdorff space X with the following properties:

(a) X has a dense hereditarily Lindelöf subspace;

(b) $\mathrm{ext}\,(X) > \omega$ and X is not star countable.

Proof.   Denote by μ the natural topology of $\mathbb{R}$ and let ν be the topology on $\mathbb{R}$ generated by the family $\mu \cup \{\mathbb{R} \backslash C : C \subset \mathbb{R}$ and $|C| \leq \omega\}$; then $(\mathbb{R}, \nu)$ is a hereditarily Lindelöf Hausdorff space. Denote by (Y, ξ) the Katetov extension of $(\mathbb{R}, \nu)$. Each open ultrafilter $\mathcal{G}$ on $(\mathbb{R}, \mu)$ is contained in some open ultrafilter $\mathcal{F}_\mathcal{G}$ on $(\mathbb{R}, \nu)$. Since distinct open ultrafilters on $(\mathbb{R}, \mu)$ have disjoint elements, the map $\mathcal{G} \to \mathcal{F}_\mathcal{G}$ is injective, so there are at least $2^{\mathfrak{c}}$-many open ultrafilters on $(\mathbb{R}, \nu)$. In particular, the space $X = (Y, \xi)$ has uncountable extent and has a dense hereditarily Lindelöf subspace $(\mathbb{R}, \nu)$.

To see that X is not star countable, let $D = Y \setminus \mathbb{R}$ and fix a surjective function $\varphi \colon D \to [\mathbb{R}]^{\leq \omega}$ such that $\varphi^{-1}(B)$ is uncountable for all $B \in [\mathbb{R}]^{\leq \omega}$. Observe that $\mathbb{R} \setminus \varphi(y) \in y$ so $U_y = \{y\} \cup (\mathbb{R} \setminus \varphi(y)) \in \tau(y, X)$ for each $y \in D$. Consider the open cover $\mathcal{U} = \{U_y : y \in D\} \cup \{\mathbb{R}\}$ of the space X. If $A \subset X$ is countable, then there exists $y \in D \setminus A$ with $\varphi(y) = B = A \cap \mathbb{R}$. Therefore $U_y \cap A = \emptyset$ and hence $y \notin \mathrm{St}\,(A, \mathcal{U})$, i.e., X is not star countable. $\qquad\square$

Example 1.18.
Under CH there exists a Tychonoff space X with the following properties:

(a) X has a dense hereditarily Lindelöf subspace;

(b) $\mathrm{ext}\,(X) > \omega$ and X is not star countable.

Proof.   Let $S = \left\{x \in \mathbb{D}^{\omega_1} : |x^{-1}(1)| \leq \omega\right\}$ be the Σ-product of the Cantor cube $\mathbb{D}^{\omega_1}$. It is known, see [2, Theorem 1.6.5], that under CH, the space S contains a dense Luzin subspace L; it is evident that $|L| = \omega_1$ so we can fix an enumeration $\{y_\alpha : \alpha < \omega_1\}$ of the set L. For every $\alpha < \omega_1$ let $d_\alpha(\beta) = 1$ if $\beta \neq \alpha$ and $d_\alpha(\alpha) = 0$. It is easy to see that $D = \{d_\alpha : \alpha < \omega_1\} \subset \mathbb{D}^{\omega_1} \setminus L$ and the unique cluster point of D is the element of $\mathbb{D}^{\omega_1}$ which takes the value 1 at every $\alpha < \omega_1$.

Thus the space $X = D \cup L$ has a dense hereditarily Lindelöf subspace L, while D is a closed discrete subspace of X so $\mathrm{ext}\,(X) > \omega$. Observe that $\mathrm{cl}_X(A) \subset L$ for every countable set $A \subset L$, so the space X is ω-monolithic. If X is star countable then it follows from $|X| = \omega_1$ that we can apply Proposition 1.16 to see that $\mathrm{ext}\,(X) \leq \omega$, which is a contradiction. $\qquad\square$

Ikenaga proved in [8, Theorem 4] that any star countable Moore space is separable. The following result generalizes this fact.

Theorem 1.19.
For a Moore space X the following are equivalent:

(1) *the space X is separable;*

(2) *X is star countable;*

(3) *X is star Lindelöf.*

Proof. It is immediate that (1)⇒(2)⇒(3), so assume that X is a star Lindelöf Moore space; hence we can fix a development $\{\mathcal{U}_n : n \in \omega\}$ in X. Choose a Lindelöf kernel L_n for the cover $\mathcal{U}_n$ for each $n \in \omega$. Observe that every L_n is a Lindelöf (and hence collectionwise normal) Moore space, so L_n is metrizable (and hence second countable) by Bing's metrization criterion [7, Theorem 5.4.1], whence we can find a countable dense subset D_n in the space L_n.

To see that the set $D = \bigcup \{D_n : n \in \omega\}$ is dense in X, fix any point $x \in X$ and $U \in \tau(x, X)$. There exists $m \in \omega$ such that $\operatorname{St}(x, \mathcal{U}_m) \subset U$. The set L_m being a kernel of $\mathcal{U}_m$, we can find $z \in L_m$ and $V \in \mathcal{U}_m$ such that $\{x, z\} \subset V$. Observe that $V \subset \operatorname{St}(x, \mathcal{U}_m) \subset U$; besides, it follows from $V \cap L_m \neq \emptyset$ that $V \cap D_m \neq \emptyset$ which shows that $U \cap D \neq \emptyset$ and hence D is dense in X. □

If L is a linearly ordered topological space and $X \subset L$, then X is collectionwise normal and hence Corollary 1.6 is applicable to see that star countability of X implies $\operatorname{ext}(X) \leq \omega$. However, even subspaces of finite products of ordinals can fail to be normal, so other methods are needed to see what happens with the extent of star countable subspaces of such products.

Theorem 1.20.
Suppose that L_i is a scattered LOTS for any $i = 0, \ldots, n$. A space $X \subset L = \prod \{L_i : i \leq n\}$ is star countable if and only if $\operatorname{ext}(X) \leq \omega$.

Proof. Suppose that X is star countable and $D \subset X$ is a closed discrete set with $|D| = \omega_1$. Let $p_i \colon L \to L_i$ be the projection on the i-th factor, $i \leq n$. There exists $j \leq n$ such that $E = p_j(D)$ is uncountable; denote by I the set of isolated points of E. The space L_j being scattered, we have $E \subset \bar{I}$. If I is countable then $\operatorname{hl}(\bar{I}) = d(\bar{I}) = \omega$; since $\bar{I}$ is scattered and hence right–separated, we conclude that $|\bar{I}| = \omega$ which shows that $|E| \leq |\bar{I}| = \omega$, a contradiction.

Therefore $|I| = \omega_1$; apply the hereditary collectionwise normality of L_j to find an open disjoint expansion $\{W_x : x \in I\}$ of the set I. Choose a point $z_x \in D$ such that $p_j(z_x) = x$ for every $x \in I$. It is immediate that $\{p_j^{-1}(W_x) : x \in I\}$ is an open disjoint expansion of an uncountable subset $\{z_x : x \in I\}$ of the set D, so X is not star countable by Lemma 1.4. This contradiction shows that $\operatorname{ext}(X) \leq \omega$. □

Corollary 1.21.
If X is a star countable subspace of λ^n for some ordinal λ and $n \in \mathbb{N}$, then $\operatorname{ext}(X) \leq \omega$.

Regarding Corollary 1.21, it is worth mentioning that the space λ^n is star countable because it has countable extent for any ordinal λ.

Corollary 1.22.
A subspace $X \subset \omega_1^\omega$ is star Lindelöf if and only if $\operatorname{ext}(X) \leq \omega$.

Proof. For each $n \in \omega$ let $p_n \colon \omega_1^\omega \to \omega_1$ be the projection onto the n-th factor. Assume that X is star Lindelöf; observe first that

$$\text{any Lindelöf subspace of } X \text{ is second countable,} \qquad (\star)$$

because if $Y \subset X$ is Lindelöf then $p_n(Y)$ is countable for any $n \in \omega$. Therefore X is star second countable and hence star countable. Let D be an uncountable closed discrete subset of X. If $p_n(D)$ is countable for every $n \in \omega$, then $D \subset \prod_{n \in \omega} p_n(D)$ is second countable which is a contradiction. Therefore $|p_n(D)| = \omega_1$ for some $n \in \omega$. If I is the set of isolated points of $E = p_n(D)$ then $|E| = \operatorname{hl}(E) = c(E) = |I|$, so the set I has to be uncountable.

Apply the hereditary collectionwise normality of the space ω_1 to find an open disjoint expansion $\{W_x : x \in I\}$ of the set I. Choose a point $z_x \in D$ such that $p_n(z_x) = x$ for every $x \in I$. It is immediate that $\{p_n^{-1}(W_x) : x \in I\}$ is an open disjoint expansion of an uncountable subset $\{z_x : x \in I\}$ of the set D, so X is not star countable by Lemma 1.4. This contradiction shows that $\operatorname{ext}(X) \leq \omega$. □

Example 1.23.

There exists a star Lindelöf subspace $X \subset (\omega_1 + 1) \times (\omega + 1)$ such that $\text{ext}(X) = \omega_1$ and hence X is not star countable.

Proof.   Consider the subspace $X = \big((\omega_1 + 1) \times \omega\big) \cup (S \times \{\omega\})$ where S is the set of all isolated points of ω_1. Since $(\omega_1 + 1) \times \omega$ is a dense σ-compact subspace of X, the space X is even star σ-compact. The set $S \times \{\omega\}$ is closed, discrete and uncountable, so $\text{ext}(X) = \omega_1$.   □

The next group of results is aimed to show that big powers of the real line are not star countable. We also define a property of open expansions of discrete sets which seems to be interesting in itself.

Definition 1.24.

Given a space X and a set $A \subset X$, say that an open expansion $\mathcal{U} = \{U_a : a \in A\}$ of A is separable if there exists a countable set $E \subset X$ such that $E \cap U_a \neq \emptyset$ for any $a \in A$.

Proposition 1.25.

If X is a star countable space and D is a closed discrete subspace of X then all open expansions of D are separable.

Proof.   If $\mathcal{U} = \{U_d : d \in D\}$ is an open expansion of D then we can find a set $V_d \in \tau(d, X)$ such that $V_d \subset U_d$ and $V_d \cap D = \{d\}$ for every $d \in D$. The family $\mathcal{V} = \{X \setminus D\} \cup \{V_d : d \in D\}$ is an open cover of X, so there exists a countable set $A \subset X$ such that $\text{St}(A, \mathcal{V}) = X$. Clearly, A witnesses that the expansion $\mathcal{U}$ of D is separable.   □

Proposition 1.26.

Suppose that X is a space and some $Y \subset X$ is star countable. If, additionally, there exists a set $D \subset X$ such that $X = Y \cup D$ and all open expansions of D in X are separable, then the space X is also star countable.

Proof.   Take any open cover $\mathcal{U}$ of the space X; there exists a countable set $A \subset Y$ such that $Y \subset \text{St}(A, \mathcal{U})$. For each $d \in D$ choose a set $U_d \in \mathcal{U}$ such that $d \in U_d$; then $\mathcal{V} = \{U_d : d \in D\}$ is an open expansion of D. Take a countable set $B \subset X$ such that $B \cap U_d \neq \emptyset$ for every $d \in D$. It is evident that the set $C = A \cup B$ is countable and $\text{St}(C, \mathcal{U}) = X$, so X is star countable.   □

Proposition 1.27.

Suppose that X is a space and all open expansions of a set $A \subset X$ are separable. If Y is ω-dense in X in the sense that $X = \bigcup \big\{\overline{B} : B \in [Y]^{\leq \omega}\big\}$, then all open expansions of A in $Z = Y \cup A$ are also separable.

Proof.   Let $\mathcal{V} = \{V_a : a \in A\}$ be an open expansion of A in Z. Pick a set $U_a \in \tau(X)$ such that $U_a \cap Z = V_a$ for all $a \in A$. The family $\mathcal{U} = \{U_a : a \in A\}$ is an open expansion of A in X, so we can find a countable set $P \subset X$ such that $P \cap U_a \neq \emptyset$ for every $a \in A$. For every $x \in P$ fix a countable set $Q_x \subset Y$ with $x \in \overline{Q}_x$. The set $Q = \bigcup \{Q_x : x \in P\}$ is countable and it is easy to see that $Q \cap V_a \neq \emptyset$ for any $a \in A$, i.e., the expansion $\mathcal{V}$ is separable.   □

Matveev proved in [13] that there exist star countable pseudocompact spaces of arbitrary extent. Using his idea we will also construct a star countable space X of arbitrarily large extent, but our space X will be the union of a σ-compact subspace and a closed discrete set; besides, X is a dense subset of a Cantor cube. We will show that our space is not realcompact either, so the question of the referee of [13] remains open.

The following fact can be extracted from the proof of Theorem 1 of the paper [13]. For the reader's convenience we reproduce its proof here.

Proposition 1.28.

Suppose that κ is an uncountable cardinal and define a point $d_\alpha \in \mathbb{D}^\kappa$ by the equalities $d_\alpha(\alpha) = 1$ and $d_\alpha(\beta) = 0$ for all $\beta \neq \alpha$. Then every open expansion of the set $D = \{d_\alpha : \alpha < \kappa\}$ in $\mathbb{D}^\kappa$ is separable.

Proof. Take any open expansion $\mathcal{U} = \{U_\alpha : \alpha < \kappa\}$ of the set D. For every $\alpha < \kappa$ there exists a finite set $Q_\alpha \subset \kappa \backslash \{\alpha\}$ such that the set $V_\alpha = \{x \in \mathbb{D}^\kappa : x(\alpha) = 1 \text{ and } x(\beta) = 0 \text{ for all } \beta \in Q_\alpha\}$ is contained in U_α.

Define a map $\varphi : \kappa \to [\kappa]^{<\omega}$ by the equality $\varphi(\alpha) = Q_\alpha$ for each ordinal $\alpha < \kappa$. Since $\alpha \notin \varphi(\alpha)$ for every $\alpha < \kappa$, we can apply a theorem of Fodor [17, Theorem 3.1.5] to conclude that there exists a family $\{K_n : n \in \omega\}$ of subsets of κ such that $\kappa = \bigcup\{K_n : n \in \omega\}$ and every K_n is φ-free, i.e., $K_n \cap \bigcup\{\varphi(x) : x \in K_n\} = \emptyset$.

Given $n \in \omega$ let $x_n(\alpha) = 1$ for all $\alpha \in K_n$; if $\alpha \notin K_n$ then $x_n(\alpha) = 0$. The set $A = \{x_n : n \in \omega\} \subset \mathbb{D}^\kappa$ is countable. Given any $\alpha < \kappa$ there exists $n \in \omega$ such that $\alpha \in K_n$ and hence $x_n(\alpha) = 1$. Besides, $Q_\alpha = \varphi(\alpha)$ does not meet K_n and hence $x_n(\beta) = 0$ for all $\beta \in Q_\alpha$. Therefore $x_n \in V_\alpha \subset U_\alpha$, i.e., we proved that $A \cap U_\alpha \neq \emptyset$ for every $\alpha < \kappa$, i.e., the expansion $\mathcal{U}$ is separable. $\qquad\square$

Theorem 1.29.

For every uncountable cardinal κ there exists a star countable space X with the following properties:

(a) *X is a dense subspace of $\mathbb{D}^\kappa$;*

(b) *X has a dense σ-compact subspace Y such that $D = X \setminus Y$ is closed and discrete in X, and $|D| = \kappa$.*

Proof. For every $\alpha < \kappa$ define a point $d_\alpha \in \mathbb{D}^\kappa$ by the equalities $d_\alpha(\alpha) = 1$ and $d_\alpha(\beta) = 0$ for all $\beta \neq \alpha$. The set $K_n = \{x \in \mathbb{D}^\kappa : x(n) = 1\}$ is easily seen to be clopen in $\mathbb{D}^\kappa$ and hence compact. We claim that the set $Y = \bigcup\{K_n : n \in \omega\}$ is ω-dense in $\mathbb{D}^\kappa$. Indeed, if $x \in \mathbb{D}^\kappa$ and $n \in \omega$ then let $x_n(n) = 1$ and $x_n(\alpha) = x(\alpha)$ for any $\alpha \neq n$. It is immediate that $\{x_n : n \in \omega\} \subset Y$ and $x_n \to x$, so x is in the ω-closure of the set Y.

Every open expansion of the set $D' = \{d_\alpha : \alpha < \kappa\}$ is separable in the space $\mathbb{D}^\kappa$ by Proposition 1.28. Therefore every open expansion of the set D' in the space $X = Y \cup D'$ is also separable by Proposition 1.27. Observe that the closure of the set D' is the one-point compactification of a discrete set which converges to the zero function in $\mathbb{D}^\kappa$. Therefore the set D' is closed and discrete in X and hence the set $D = X \setminus Y = \{d_\alpha : \omega \leq \alpha < \kappa\}$ is also closed, discrete and has $|D| = \kappa$.

Finally note that Y is σ-compact and hence star countable; since all open expansions of the set D' are separable in X, we can apply Proposition 1.26 to see that X is star countable. $\qquad\square$

Observation 1.30.

The space X constructed in Theorem 1.29 is not realcompact. Indeed, let u be the zero function of $\mathbb{D}^\kappa$; then $D \cup \{u\}$ is a one-point compactification of an uncountable discrete space D. If $f : X \to \mathbb{R}$ is a continuous function then it depends on countably many coordinates, i.e., there exists a countable set $A \subset \kappa$ such that $f = g \circ (\pi_A \restriction X)$, where $\pi_A : \mathbb{D}^\kappa \to \mathbb{D}^A$ is the natural projection and $g : \pi_A(X) \to \mathbb{R}$ is a continuous function. Take any $\alpha \in \kappa \setminus (\omega \cup A)$; then $d_\alpha \in D$ and $\pi_A(d_\alpha) = \pi_A(u)$. This shows that $\pi_A(u) \in \pi_A(D) \subset \pi_A(X)$ and hence $g \circ \pi_A$ is a continuous extension of f to $X \cup \{u\}$. Thus, X is not realcompact being C-embedded in $X \cup \{u\}$.

Observation 1.31.

The referee noted that we can add a Σ-product to our space X from Theorem 1.29 to obtain a pseudocompact star countable space which is still dense in a Cantor cube but has large extent.

Proposition 1.28 shows that there are discrete subspaces of $\mathbb{D}^\kappa$ all of whose open expansions are separable. However, this is not true for all discrete subspaces of $\mathbb{D}^\kappa$ as the following example shows.

Theorem 1.32.

For any cardinal $\kappa > \mathfrak{c}$ there exists a discrete subspace E of the space $\{-1, 0, 1\}^\kappa \subset \mathbb{R}^\kappa$ with an open expansion which is not separable in $\mathbb{R}^\kappa$.

Proof. Consider the discrete space $D_\alpha = \{-1, 0, 1\}$ for any $\alpha < \kappa$; it is clear that the space $D = \prod\{D_\alpha : \alpha < \kappa\}$ is homeomorphic to $\mathbb{D}^\kappa$. Given any pair $(\alpha, \beta) \in \kappa \times \kappa$ with $\alpha < \beta$ consider the point $x_{\alpha,\beta} \in D$ defined by the equalities $x_{\alpha,\beta}(\alpha) = -1$, $x_{\alpha,\beta}(\beta) = 1$, and $x_{\alpha,\beta}(\gamma) = 0$ for any $\gamma \in \kappa \setminus \{\alpha, \beta\}$.

To see that $E = \{x_{\alpha,\beta} : \alpha < \beta < \kappa\}$ is a discrete subspace of D, consider the set $V_{\alpha,\beta} = \{x \in \mathbb{R}^{\kappa} : x(\alpha) < 0 \text{ and } x(\beta) > 0\}$ for all $\alpha, \beta < \kappa$ with $\alpha < \beta$. It is evident that the family $\mathcal{U} = \{V_{\alpha,\beta} : \alpha < \beta < \kappa\}$ is an open expansion of the set E in $\mathbb{R}^{\kappa}$ and $V_{\alpha,\beta} \cap E = \{x_{\alpha,\beta}\}$ whenever $\alpha < \beta < \kappa$; this shows that E is a discrete subspace of D.

The family $\mathcal{U}$ is not separable even in $\mathbb{R}^{\kappa}$; to see it, take an arbitrary countable set $A \subset \mathbb{R}^{\kappa}$ and consider the family $\mathcal{A} = \{x^{-1}((p,q)) : x \in A \text{ and } p, q \in \mathbb{Q}\}$. The family $\mathcal{A}$ being countable cannot T_0-separate the points of $\kappa > \mathfrak{c}$. Indeed, let $\mathcal{A}' = \mathcal{A} \cup \{\kappa \setminus B : B \in \mathcal{A}\}$ and assume that the family $\mathcal{A}$ is T_0-separating on κ. Then the map $\alpha \to \{B \in \mathcal{A}' : \alpha \in B\}$ is an injection of κ into the power set $\exp(\mathcal{A}')$ of $\mathcal{A}'$. Since $|\exp(\mathcal{A}')| \leq \mathfrak{c}$, this is a contradiction.

Therefore we can find ordinals $\alpha, \beta \in \kappa$ such that $\alpha < \beta$ and either $\{\alpha, \beta\} \subset B$ or $\{\alpha, \beta\} \cap B = \emptyset$ for every $B \in \mathcal{A}$. If $x \in A$ and $x(\alpha) \neq x(\beta)$ then there exist $p, q \in \mathbb{Q}$ such that $p < q$ and $x(\alpha) \in (p,q)$ while $x(\beta) \notin (p,q)$. The set $B = x^{-1}((p,q))$ belongs to the family $\mathcal{A}$ and separates the points α and β which is a contradiction. Therefore $x(\alpha) = x(\beta)$ and hence $x \notin V_{\alpha,\beta}$ for any $x \in A$; this shows that $\mathcal{U}$ is not separable. $\qquad\square$

Theorem 1.33.
Suppose that X is a Tychonoff space and $D \subset X$ is a discrete C^*-embedded subset of X such that $|D| > \mathfrak{c}$. Then D has a non-separable open expansion in X.

Proof. Let $\kappa = |D|$; it follows from Theorem 1.32 that we can find a discrete subspace $E \subset \mathbb{I}^{\kappa}$ such that $|E| = \kappa$ and there exists a non-separable open expansion $\mathcal{V} = \{V_y : y \in E\}$ of the set E in $\mathbb{I}^{\kappa}$. Let $f : D \to E$ be a bijection; the map f is continuous because D is discrete. The set D being C^*-embedded in X, there exists a continuous map $g : X \to \mathbb{I}^{\kappa}$ such that $g \restriction D = f$. For any $x \in D$ let $U_x = g^{-1}(V_{f(x)})$; it is evident that $\{U_x : x \in D\}$ is a non-separable open expansion of D. $\qquad\square$

Corollary 1.34.
For any $\kappa \geq 2^{\mathfrak{c}^+}$ the space $\mathbb{R}^{\kappa}$ is not star countable.

Proof. Let $\mu = 2^{\mathfrak{c}^+}$; if $\kappa \geq \mu$ then $\mathbb{R}^{\mu}$ is a continuous image of $\mathbb{R}^{\kappa}$, so it suffices to show that $\mathbb{R}^{\mu}$ is not star countable. Take a discrete space D with $|D| = \mathfrak{c}^+$ and let $A = \mathbb{R}^D$; it is clear that $|A| = \mu$. Denote by φ the diagonal product of A; then $\varphi : D \to \mathbb{R}^{\mu}$. It is standard that $\varphi : D \to E = \varphi(D)$ is a homeomorphism and E is C-embedded in $\mathbb{R}^{\mu}$. The cardinal μ is not measurable so E is a realcompact space; as an immediate consequence, E is closed in $\mathbb{R}^{\mu}$. Applying Theorem 1.33, we can see that E has a non-separable open expansion in $\mathbb{R}^{\mu}$, so $\mathbb{R}^{\mu}$ is not star countable by Proposition 1.25. $\qquad\square$

Quite a few results of this paper were motivated by a problem of Arkhangel'skii [4, Question 2.2.4] which asks whether a star countable $C_p(X)$ must be Lindelöf whenever X is compact. We will show that this question has a positive answer in the class of ω_1-monolithic compact spaces.

It is known that if X is a dyadic compact space and $C_p(X)$ is Lindelöf, then X is metrizable [3, Corollary IV.11.8]. We will prove that star countability of $C_p(X)$ is sufficient for such an X to be metrizable. An analogous consistent result will be proved for a linearly ordered compact space X.

The three theorems that follow make use of a result of Dow, Junnila and Pelant [6, Theorem 1.2] which states that, for any compact space K with $w(K) \leq \omega_1$, the space $C_p(K)$ is hereditarily metalindelöf.

Theorem 1.35.
Suppose that X is an ω_1-monolithic compact space and $C_p(X)$ is star countable. Then $C_p(X)$ is Lindelöf.

Proof. Baturov proved [3, Theorem III.6.1] that $\mathrm{ext}(C_p(K)) = l(C_p(K))$ for any compact space K, so it suffices to establish that $\mathrm{ext}(C_p(X)) \leq \omega$. Striving for a contradiction assume that D is a closed discrete subset of $C_p(X)$ and $|D| = \omega_1$.

For every $f \in D$ there exists a finite set $Q_f \subset X$ and $\varepsilon_f > 0$ such that, for the set $U_f = \{g \in C_p(X) : |g(x) - f(x)| < \varepsilon_f$ for every $x \in Q_f\}$, we have $U_f \cap D = \{f\}$. By ω_1-monolithity of X the set $Y = \overline{\bigcup\{Q_f : f \in D\}}$ has weight not exceeding

ω_1 and hence $\text{nw}\,(C_p(Y)) = \text{nw}\,(Y) \leq \omega_1$. The restriction map $\pi\colon C_p(X) \to C_p(Y)$ is continuous and surjective. It follows from our choice of Y that $\pi{\upharpoonright}D$ is an injective map and $E = \pi(D)$ is a discrete (but maybe not closed) subset of $C_p(Y)$.

Applying a theorem of Dow, Junnila and Pelant [6, Theorem 1.2] we can see that the space $C_p(Y)$ is hereditarily metalindelöf, so the set E has a point-countable open expansion $\{W_h : h \in E\}$ in $C_p(Y)$. Then $\{\pi^{-1}(W_{\pi(f)}) : f \in D\}$ is a point-countable open expansion of D in $C_p(X)$ which, together with Lemma 1.4 gives us a contradiction. $\qquad\square$

Corollary 1.36.
If $C_p(X)$ is star countable and X is an ω-monolithic compact space with $t(X) \leq \omega$, then $C_p(X)$ is Lindelöf.

Proof. It is an easy exercise to show that any ω-monolithic space of countable tightness is ω_1-monolithic, so Theorem 1.35 applies. $\qquad\square$

Theorem 1.37.
If X is a dyadic compact space and $C_p(X)$ is star Lindelöf, then X is metrizable.

Proof. Suppose that X is a dyadic compact space and $C_p(X)$ is star Lindelöf. Every Lindelöf subspace of $C_p(X)$ is cosmic by [3, Corollary IV.11.8], so $C_p(X)$ is star cosmic and hence star countable.

If the space X is not metrizable, then some $Y \subset X$ is homeomorphic to the Cantor cube $\mathbb{D}^{\omega_1}$ [2, Theorem 3.1.6]. Let $\pi_Y\colon C_p(X) \to C_p(Y)$ be the restriction map, i.e., $\pi_Y(f) = f{\upharpoonright}Y$ for each $f \in C_p(X)$. Since star countability is preserved in continuous images, the space $C_p(Y)$ is also star countable. Besides, the space $C_p(Y)$ is metalindelöf by [6, Theorem 1.2], so it has to be Lindelöf by Corollary 1.5. Therefore, $\omega_1 = t(\mathbb{D}^{\omega_1}) = t(Y) \leq l(C_p(Y)) = \omega$ which is a contradiction. $\qquad\square$

Theorem 1.38.
Under CH, if L is a linearly ordered compact space such that $C_p(L)$ is star countable then L is metrizable.

Proof. If $t(L) > \omega$ then there exists a subspace $Y \subset L$ which is homeomorphic to $\omega_1 + 1$. Let $\pi_Y\colon C_p(X) \to C_p(Y)$ be the restriction map; since star countability is preserved by continuous images, the space $C_p(Y)$ is also star countable. Besides, the space $C_p(Y)$ is metalindelöf by [6, Theorem 1.2], so it must be Lindelöf by Corollary 1.5. Therefore $\omega_1 = t(\omega_1 + 1) = t(Y) \leq l(C_p(Y)) = \omega$, a contradiction. Thus, $\chi(L) = t(L) = \omega$ so we have $w(L) \leq |L| \leq \mathfrak{c} = \omega_1$. Applying [6, Theorem 1.2] once more, we conclude that the space $C_p(L)$ is metalindelöf and hence Lindelöf by Corollary 1.5. Therefore L is metrizable by [3, Theorem IV.10.1]. $\qquad\square$

2.   Open questions

The first results on star countable and star Lindelöf spaces were published in the 1980's. However, this topic is still a challenge for a researcher as can be seen from the following list of open questions.

Question 2.1.
Suppose that X is a monotonically monolithic [16] star countable space. Must X be Lindelöf?

Question 2.2.
Suppose that X is strongly monotonically monolithic [16] star countable space. Must X be Lindelöf?

Question 2.3.
Is $\mathbb{R}^{\mathfrak{c}^+}$ is star countable?

Question 2.4.

Suppose that X is a space and some discrete subspace $D \subset X$ is C^*-embedded in X, and $|D| = \mathfrak{c}$. Is it true that D must have a non-separable open expansion?

Question 2.5.

Does there exist an uncountable regular hereditarily Lindelöf space X such that every countable set $A \subset X$ is closed in X?

Question 2.6.

Suppose that K is a linearly ordered compact space such that $C_p(K)$ is star countable. Is it true in ZFC that K is metrizable?

Question 2.7.

Suppose that X is a compact space and $C_p(X)$ is star countable. Is it true that $t(X) \le \omega$?

Acknowledgements

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