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Pattern avoiding partitions and Motzkin left factors

Research Article

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Abstract: Let L_n, $n \geq 1$, denote the sequence which counts the number of paths from the origin to the line $x = n-1$ using $(1,1)$, $(1,-1)$, and $(1,0)$ steps that never dip below the x-axis (called *Motzkin left factors*). The numbers L_n count, among other things, certain restricted subsets of permutations and Catalan paths. In this paper, we provide new combinatorial interpretations for these numbers in terms of finite set partitions. In particular, we identify four classes of the partitions of size n, all of which have cardinality L_n and each avoiding a set of two classical patterns of length four. We obtain a further generalization in one of the cases by considering a pair of statistics on the partition class. In a couple of cases, to show the result, we make use of the kernel method to solve a functional equation arising after a certain parameter has been introduced.

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1. Introduction

Let $\mathcal{M}_n$ be the set of all paths from $(0,0)$ to $(n,0)$ using $(1,1)$, $(1,-1)$ and $(1,0)$ steps, which we will denote by u, d, and ℓ, respectively, with no steps lying below the x-axis (called *Motzkin paths*). The cardinality of $\mathcal{M}_n$ defines the *Motzkin number* M_n, which has been widely studied (see, e.g., [16, A001006] and references therein). Let $\mathcal{L}_n$ denote the set of all paths of length n using u, d and ℓ steps starting from the origin and not dipping below the x-axis. Such paths are called *Motzkin left factors*; see, e.g., [1, p. 111] or [9, p. 9]. Let $L_n = |\mathcal{L}_{n-1}|$ if $n \geq 1$, with $L_0 = 1$. The L_n are also given by the generating function

$$\sum_{n\geq 0} L_n x^n = \frac{1-3x+\sqrt{1-2x-3x^2}}{2(1-3x)} \tag{1}$$

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and satisfy the relation

$$L_{n+1} = M_n + \sum_{k=0}^{n-1} M_k L_{n-k}, \qquad n \geq 1, \tag{2}$$

with $L_0 = L_1 = 1$.

Among the other lattice path interpretations of the numbers L_n is the fact that they count the symmetric Catalan paths of semilength $2n-1$ with no peaks at even level as well as the Catalan paths of semilength n with no occurrence of $duuu$. The L_n also enumerate a variety of other structures, ranging from the set of directed animals [6] of size n to the permutations of $\{1,2,\ldots,n\}$ simultaneously avoiding 321 and the barred pattern $4\bar{1}523$, see [3], to the set of base 3 n-digit numbers whose digit sum is also n. See [16, A005773] for further information on these numbers. Here, we provide new combinatorial interpretations for the L_n in terms of finite set partitions, showing, in particular, that they enumerate certain two-pattern avoidance classes.

If $n \geq 1$, then a *partition* of $[n] = \{1,2,\ldots,n\}$ is any collection of non-empty, pairwise disjoint subsets, called *blocks*, whose union is $[n]$. (If $n = 0$, then there is a single empty partition which has no blocks.) Throughout, we will use the term *partition* when referring to a partition of a *set*. A partition Π having exactly k blocks is also called a k-*partition* and will be denoted by $\Pi = B_1/B_2/\cdots/B_k$, where the blocks are arranged in ascending order according to the size of the smallest elements. We will denote the set of k-partitions of $[n]$ by $P_{n,k}$ and the set of all partitions of $[n]$ by P_n. One can represent the partition $\Pi = B_1/B_2/\cdots/B_k \in P_{n,k}$, equivalently, by the *canonical sequential form* $\pi = \pi_1\pi_2\cdots\pi_n$, wherein $j \in B_{\pi_j}$, $1 \leq j \leq n$, and in such case we will write $\Pi = \pi$. For example, the partition $\Pi = 1,5,7/2,3/4,8/6 \in P_{8,4}$ has the canonical sequential form $\pi = 12231413$. Note that $\pi = \pi_1\pi_2\cdots\pi_n \in P_{n,k}$ is a *restricted growth function* from $[n]$ to $[k]$ (see, e.g., [12] for details), meaning that it satisfies the following three properties: (i) $\pi_1 = 1$, (ii) π is onto $[k]$, and (iii) $\pi_{i+1} \leq \max\{\pi_1,\pi_2,\ldots,\pi_i\} + 1$ for all i, $1 \leq i \leq n-1$. In what follows, we will represent set partitions as words using their canonical sequential forms and consider some particular cases of the general problem of counting the members of a partition class having various restrictions imposed on the order of the letters.

A *classical pattern* τ is a member of $[\ell]^m$ which contains all of the letters in $[\ell]$. We say that a word $\sigma \in [k]^n$ *contains* the classical pattern τ if σ contains a subsequence isomorphic to τ. Otherwise, we say that σ *avoids* τ. For example, a word $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ avoids the pattern 132 if it has no subsequence $\sigma_i\sigma_j\sigma_k$ with $i<j<k$ and $\sigma_i < \sigma_k < \sigma_j$ and avoids the pattern 1212 if it has no subsequence $\sigma_i\sigma_j\sigma_k\sigma_\ell$ with $\sigma_i = \sigma_k < \sigma_j = \sigma_\ell$. The pattern avoidance question has been the topic of many papers in enumerative combinatorics, starting with Knuth [11] and Simion and Schmidt [15] on permutations and considered, more recently, on words, compositions, and finite set partitions. For the avoidance problem on partitions, we refer the reader to the papers by Klazar [10], Sagan [13], and Jelínek and Mansour [8] and to the references therein.

We will use the following notation. If $\{w_1,w_2,\ldots\}$ is a set of classical patterns, then let $P_n(w_1,w_2,\ldots)$ and $P_{n,k}(w_1,w_2,\ldots)$ denote, respectively, the subsets of P_n and $P_{n,k}$ which avoid all of the patterns. We will denote the cardinalities of $P_n(w_1,w_2,\ldots)$ and $P_{n,k}(w_1,w_2,\ldots)$ by $p_n(w_1,w_2,\ldots)$ and $p_{n,k}(w_1,w_2,\ldots)$, respectively.

In this paper, we identify four classes of partitions each avoiding a pair of classical patterns of length four and each enumerated by the number L_n. In addition to providing new interpretations for the numbers L_n, this addresses specific cases of a general question raised by Goyt in the final section of [7] concerning the avoidance by set partitions of two or more patterns of length four. Our main result is the following theorem which we prove in the next section as a series of propositions.

Theorem 1.1.

If $n \geq 0$, then $p_n(u,v) = L_n$ for the following sets (u,v):

$$(i)\quad (1222,1212), \qquad (ii)\quad (1112,1212), \qquad (iii)\quad (1211,1221), \qquad (iv)\quad (1222,1221).$$

We remark that, in the first two cases, our proofs are more or less combinatorial, while, in the last two, they are algebraic and involve applications of the kernel method [2] to solve the functional equations that arise once a certain parameter has been introduced. In addition, we prove a refinement of Theorem 1.1 as well as obtain p,q-generalizations of the numbers M_n and L_n by considering pairs of statistics on the sets $P_n(111,1212)$ and $P_n(1222,1212)$, respectively.

2. Proof of the main result

Theorem 1.1 will follow from combining the propositions in the sections below. We first consider the patterns $\{1222, 1212\}$ and $\{1112, 1212\}$.

2.1. The cases $\{1222, 1212\}$ and $\{1112, 1212\}$

Throughout, we will denote the sets $P_n(1222, 1212)$ and $P_n(1112, 1212)$ by $\mathcal{A}_n$ and $\mathcal{B}_n$, respectively. Let $\mathcal{R}_n$ denote the set $P_n(111, 1212)$.

Proposition 2.1.
If $n \geq 0$, then $p_n(1222, 1212) = L_n$.

Proof. We will define an explicit bijection between $\mathcal{A}_n$ and $\mathcal{L}_{n-1}$ for all $n \geq 1$. First observe that any member $\pi \in \mathcal{A}_n$ can be expressed as $\pi = 1\pi_1 1\pi_2 \cdots 1\pi_r$ for some $r \geq 1$, where π_i does not contain 1 and is such that $\operatorname{stan}\pi_i$ belongs to $P_{n_i}(111, 1212)$ for some $n_i \geq 0$ for all i (by $\operatorname{stan}\pi_i$ we mean the equivalent partition on the letters $\{1, 2, \dots\}$, called the *standardization*, obtained by replacing the j-th smallest letter of π_i with j). Furthermore, note that it must be the case that every letter of π_j is larger than every letter of π_i if $j > i$ in order to avoid 1212.

We now define, in a recursive fashion, a bijection between $\mathcal{R}_m$ and $\mathcal{M}_m$ for all $m \geq 0$, where $f_0(\varnothing) = \varnothing$ and $f_1(1) = \ell$. If $m \geq 2$ and $\lambda \in P_m(111, 1212)$, then either

$$(i) \quad \lambda = 1\lambda' \qquad \text{or} \qquad (ii) \quad \lambda = 1\lambda' 1\lambda'',$$

where 1 does not belong to λ' or λ'' and all of the letters of λ'' are larger than all of those in λ', with λ' and λ'' both avoiding the patterns 111 and 1212. If $m \geq 2$, we define f_m, recursively, by setting $f_m(\lambda) = \ell f_{m-1}(\lambda')$ in the first case and by setting $f_m(\lambda) = u f_{m_1}(\lambda') d f_{m_2}(\lambda'')$ in the second, where m_1 and m_2 denote the respective lengths of λ' and λ''. The bijection is reversed upon considering whether or not a path in $\mathcal{M}_m$ starts with ℓ or u, and in the latter case, considering the position of the first d returning to the x-axis. In what follows, we will write f to denote f_m, suppressing the subscript whenever the cardinality of the underlying structure is understood.

We now define a bijection g between $\mathcal{A}_n$ and $\mathcal{L}_{n-1}$, which will give the result. If $\pi = 1\pi_1 1\pi_2 \cdots 1\pi_r$ is as above, then let

$$g(\pi) = f(\pi_1)\, u f(\pi_2) \,\cdots\, u f(\pi_r).$$

To reverse g, suppose $\alpha \in \mathcal{L}_{n-1}$ terminates at the point $(n-1, r-1)$ for some $r \geq 1$. Given $0 \leq i \leq r-1$, let s_i denote the rightmost step of α which either lies along the line $y = i$ as an ℓ or touches it from above as a d or touches it from below as a u (in the case when $i = 0$, only the first two conditions would apply). Decompose α as $\alpha = \alpha_0\alpha_1 \cdots \alpha_{r-1}$, where α_0 counts all steps of α up to and including s_0 and α_i, $1 \leq i \leq r-1$, is the sequence of steps starting with the u directly following step s_{i-1} and ending at step s_i. Note that $\alpha_i = u\alpha_i'$ if $i \geq 1$, with α_i' and α_0 possibly empty Motzkin paths. Then define $g^{-1}(\alpha)$ by

$$g^{-1}(\alpha) = 1f^{-1}(\alpha_0)\, 1f^{-1}(\alpha_1) \,\cdots\, 1f^{-1}(\alpha_{r-1}). \qquad \square$$

Figure 1 below illustrates the path $g(\pi)$ corresponding to $\pi = 12334215511617898 \in A_{17}$.

One can give a full bijection between $\mathcal{R}_m$ and $\mathcal{M}_m$ as follows. First recall the equivalence between Catalan paths of semilength m and perfect matchings of $[2m]$ that avoid the pattern 1212 (called *non-crossing matchings*, see, e.g., [8]) obtained by drawing horizontal lines to the right of each up step in a Catalan path, noting the position of the first down step encountered, and partitioning the steps into the m position pairs. The resulting perfect matching on $[2m]$ avoids 1212 and, conversely, starting with such a matching, one can construct a Catalan path of semilength m whose paired up and down steps correspond to the blocks of the matching.

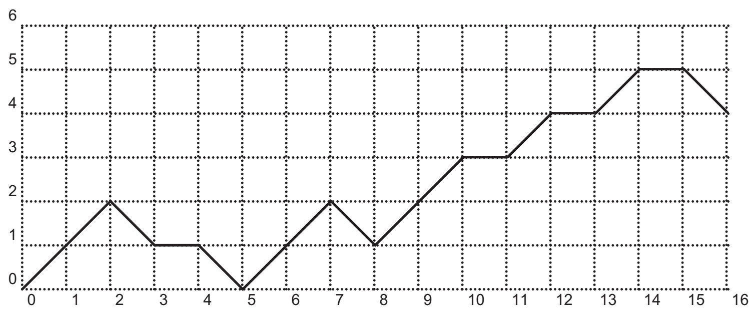

Figure 1. The Motzkin left factor $g(\pi) \in \mathcal{L}_{16}$.

Now suppose $\pi \in \mathcal{R}_m$ and let $S \subset [m]$ comprise the set of singletons of π with $k = |S|$. Let π' denote the standardization of the partition $\pi \cap ([m]{-}S)$. Since π' is a perfect matching which avoids 1212, one may construct a Catalan path $p(\pi')$ of semilength $(m-k)/2$ as described above. Then insert level steps ℓ into $p(\pi')$ such that their positions correspond to the elements of S to yield a Motzkin path of length m. This process is seen to be reversible.

We now consider avoidance of the patterns 1112 and 1212.

Proposition 2.2.

The generating function for the number of partitions of $[n]$*,* $n \geq 0$*, that avoid the patterns* 1112 *and* 1212 *is given by*

$$\frac{1-3x+\sqrt{1-2x-3x^2}}{2(1-3x)}.$$

Proof. If $n \geq 3$ and $\pi \in \mathcal{B}_n$, then we can decompose π as either

$$(i) \quad \pi = 1\alpha\underbrace{11\cdots1}_{s \text{ times}} \qquad \text{or} \qquad (ii) \quad \pi = 1\beta1\gamma\underbrace{11\cdots1}_{s \text{ times}},$$

where $s \geq 0$, α is a possibly empty partition on the letters $\{2, 3, \ldots\}$ avoiding the patterns 1112 and 1212, β is a possibly empty partition on $\{2, 3, \ldots, i\}$ for some i avoiding 111 and 1212, and γ is a non-empty partition on the letters $\{i+1, i+2, \ldots\}$ avoiding 1112 and 1212. Note that β must avoid 111 since γ is assumed non-empty. Let $M(x)$ denote the generating function for the Motzkin numbers M_n, i.e., $M(x) = \sum_{n\geq0} M_n x^n$. If $h(x) = \sum_{n\geq0} |\mathcal{B}_n| x^n$, then we see from the prior proof and the above decompositions that it must satisfy

$$h(x) = 1 + \frac{x}{1-x}h(x) + \frac{x^2}{1-x}M(x)(h(x)-1),$$

i.e.,

$$h(x) = \frac{1-x-x^2M(x)}{1-2x-x^2M(x)}.$$

Upon simplifying, the required result now follows from the last equation and the fact that

$$M(x) = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}. \qquad \square$$

Remark.

It is possible to construct a bijection between $\mathcal{B}_n$ and $\mathcal{L}_{n-1}$. Note that the members of $\mathcal{B}_n$ of the form in the first decomposition above, i.e., in the case (i), correspond to paths in $\mathcal{L}_{n-1}$ starting with $\ell^s u$ and not returning to the x-axis when α is non-empty and to the path ℓ^{n-1} when α is empty. Members of $\mathcal{B}_n$ of the form in (ii) correspond to paths in $\mathcal{L}_{n-1}$ of the form $\ell^s u\lambda' d\lambda''$, where λ' and λ'' denote a Motzkin path and a Motzkin left factor, respectively.

2.2. The case $\{1211, 1221\}$

Let $\pi = \pi_1\pi_2\cdots\pi_n$ denote a partition of $[n]$, represented canonically. Recall that empty sums take the value zero, by convention. To establish this case, we divide up the set of partitions in question according to a certain statistic, namely, the one which records the length of the maximal increasing initial run. To do so, given $k \geq 1$, let $f_k(x)$ denote the generating function for the number of partitions $\pi = \pi_1\pi_2\cdots\pi_n \in P_n(1211, 1221)$, where $n \geq k$, such that $\pi_1\pi_2\cdots\pi_k = 12\cdots k$ with $\pi_{k+1} \leq k$ (if there is a $(k+1)$-st letter). We have the following relation involving the generating functions $f_k(x)$.

Lemma 2.3.

If $k \geq 1$, then

$$f_k(x) = x^k + x^k\overline{f}_1(x) + \sum_{j=1}^{k-1} x^{j+1}\overline{f}_{k-j}(x), \tag{3}$$

with initial condition $f_0(x) = 1$, where $\overline{f}_k(x) = \sum_{i\geq k} f_i(x)$.

Proof. Note that $f_1(x) = x + x\overline{f}_1(x)$, since a partition in this case can just have one letter or start with 11. Thus (3) holds in the case $k = 1$ since empty sums take value zero, by convention. If $k \geq 2$, then a partition π enumerated by $f_k(x)$ must be of one of the following three forms:

$$(i)\quad 12\cdots k, \qquad (ii)\quad 12\cdots kj\pi',\ \ 1 \leq j \leq k-1, \qquad (iii)\quad 12\cdots kk\pi''.$$

The first case contributes x^k. Note that in the second case, the word π' contains no letters in $[j]$, for otherwise if it contained a letter in $[j-1]$, then there would be an occurrence of 1221 (with $ijji$ for some $i < j$) and if it contained the letter j, then there would be an occurrence of 1211 (with $jkjj$). Thus, the letters $(j+1)(j+2)\cdots k\pi'$, taken together, comprise a partition of the form enumerated by $\overline{f}_{k-j}(x)$. One may then safely delete from π the letters in $[j-1]$ as well as both copies of the letter j since they are seen to be extraneous concerning possible occurrences of 1211 or 1221. Thus, the contribution in this case towards the generating function $f_k(x)$ is $x^{j+1}\overline{f}_{k-j}(x)$. Similar reasoning in the third case yields a contribution of $x^k\overline{f}_1(x)$ since π'' can contain no letters in $[k-1]$, whence the letters in $[k-1]$ as well as the second k are extraneous. Combining the three cases yields (3). ☐

We now prove the third case in Theorem 1.1 above.

Proposition 2.4.

The generating function for the number of partitions of $[n]$, $n \geq 0$, that avoid the patterns 1211 and 1221 is given by

$$\frac{1 - 3x + \sqrt{1 - 2x - 3x^2}}{2(1-3x)}.$$

Proof. Define the generating function $f(x, y) = \sum_{k\geq 0} f_k(x)y^k$. Multiplying (3) by y^k and summing over $k \geq 1$ yields

$$\begin{aligned}
f(x,y) &= 1 + \frac{xy}{1-xy} + \frac{xy}{1-xy}\overline{f}_1(x) + \sum_{k\geq 2}\left(\sum_{j=1}^{k-1} x^{j+1}\overline{f}_{k-j}(x)\right)y^k \\
&= 1 + \frac{xy}{1-xy} + \frac{xy}{1-xy}(f(x,1)-1) + \sum_{j\geq 1} x^{j+1}\sum_{k\geq j+1}\overline{f}_{k-j}(x)y^k = 1 + \frac{xy}{1-xy}f(x,1) + \sum_{j\geq 1} x^{j+1}y^j\sum_{k\geq 1}\overline{f}_k(x)y^k \\
&= 1 + \frac{xy}{1-xy}f(x,1) + \frac{x^2y}{1-xy}\sum_{k\geq 1} y^k\sum_{i\geq k} f_i(x) = 1 + \frac{xy}{1-xy}f(x,1) + \frac{x^2y}{1-xy}\sum_{i\geq 1} f_i(x)\sum_{k=1}^{i} y^k \\
&= 1 + \frac{xy}{1-xy}f(x,1) + \frac{x^2y^2}{(1-xy)(1-y)}\sum_{i\geq 1} f_i(x)(1-y^i) = 1 + \frac{xy}{1-xy}f(x,1) + \frac{x^2y^2}{(1-xy)(1-y)}(f(x,1) - f(x,y)),
\end{aligned}$$

which implies

$$\left(1+\frac{x^2y^2}{(1-xy)(1-y)}\right)f(x,y)=1+\left(\frac{xy}{1-xy}+\frac{x^2y^2}{(1-xy)(1-y)}\right)f(x,1). \tag{4}$$

This type of functional equation can be solved systematically using the *kernel method* [2]. In this case, if we assume that $y=y_0$ in (4), where y_0 satisfies

$$1+\frac{x^2y_0^2}{(1-xy_0)(1-y_0)}=0, \qquad \text{i.e.,} \qquad y_0=\frac{1+x-\sqrt{1-2x-3x^2}}{2x(1+x)},$$

then

$$\sum_{n\geq 0}P_n(1211,1221)x^n=f(x,1)=\frac{1}{1-xy_0/(1-xy_0)}=\frac{1-3x+\sqrt{1-2x-3x^2}}{2(1-3x)},$$

as required. (Note that $f(0,1)=1$ dictates our choice of root for y_0.) □

Remark.

Substituting the expression above for $f(x,1)$ into (4) recovers the expression for $f(x,y)$, from which one can compute an explicit formula for the coefficient of x^ny^k.

2.3. The case $\{1222, 1221\}$

We once again divide up the set of partitions in question according to the statistic which records the length of the maximal increasing initial run. If $k\geq 1$, then let $f_k(x)$ (respectively, $g_k(x)$) denote the generating function for the number of partitions π of $[n]$ having at least k letters and avoiding the patterns 1222 (respectively, 111) and 1221 such that $\pi_1\pi_2\cdots\pi_k=12\cdots k$ with $\pi_{k+1}\leq k$ (if there is a $(k+1)$-st letter). We have the following relations involving the generating functions $g_k(x)$ and $f_k(x)$.

Lemma 2.5.

If $k\geq 1$, then

$$g_k(x)=x^k+\sum_{j=1}^{k}x^{j+1}\overline{g}_{k-j}(x), \tag{5}$$

with initial condition $g_0(x)=1$, where $\overline{g}_k(x)=\sum_{i\geq k}g_i(x)$.

Proof. If $k\geq 1$, then a partition π enumerated by $g_k(x)$ must be of one of the following two forms:

$$(i)\quad 12\cdots k, \qquad (ii)\quad 12\cdots kj\pi',\quad 1\leq j\leq k.$$

The first case contributes x^k. Note that in the second case, the word π' contains no letters in $[j]$, for otherwise there would be an occurrence of 1221 if it contained a letter in $[j-1]$ or an occurrence of 111 if it contained the letter j. Thus, the letters $(j+1)(j+2)\cdots k\pi'$, taken together, comprise a partition of the form enumerated by $\overline{g}_{k-j}(x)$, which implies the contribution in this case is $x^{j+1}\overline{g}_{k-j}(x)$. Combining the two cases yields (5). □

Lemma 2.6.

If $k\geq 1$, then

$$f_k(x)=x^k+x\overline{f}_k(x)+\sum_{j=2}^{k}x^{j+1}\overline{g}_{k-j}(x), \tag{6}$$

with initial condition $f_0(x)=1$, where $\overline{f}_k(x)=\sum_{i\geq k}f_i(x)$.

Proof. Note that $f_1(x) = x + x\overline{f}_1(x)$, since a partition in this case can have just one letter or start with 11. If $k \geq 2$, then a partition π enumerated by $f_k(x)$ must be of one of the following three forms:

$$(i)\quad 12\cdots k, \qquad (ii)\quad 12\cdots k1\pi', \qquad (iii)\quad 12\cdots kj\pi'',\quad 2 \leq j \leq k.$$

The second case contributes $x\overline{f}_k(x)$, as we can safely delete the second 1 from π since it is superfluous concerning possible occurrences of 1222 or 1221, with the resulting partition $12\cdots k\pi'$ of the form enumerated by $\overline{f}_k(x)$. Note that in the third case, the word π'' contains no letter in $[j]$, for otherwise there would be an occurrence of 1221 if it contained a letter in $[j-1]$ or an occurrence of 1222 if it contained the letter j. Thus, the letters $(j+1)(j+2)\cdots k\pi''$, taken together, comprise a partition of the form enumerated by $\overline{g}_{k-j}(x)$ since it must avoid $\{111, 1221\}$, which implies the contribution in this case is $x^{j+1}\overline{g}_{k-j}(x)$, upon including the letters in $[j-1]$ as well as the two copies of j. Combining the three cases yields (6). $\square$

The final case of Theorem 1.1 now follows from the two previous lemmas.

Proposition 2.7.

The generating function for the number of partitions of $[n]$, $n \geq 0$, that avoid the patterns 1222 and 1221 is given by

$$\frac{1 - 3x + \sqrt{1 - 2x - 3x^2}}{2(1 - 3x)}.$$

Proof. Define the generating functions $g(x, y) = \sum_{k\geq 0} g_k(x)y^k$ and $f(x, y) = \sum_{k\geq 0} f_k(x)y^k$. Multiplying (5) and (6) by y^k and summing over $k \geq 1$ yields

$$\begin{aligned}
g(x,y) &= \frac{1}{1-xy} + \sum_{k\geq 1} y^k \sum_{j=1}^{k} x^{j+1}\overline{g}_{k-j}(x) = \frac{1}{1-xy} + \frac{x^2y}{1-xy}\sum_{k\geq 0} y^k\overline{g}_k(x)\\
&= \frac{1}{1-xy} + \frac{x^2y}{(1-xy)(1-y)}\left(g(x,1) - yg(x,y)\right),\\
f(x,y) &= \frac{1}{1-xy} + x\sum_{k\geq 1} y^k\overline{f}_k(x) + \sum_{k\geq 2} y^k \sum_{j=2}^{k} x^{j+1}\overline{g}_{k-j}(x)\\
&= \frac{1}{1-xy} + \frac{xy}{1-y}\sum_{i\geq 1}\left(1-y^i\right)f_i(x) + \frac{x^3y^2}{(1-xy)(1-y)}\sum_{i\geq 0}\left(1-y^{i+1}\right)g_i(x)\\
&= \frac{1}{1-xy} + \frac{xy}{1-y}\left(f(x,1) - f(x,y)\right) + \frac{x^3y^2}{(1-xy)(1-y)}\left(g(x,1) - yg(x,y)\right).
\end{aligned}$$

This implies

$$\left(1 + \frac{x^2y^2}{(1-xy)(1-y)}\right)g(x,y) = \frac{1}{1-xy} + \frac{x^2y}{(1-xy)(1-y)}g(x,1), \tag{7}$$

$$\left(1 + \frac{xy}{1-y}\right)f(x,y) = \frac{1}{1-xy} + \frac{xy}{1-y}f(x,1) + \frac{x^3y^2}{(1-xy)(1-y)}\left(g(x,1) - yg(x,y)\right). \tag{8}$$

To solve these functional equations, we use the kernel method. In this case, if we assume that $y = y_0$ in (7), where y_0 satisfies

$$1 + \frac{x^2y_0^2}{(1-xy_0)(1-y_0)} = 0, \qquad \text{i.e.,} \qquad y_0 = \frac{1 + x - \sqrt{1 - 2x - 3x^2}}{2x(1+x)},$$

then

$$\sum_{n\geq 0} P_n(111, 1221)x^n = g(x,1) = \frac{y_0}{1 - xy_0} = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.$$

Moreover, (7) gives

$$g\left(x, \frac{1}{1-x}\right) = \frac{1-x}{1-3x}(1 - xg(x,1)). \tag{9}$$

Now, if we assume that $y = y_1 = 1/(1-x)$ in (8), then

$$f(x,1) = \frac{1-x}{1-2x} - \frac{x^2}{1-2x}\left(g(x,1) - \frac{1}{1-x}\, g\left(x, \frac{1}{1-x}\right)\right),$$

which, by (9), implies

$$f(x,1) = \frac{1-3x+\sqrt{1-2x-3x^2}}{2(1-3x)},$$

as required. □

3. Other results

In this section, we prove some results related to Theorem 1.1.

3.1. A refinement and an identity

One can refine Theorem 1.1 by adding a parameter y which records the number of blocks of a partition.

Theorem 3.1.
If $n, k \geq 0$, then $p_{n,k}(u,v)$ is the same for the following pairs (u,v):

$$(i) \quad (1222, 1212), \qquad (ii) \quad (1112, 1212), \qquad (iii) \quad (1211, 1221), \qquad (iv) \quad (1222, 1221).$$

Furthermore, the common generating function $h(x,y) = \sum_{n,k\geq 0} p_{n,k}(u,v)x^n y^k$ is given by

$$h(x,y) = \frac{(2-y) + x\left(y^2 - 2y - 2\right) + y\sqrt{(1-xy)^2 - 4x^2y}}{2(1-(2y+1)x)}. \tag{10}$$

Proof. We compute $h(x,y)$ in the first case. For this, we consider the generating function $M(x,y)$ defined by

$$M(x,y) = \sum_{n,k\geq 0} p_{n,k}(111, 1212)x^n y^k.$$

From the reasoning in the second paragraph of the proof of Proposition 2.1, we see that it must satisfy

$$M(x,y) = 1 + xyM(x,y) + x^2yM(x,y)^2$$

and is thus given by

$$M(x,y) = \frac{1 - xy - \sqrt{(1-xy)^2 - 4x^2y}}{2x^2y}.$$

From the proof of Proposition 2.1, we also see that $h(x,y)$ in the first case is given by

$$1 + y\sum_{i\geq 1}\left(xM(x,y)\right)^i = 1 + \frac{xyM(x,y)}{1 - xM(x,y)},$$

which yields (10), upon substituting the expression for $M(x,y)$ and simplifying. A similar proof applies in the second case, upon adjusting the argument given for Proposition 2.2. The last two cases follow from the proofs of Propositions 2.4 and 2.7, upon adding an extra parameter recording the number of blocks in a partition. □

Remark.
Substituting $y = 1$ in (10) yields (1).

Using the interpretation for the numbers L_n given in Theorem 1.1 and of the Motzkin numbers given in the proof of Proposition 2.1, one can perhaps supply combinatorial proofs (in the sense of [4]) of certain identities involving these numbers more easily. We give one such example below. We have not been able to find the following identity in the literature. It follows easily from the generating functions, once stated.

Proposition 3.2.
The numbers L_n and M_n satisfy the relation

$$L_n = 3L_{n-1} - M_{n-2}, \qquad n \geq 2, \tag{11}$$

with $L_0 = L_1 = 1$.

Proof. By Theorem 1.1, the left side of (11) counts the members of $\mathcal{A}_n$. To show that the right side also achieves this, first note that there are L_{n-1} members of $\mathcal{A}_n$ that end in 1 as well as L_{n-1} members whose final letter occurs nowhere else. So we must show that the members of $\mathcal{A}_n$ whose final letter is greater than 1 and occurs one other time number $L_{n-1} - M_{n-2}$. We will denote this subset by $\mathcal{A}'_n$. Let $\mathcal{A}^*_n \subseteq \mathcal{A}_n$ consist of those partitions having at least two occurrences of 1. Note that $|\mathcal{A}^*_n| = L_n - M_{n-1}$, upon subtracting the members of $\mathcal{A}_n$ having a single 1, of which there are M_{n-1} (note that they are of the form 1α, where $\alpha \in \mathcal{R}_{n-1}$).

To complete the proof, it suffices to define a bijection between $\mathcal{A}'_n$ and $\mathcal{A}^*_{n-1}$. Note that $\pi \in \mathcal{A}'_n$ implies that it can be expressed as $\pi = \pi' a \pi'' a$, where $a > 1$, each element of $[a-1]$ occurs in π', and all of the letters of π'' are greater than a. Define $f\colon \mathcal{A}'_n \to \mathcal{A}^*_{n-1}$ by $f(\pi) = \pi' 1 \pi''$. One can verify that f is a bijection; note that f is well-defined since $a > 1$ implies π' is non-empty. $\square$

3.2. Statistics on partitions and paths

In this section, we consider statistics on the set of partitions which avoid $\{111, 1212\}$ as well as on the set avoiding $\{1222, 1212\}$. Recall that $P_n(111, 1212)$ is denoted by $\mathcal{R}_n$. In Section 2.1 we saw the equivalence of the set of partitions $\mathcal{R}_n$ and the set of Motzkin paths $\mathcal{M}_n$. Here, we take this equivalence a step further and consider a pair of equally distributed statistics on the two sets. We also consider extensions of these statistics to the sets $\mathcal{A}_n = P_n(1222, 1212)$ and $\mathcal{L}_{n-1}$, thereby obtaining a p,q-analogue of the sequence L_n.

Given $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{R}_n$, expressed canonically, let $\operatorname{des}\pi$ denote the number of *descents* of π, i.e., the number of indices i, $1 \leq i \leq n-1$, such that $\pi_i > \pi_{i+1}$, and let $\operatorname{inv}\pi$ denote the number of *inversions* of π, i.e., the number of ordered pairs (i,j) with $1 \leq i < j \leq n$ and $\pi_i > \pi_j$. Define the distribution polynomial $M_n(p,q)$ by

$$M_n(p,q) = \sum_{\pi \in \mathcal{R}_n} p^{\operatorname{des}\pi} q^{\operatorname{inv}\pi}, \qquad n \geq 1,$$

with $M_0(p,q) = 1$. For example, if $n = 4$, then

$$\mathcal{R}_4 = \big\{1122, 1123, 1223, 1233, 1234, 1213, 1232, 1221, 1231\big\},$$

which implies $M_4(p,q) = 5 + 2pq + 2pq^2$.

Note that $\pi \in \mathcal{R}_n$ can be expressed as either (i) 1α or (ii) $1\beta1\gamma$, where α, β, and γ do not contain 1 and all of the letters of γ are greater than all of the letters of β. Conditioning on the length i of β, where we differentiate between the $i = 0$ and $i > 0$ cases, we obtain the recurrence

$$M_n(p,q) = M_{n-1}(p,q) + (1-p)M_{n-2}(p,q) + p\sum_{i=0}^{n-2} q^i M_i(p,q) M_{n-2-i}(p,q), \qquad n \geq 2, \tag{12}$$

with the initial conditions $M_0(p,q) = M_1(p,q) = 1$.

Given $\lambda \in \mathcal{M}_n$, consider the matching u, d pairs obtained by drawing a horizontal line to the right of each u in λ and noting the d where it first intersects λ again. For each such pair, consider the number of steps of λ (including ℓ's) strictly between the u and the d and then add up the resulting numbers for all of the pairs to obtain a value which we will denote by $\operatorname{sum}\lambda$. Let $\operatorname{num}\lambda$ denote the number of matching pairs in which the u and d are separated by at least one step. We leave it as an exercise for the reader to verify that the joint distribution of (num, sum) on $\mathcal{M}_n$ is equal to the distribution of (des, inv) on $\mathcal{R}_n$. The next proposition summarizes the above observations.

Proposition 3.3.

Let $M_n(p,q)$ be given by recurrence (12). Then for all $n \geq 0$, we have

$$M_n(p,q) = \sum_{\pi \in \mathcal{R}_n} p^{\operatorname{des}\pi} q^{\operatorname{inv}\pi} = \sum_{\lambda \in \mathcal{M}_n} p^{\operatorname{num}\lambda} q^{\operatorname{sum}\lambda}.$$

Remark.

The statistic num is seen to give the number of occurrences of ℓd or dd within a member of $\mathcal{M}_n$. Thus, num (and hence des on $\mathcal{R}_n$) is seen to be equivalent to the "right double fall" statistic considered in [14], upon comparing the definitions. On the other hand, we were unable to find in the literature the sum statistic on $\mathcal{M}_n$. Also, our formula concerning the total num on $\mathcal{M}_n$ as well as the extension of num to $\mathcal{L}_n$ seem to be new.

When $p = 0$ or $q = 0$, note that (12) reduces to the Fibonacci recurrence and we have $M_n(0,q) = M_n(p,0) = F_n$ for all $n \geq 0$. This can be realized combinatorially by observing that members of $\mathcal{R}_n$ having either no descents or no inversions are precisely those in which each block is of the form $\{i\}$ or $\{i, i+1\}$ for some i. Such partitions are clearly synonymous with square-and-domino tilings of length n and thus they are counted by F_n. A similar interpretation can be given for this using Motzkin paths.

Let $M(x;p,q) = \sum_{n \geq 0} M_n(p,q)x^n$. Multiplying (12) by x^n and summing over $n \geq 2$ yields the relation

$$M(x;p,q) = 1 + x(1+(1-p)x)M(x;p,q) + px^2 M(qx;p,q)M(x;p,q). \tag{13}$$

While it does not seem possible to find an explicit expression for $M(x;p,q)$ for general p and q, one can give the following continued fraction expansion.

Proposition 3.4.

We have

$$M(x;p,q) = \cfrac{1}{1 - x(1+(1-p)x) - \cfrac{px^2}{1 - qx(1+(1-p)qx) - \cfrac{pq^2x^2}{1 - q^2x(1+(1-p)q^2x) - \cfrac{pq^4x^2}{\ddots}}}}.$$

Proof. By (13), we have

$$M(x;p,q) = \frac{1}{1 - x(1+(1-p)x) - px^2 M(qx;p,q)}.$$

Applying this recurrence an infinite number of times yields the required result. □

Remark.

An explicit formula for $M_n(p,q)$ in the sense of [5, Proposition 3A] can be given using the above continued fraction expansion, though it involves multiple sums. When $q = 1$, one can solve (13) explicitly to get

$$M(x;p,1) = \frac{1 - x + (p-1)x^2 - \sqrt{(1 - x + (p-1)x^2)^2 - 4px^2}}{2px^2}. \tag{14}$$

We now derive an explicit formula for the number of members of $P_{n+m}(111, 1212)$ having exactly m descents.

Proposition 3.5.
The members of $P_{n+m}(111, 1212)$ *having exactly* m *descents number*

$$\sum_{j=m+1}^{n} \frac{1}{j} \binom{j}{m+1} \binom{j}{m} \binom{j-m}{n-j}$$

for all $n \geq 1$ *and* $m \geq 0$.

Proof. We first rewrite equation (13) when $q = 1$ in the form

$$\widetilde{M} = x(1+\widetilde{M})(1+x+px\widetilde{M}),$$

where $\widetilde{M} = M(x; p, 1) - 1$. We consider a more general equation

$$\overline{M} = yx(1+\overline{M})(1+x+px\overline{M})$$

and find the coefficient of y^j in $\overline{M}$. By the Lagrange inversion formula and the binomial theorem, we have

$$\begin{aligned}
[y^j](\overline{M}) &= \frac{x^j}{j}\left[z^{j-1}\right]\left((1+z)^j(1+x+pxz)^j\right) = \frac{x^j}{j}\left[z^{j-1}\right]\sum_{i_1=0}^{j}\sum_{i_2=0}^{j}\binom{j}{i_1}\binom{j}{i_2}(px)^{i_2}(1+x)^{j-i_2}z^{i_1+i_2} \\
&= \frac{x^j}{j}\sum_{i_2=0}^{j-1}\binom{j}{j-1-i_2}\binom{j}{i_2}(px)^{i_2}\sum_{i_3=0}^{j-i_2}\binom{j-i_2}{i_3}x^{i_3},
\end{aligned}$$

which implies

$$\widetilde{M} = \overline{M}\Big|_{y=1} = \sum_{j\geq 1}\sum_{i_2=0}^{j-1}\sum_{i_3=0}^{j-i_2}\frac{1}{j}\binom{j}{i_2+1}\binom{j}{i_2}\binom{j-i_2}{i_3}p^{i_2}x^{j+i_2+i_3}.$$

Thus, we have

$$[p^m](\widetilde{M}) = \sum_{j\geq m+1}\sum_{i_3=0}^{j-m}\frac{1}{j}\binom{j}{m+1}\binom{j}{m}\binom{j-m}{i_3}x^{j+m+i_3},$$

and, letting $n = j + i_3$, we get

$$[x^{n+m}p^m](\widetilde{M}) = \sum_{j=m+1}^{n}\frac{1}{j}\binom{j}{m+1}\binom{j}{m}\binom{j-m}{n-j},$$

which completes the proof. □

See [14] for a similar formula for the number of paths in $\mathcal{M}_n$ having a prescribed number of levels and right double falls.

Let b_n denote the total number of descents in all of the members of $P_n(111, 1212)$. Differentiating both sides of (14) with respect to p, and letting $p = 1$, implies

$$\frac{d}{dp}M(x; p, 1)\Big|_{p=1} = \frac{x^2 + x - 1 + (x^2 - 3x + 1)\sqrt{(1+x)(1-3x)}}{2x^2},$$

which is known to be the generating function for the number of compact-rooted directed animals of size n having three source points; see, e.g., [6] or [16, A005775]. Thus, this number equals b_n for all $n \geq 3$ and it would be interesting to find a combinatorial proof.

On the other hand, when $p = 1$, it does not seem that (13) can be solved explicitly for general q nor are we able to find a closed form for $M_n(1, q)$. However, we do have the following result when $q = -1$.

Proposition 3.6.
If $n \geq 0$, then

$$M_{2n}(1,-1) = M_{2n+1}(1,-1) = \sum_{k=0}^{n} \binom{n}{k} C_k, \tag{15}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ denotes the k-th Catalan number.

Proof. We supply both algebraic and combinatorial proofs of this result. Taking $p = 1$ and $q = -1$ in (13) gives

$$M(x;1,-1) = 1 + xM(x;1,-1) + x^2M(-x;1,-1)M(x;1,-1).$$

Replacing x with $-x$ in this equation, solving the resulting system in the variables $M(x;1,-1)$ and $M(-x;1,-1)$, and noting $M(0;1,-1) = 1$ yields

$$M(x;1,-1) = \frac{1 - x^2 - \sqrt{(1-x^2)^2 - 4x^2(1-x^2)}}{2x^2(1-x)},$$

which can be rewritten as

$$M(x;1,-1) = \frac{1}{1-x}\,\frac{1-\sqrt{1-4u}}{2u},$$

where $u = x^2/(1-x^2)$. The result now follows from a short calculation using the facts that

$$\sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x} \qquad \text{and} \qquad \sum_{n\geq k}\binom{n}{k}x^k = \frac{x^k}{(1-x)^{k+1}}.$$

To give a bijective proof of (15), let $\mathcal{R}_n^+$ and $\mathcal{R}_n^-$ denote the subsets of $\mathcal{R}_n$ whose members have even and odd inv-parity, respectively. It suffices to identify a subset $\mathcal{R}_n^*$ of $\mathcal{R}_n^+$ having cardinality

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{\lfloor n/2\rfloor}{k}C_k,$$

along with an inv-parity changing involution of $\mathcal{R}_n - \mathcal{R}_n^*$.

Let us first consider the even case. Let $\mathcal{R}_{2n}^*$ comprise those partitions $\pi = \pi_1\pi_2\cdots\pi_{2n}$ such that for all $1 \leq i \leq n$ either π_{2i-1} and π_{2i} are both the only letters of their kind in π or neither of them are. For example, if $n = 5$, then 1123456754 and 1234556721 both belong to R_{10}^*, but 1123442567 does not since $\pi_3\pi_4 = 23$, with 3 the only letter of its kind, but not 2 (and likewise for $\pi_7\pi_8$). It is seen that all members of $\mathcal{R}_{2n}^*$ have an even number of inversions and their cardinality is the right-hand side of (15), upon choosing the $n-k$ indices i such that both π_{2i-1} and π_{2i} correspond to singletons.

We now define an inv-parity involution of $\mathcal{R}_{2n} - \mathcal{R}_{2n}^*$. Suppose that $\pi = \pi_1\pi_2\cdots\pi_{2n} \in \mathcal{R}_{2n} - \mathcal{R}_{2n}^*$ and that i_0 is the *smallest* index i such that either π_{2i-1} or π_{2i} is the only letter of its kind in π, but not both. Let $\pi_{2i_0-1}\pi_{2i_0} = ab$ and let us first assume that there is an additional a or b occurring to the right of b in π. We change that letter to the other option, noting that this changes the inv-parity since $b = a+1$ in this case. Otherwise, we must have either (i) $a < b$, where there is a second a to the left of this one, or (ii) $a > b$, where there is a second b to the left of the a. If (i) or (ii) occurs, then switch the order of the letters π_{2i_0-1} and π_{2i_0}, leaving the rest of π undisturbed. Using the appropriate mapping of the two described yields the desired involution of $\mathcal{R}_{2n} - \mathcal{R}_{2n}^*$.

For the odd case, apply the involution described above in the even case to the first $2n$ letters of $\pi = \pi_1\pi_2\cdots\pi_{2n+1} \in \mathcal{R}_{2n+1}$ if π_{2n+1} is the only letter of its kind or if it does not equal $a = \pi_{2i_0-1}$ or $b = \pi_{2i_0}$. Extend this involution by changing π_{2n+1} to the other option if it happens that it equals either a or b. The set of survivors are precisely those partitions of the form $\pi'(t+1)$ for some t, where $\pi' \in \mathcal{R}_{2n}^*$ has exactly t distinct letters. □

One can also consider statistics on the set $\mathcal{A}_n = P_n(1222, 1212)$, or, equivalently, statistics on $\mathcal{L}_{n-1}$, where $n \geq 1$, and thereby obtain polynomial generalizations of L_n. Let $\operatorname{des}\pi$ count the number of descents of $\pi = \pi_1\pi_2\cdots\pi_n \in \mathcal{A}_n$ and let $\operatorname{inv}^*\pi$ count the number of ordered pairs (i, j), $1 \leq i < j \leq n$, with $\pi_i > \pi_j > 1$. One can also give equivalent statistics on $\mathcal{L}_{n-1}$, though their descriptions are lengthier than those given above for num and sum on $\mathcal{M}_n$. Define the polynomials $L_n(p, q)$ by

$$L_n(p,q) = \sum_{\pi\in\mathcal{A}_n} p^{\operatorname{des}\pi} q^{\operatorname{inv}^*\pi}, \qquad n \geq 1,$$

with $L_0(p, q) = 1$, and the generating function $L(x; p, q)$ by

$$L(x;p,q) = \sum_{n\geq 0} L_n(p,q)x^n.$$

The following result generalizes Proposition 2.1 and is equivalent to it in the case $p = q = 1$.

Theorem 3.7.

Let $M(x; p, q)$ be given by (13). We have

$$L(x;p,q) = 1 + \frac{xM(x;p,q)}{1 - x(1 - p + pM(x;p,q))}. \tag{16}$$

Proof. We decompose non-empty $\pi \in \mathcal{A}_n$ as $\pi = 1(\pi_1 1)(\pi_2 1)\cdots(\pi_{r-1}1)\pi_r$, where $r \geq 1$ and the π_i contain no 1's and avoid the patterns 111 and 1212. Considering whether or not π is empty, we see from Proposition 3.3 that each section $\pi_i 1$ has the same generating function $x(1 - p + pM(x; p, q))$ for $1 \leq i \leq r - 1$. The numerator $xM(x; p, q)$ accounts for the remaining letters, namely, $1\pi_r$. □

Remark.

One can also show (16) by first arguing directly that $L_n(p, q)$ satisfies the relation

$$L_{n+1}(p,q) = M_n(p,q) + L_n(p,q) + p\sum_{i=1}^{n-1} M_i(p,q)L_{n-i}(p,q), \qquad n \geq 1, \tag{17}$$

which generalizes (2). Multiplying (17) by x^n, summing over $n \geq 1$, and solving for $L(x; p, q)$ then yields (16).

Remark.

Using (14) and (16), one can give an explicit formula for $L(x; p, 1)$.

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