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A new method of proof of Filippov's theorem based on the viability theorem

Research Article

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Abstract: Filippov's theorem implies that, given an absolutely continuous function $y\colon [t_0, T] \to \mathbb{R}^d$ and a set-valued map $F(t,x)$ measurable in t and $l(t)$-Lipschitz in x, for any initial condition x_0, there exists a solution $x(\cdot)$ to the differential inclusion $x'(t) \in F(t,x(t))$ starting from x_0 at the time t_0 and satisfying the estimation

$$|x(t) - y(t)| \le r(t) = |x_0 - y(t_0)|\, e^{\int_{t_0}^t l(s)ds} + \int_{t_0}^t \gamma(s)\, e^{\int_s^t l(\tau)d\tau}\, ds,$$

where the function $\gamma(\cdot)$ is the estimation of $\operatorname{dist}(y'(t), F(t,y(t))) \le \gamma(t)$. Setting $P(t) = \{x \in \mathbb{R}^n : |x - y(t)| \le r(t)\}$, we may formulate the conclusion in Filippov's theorem as $x(t) \in P(t)$. We calculate the contingent derivative $DP(t,x)(1)$ and verify the tangential condition $F(t,x) \cap DP(t,x)(1) \neq \emptyset$. It allows to obtain Filippov's theorem from a viability result for tubes.

MSC: 34A60, 34A12, 49J53, 93C15

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We consider a Cauchy problem

$$\begin{cases} x'(t) \in F(t, x(t)), \\ x(t_0) = x_0. \end{cases} \tag{1}$$

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We assume that the right hand side in (1) is a set-valued map $F\colon [0,T]\times\mathbb{R}^d \rightsquigarrow \mathbb{R}^d$, i.e. $F(t,x)\subset\mathbb{R}^d$, satisfying the following regularity conditions:

$$F(t,x) \text{ is compact convex for every } (t,x)\in[t_0,T]\times\mathbb{R}^d, \tag{2}$$

$$t\rightsquigarrow F(t,x) \text{ is measurable for every } x\in\mathbb{R}^d, \tag{3}$$

$$x\rightsquigarrow F(t,x) \text{ is } l(t)\text{-Lipschitz for almost all } t\in[t_0,T],\ l\in L^1(t_0,T), \tag{4}$$

$$\|F(t,x)\|(=\sup\{|f| : f\in F(t,x)\}) \le \mu(t), \qquad \mu \text{ is integrable.} \tag{5}$$

(See [1, 2] for the definitions of measurability and continuity concepts of set-valued maps.)

We are interested in the existence of solutions to (1) satisfying constrains of the type $x(t)\in P(t)$, where $P\colon[t_0,T]\rightsquigarrow\mathbb{R}^d$ is a set-valued map (we shall call it a tube). We say that the tube P is absolutely continuous if there exists an integrable function $v\in L^1(t_0,T)$ such that $d_H(P(t_1),P(t_2))\le\int_{t_1}^{t_2}v(s)\,ds$ for every $t_1<t_2$, where d_H denotes the Hausdorff distance of sets.

Let $K\subset\mathbb{R}^d$ be a nonempty subset and $x\in K$. Then the contingent cone to K at x is defined by

$$v\in T_K(x) \qquad\Longleftrightarrow\qquad \text{there exist } h_n\to 0^+ \text{ and } x_n\in K \text{ such that } \frac{x_n-x}{h_n}\to v.$$

The contingent derivative $\mathrm{D}P(t,x)$ of P at $(t,x)\in\mathrm{Graph}(P)$ is defined as the set-valued map from $\mathbb{R}$ to $\mathbb{R}^d$ whose graph is described by

$$\mathrm{Graph}(\mathrm{D}P(t,x)) = T_{\mathrm{Graph}(P)}(t,x).$$

It is not difficult to prove, using [2, Proposition 5.1.4], that

$$v\in \mathrm{D}P(t,x)(1) \qquad\Longleftrightarrow\qquad \text{there exist } h_n\to 0^+ \text{ and } x_n\in P(t+h_n) \text{ such that } \frac{x_n-x}{h_n}\to v.$$

We shall use the following viability result for tubes, see [4, Theorem 4.2].

Theorem 1.

Assume that a closed valued tube $P\colon[t_0,T]\rightsquigarrow\mathbb{R}^d$ is absolutely continuous and $F\colon[0,T]\times\mathbb{R}^d\rightsquigarrow\mathbb{R}^d$ satisfies (2)–(5). If there exists $A\subset[t_0,T]$ of full measure such that

$$\text{for all } t\in A \text{ and } x\in P(t), \qquad F(t,x)\cap DP(t,x)(1)\neq\emptyset, \tag{6}$$

then for every $x_0\in P(t_0)$ there exists a solution to (1) such that $x(t)\in P(t)$ for every $t\in[t_0,T]$.

We shall need some calculus formula for tubes.

Lemma 2.

Suppose that a tube P is given by

$$P(t) = y(t) + r(t)\cdot K,$$

where $y\colon[t_0,T]\to\mathbb{R}^d$, $r\colon[t_0,T]\to(0,+\infty)$ are absolutely continuous functions and $K\subset\mathbb{R}^d$ is a nonempty compact set. Then there exists $A\subset[t_0,T]$ of full measure such that for every $t\in A$, $x\in P(t)$ we have

$$\mathrm{D}P(t,x)(1) = y'(t) + r'(t)\cdot k + r(t)\cdot T_K(k),$$

where $x=y(t)+r(t)\cdot k$ and $k\in K$.

Proof. Fix $t \in [t_0, T]$ such that derivatives $y'(t), r'(t)$ exist.

($\subseteq$) Let $v \in \mathrm{D}P(t,x)(1)$. There exist $h_n \to 0^+$ and $x_n \in P(t+h_n)$ such that $(x_n - x)/h_n \to v$. Since $x_n \in P(t+h_n)$ then $x_n = y(t+h_n) + r(t+h_n)k_n$, where $k_n \in K$. The set K is compact. Passing to a subsequence (denoted again by k_n) we obtain that $k_n \to a$ and $a \in K$. We have

$$y(t+h_n) = y(t) + h_n y'(t) + o_1(h_n), \qquad r(t+h_n) = r(t) + h_n r'(t) + o_2(h_n),$$

where $o_i(h_n)/h_n \to 0$ for $i = 1, 2$. So

$$\begin{aligned}\frac{x_n - x}{h_n} &= \frac{y(t+h_n) - y(t)}{h_n} + \frac{r(t+h_n)k_n - r(t)k}{h_n} = y'(t) + \frac{o_1(h_n)}{h_n} + \frac{(r(t) + r'(t)h_n + o_2(h_n))k_n - r(t)k}{h_n} \\ &= y'(t) + \frac{o_1(h_n)}{h_n} + r'(t)k_n + r(t)\frac{k_n - k}{h_n} + \frac{o_2(h_n)}{h_n}k_n.\end{aligned}$$

Thus the limit $\lim (k_n - k)/h_n$ exists and equals to $(v - y'(t) - r'(t)a)/r(t) = \omega \in T_K(k)$. Moreover $\lim k_n = k$. We obtain that

$$v \in y'(t) + r'(t)\cdot k + r(t)\cdot T_K(k).$$

($\supseteq$) Let $v = y'(t) + r'(t)\cdot k + r(t)\omega$, where $\omega \in T_K(k)$. There exist $k_n \in K$ and $h_n \to 0^+$ such that $\omega = \lim (k_n - k)/h_n$. So $k_n = k + h_n\omega + o(h_n)$, where $o(h_n)/h_n \to 0$. Setting $x_n = y(t+h_n) + r(t+h_n)k_n$ we have

$$\begin{aligned}\frac{x_n - x}{h_n} &= \frac{y(t+h_n) + r(t+h_n)k_n - y(t) - r(t)k}{h_n} = \frac{y(t+h_n) - y(t)}{h_n} + \frac{r(t+h_n) - r(t)}{h_n}k + r(t+h_n)\omega + r(t+h_n)\frac{o(h_n)}{h_n} \\ &\to y'(t) + r'(t)k + r(t)\omega = v.\end{aligned}$$

Since $\lim (x_n - x)/h_n \in \mathrm{D}P(x,t)(1)$, then $v \in \mathrm{D}P(t,x)(1)$. □

We consider a special case $K = K(0,1) = \{x : |x| \le 1\}$ and $r(t) = |x_0 - y(t_0)|e^{\int_{t_0}^t l(s)\,ds} + \int_{t_0}^t \gamma(s)e^{\int_s^t l(\tau)d\tau}\,ds$. We have $r'(t) = l(t)r(t) + \gamma(t)$ for almost all t. Moreover,

$$T_K(k) = \begin{cases}\{\omega : \omega\cdot k \le 0\} & \text{for } |k| = 1,\\ \mathbb{R}^n & \text{for } |k| < 1.\end{cases}$$

Let $x = y(t) + r(t)k \in P(t)$. If $|k| < 1$ then $\mathrm{D}P(t,x)(1) = \mathbb{R}^d$. If $|k| = 1$ then using Lemma 2 we obtain

$$\begin{aligned}\mathrm{D}P(t,x)(1) &= y'(t) + (l(t)\cdot r(t) + \gamma(t))k + r(t)T_K(k) = y'(t) + r(t)(l(t)\cdot k + T_K(k)) + \gamma(t)k \\ &= y'(t) + r(t)\{\omega : \langle\omega, k\rangle \le l(t)\} + \gamma(t)k = y'(t) + \{\omega : \langle\omega, k\rangle \le l(t)\cdot r(t) + \gamma(t)\}.\end{aligned}$$

Thus

$$\mathrm{D}P(t,x)(1) = \begin{cases} y'(t) + \{\omega : \langle\omega, k\rangle \le l(t)\cdot r(t) + \gamma(t)\} & \text{for } |k| = 1,\\ \mathbb{R}^d & \text{for } |k| < 1.\end{cases} \tag{7}$$

Now we recall Filippov's theorem, cf. [1, Theorem 2.4.1] and [3].

Theorem 3.

If $F\colon [0,T]\times\mathbb{R}^d \rightsquigarrow \mathbb{R}^d$ satisfies (2)–(5), $x_0 \in \mathbb{R}^d$ and $y\colon [t_0,T] \to \mathbb{R}^d$ is an absolutely continuous function, then there exists a solution $x\colon [t_0,T] \to \mathbb{R}^d$ to (1) such that

$$|x(t) - y(t)| \le |x_0 - y(t_0)|\,e^{\int_{t_0}^t l(s)ds} + \int_{t_0}^t \gamma(s)\,e^{\int_s^t l(\tau)d\tau}ds. \tag{8}$$

Proof. We set $P(t) = y(t) + r(t)K(0,1)$, where

$$r(t) = |x_0 - y(t_0)|\, e^{\int_{t_0}^{t} l(s)ds} + \int_{t_0}^{t} \gamma(s)\, e^{\int_s^t l(\tau)d\tau} ds.$$

We have to verify the tangential condition (6). Fix t such that the derivatives $y'(t), r'(t)$ exist and $F(t,\cdot)$ is $l(t)$-Lipschitz. Let $x = y(t) + r(t)k \in P(t)$ and $|k| = 1$. There exists $w \in F(t, y(t))$ such that $|y'(t) - w| = \gamma(t)$. There exists $v \in F(t,x)$ such that

$$|w - v| \le d_H(F(t, y(t)), F(t,x)) \le l(t)r(t).$$

We have

$$\langle v - y'(t), k\rangle = \big\langle (v - w) + (w - y'(t)),\, k\big\rangle \le l(t)r(t) + \gamma(t).$$

So $v \in DP(t,x)(1)$. By Theorem 1, there exists a solution $x(\cdot)$ to (1) such that $x(t) \in P(t)$, which implies the estimation (8). □

Remark 1.

In the viability theorem we assume that the values of the set-valued map $F(t,x)$ are convex. In Filippov's theorem this assumption is not necessary. Therefore the presented method of proof required stronger assumptions.

Remark 2.

In [4, Theorem 4.2] we assume that the set-valued map $F(t,\cdot)$ is continuous. We used in the proof of Filippov's theorem the assumption (4) ($F(t,\cdot)$ is $l(t)$-Lipschitz) to obtain the tangential condition (6). The same arguments can be applied under a weaker assumption. Instead of (4) it is sufficient to assume that for all $x, y \in \mathbb{R}^d$ and $w \in F(t,y)$ there exists $v \in F(t,x)$ such that $\langle w - v,\, y - x\rangle \le l(t)|y - x|^2$ for almost all t.

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