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The minimal displacement problem in the space l^∞

Research Article

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Abstract: We give a lower bound for the minimal displacement characteristic in the space l^∞.

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1. Introduction

In 1973 Goebel [2] introduced the notion of minimal displacement. Let C be a bounded, closed and convex subset of an infinite dimensional Banach space X. The *minimal displacement* is the number

$$d_T = \inf\{\|x - Tx\| : x \in C\}.$$

Goebel showed that d_T can be positive for lipschitzian mappings and he proved the basic property of the minimal displacement for lipschitzian mappings, i.e.

$$d_T \le \left(1 - \frac{1}{k}\right) r(C) \qquad \text{for} \quad k \ge 1,$$

where $r(C) = \inf[\sup\{\|x - y\| : y \in C\} : x \in X]$ is the Chebyshev radius of C. There are spaces and sets such that $d_T = (1 - 1/k)r(C)$. Goebel also introduced the so-called minimal displacement characteristic of X. This is a function defined for $k \ge 1$ as

$$\psi_X(k) = \sup\{d_T \mid T : B \to B,\ T \in L(k)\},$$

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where B denotes the closed unit ball and $L(k)$ the class of lipschitzian mappings with constant k. It is known that

$$\psi_X(k) \le 1 - \frac{1}{k}$$

for any space X. There exist some 'extremal' spaces for which $\psi_X(k) = 1 - 1/k$. Examples of such spaces are $C[0,1]$, c_0. Recently, Piasecki [4] proved that also c is extremal with respect to the minimal displacement problem. In the case of l^∞ it is known that $\psi_{l^\infty}(k) \ge (1 - 2/k)/2$, see [1]. This result is very weak compared with the fact that for any infinite dimensional Banach space we have $\lim_{k\to\infty} \psi_X(k) = 1$, see [3].

2. Results

Let $B_2^+(c_0)$ and $B_2^+(l^\infty)$ denote nonnegative parts of balls centered at 0 of radius 2, respectively in spaces c_0 and l^∞, i.e., sets

$$B_2^+(c_0) = \{x \in c_0 \,:\, 0 \le x_i \le 2,\ i = 1,2,3,\dots\},$$

$$B_2^+(l^\infty) = \{x \in l^\infty \,:\, 0 \le x_i \le 2,\ i = 1,2,3,\dots\}.$$

Observe that $B_2^+(l^\infty)$ is isometric to the unit ball in l^∞. We start our construction with two technical lemmas.

Lemma 2.1.

A map $T\colon B_2^+(c_0) \to B_2^+(c_0)$ defined for $k \ge 1$ and $x = (x_1, x_2, x_3, \dots)$ as

$$Tx = \big(2, \min\{kx_1, 2\}, \min\{kx_2, 2\}, \min\{kx_3, 2\}, \dots\big)$$

is lipschitzian with constant k and for any $x \in B_2^+(c_0)$

$$\|x - Tx\| > 2 - \frac{2}{k}.$$

Proof. It is easy to check that T is lipschitzian with constant k and moreover $\|x - Tx\| > 2 - 2/k$ since the reverse inequality implies $x_i \ge 2/k$ for $i = 1,2,3,\dots$ which is a contradiction. □

Lemma 2.2.

The mapping $R\colon B_2^+(l^\infty) \to B_2^+(c_0)$ defined for $x = (x_1, x_2, x_3, \dots)$ as

$$Rx_n = \begin{cases} 0 & \text{if} \quad x_n \le d(x), \\ x_n - d(x) & \text{if} \quad x_n > d(x), \end{cases}$$

where $d(x) = \operatorname{dist}(x, c_0) = \limsup_{n\to\infty} |x_n|$, is a lipschitzian retraction with constant 2.

Proof. Observe that for any $x \in B^+(c_0)$, $Rx = x$. Now we show that R is 2-lipschitzian. To prove it we consider the following cases.

Case 1. $Rx_n = Ry_n = 0$. Then obviously $|Rx_n - Ry_n| = 0$.

Case 2. $Rx_n > 0$ and $Ry_n = 0$. Then

$$|Rx_n - Ry_n| = x_n - d(x) \le x_n - d(x) + d(y) - y_n \le |x_n - y_n| + |d(x) - d(y)| \le 2\|x - y\|.$$

Case 3. $Rx_n = 0$ and $Ry_n > 0$. Proof the same as in Case 2.

Case 4. $Rx_n > 0$ and $Ry_n > 0$. Then

$$|Rx_n - Ry_n| = |x_n - d(x) + d(y) - y_n| \le |x_n - y_n| + |d(x) - d(y)| \le 2\|x - y\|.$$ □

Now we can formulate our result.

Theorem 2.3.
In the space l^∞ the following evaluation for minimal displacement characteristic holds:

$$\psi_{l^\infty}(k) \geq \begin{cases} (3-2\sqrt{2})(k-1) & \text{if} \quad 1 \leq k \leq 2+\sqrt{2}, \\ 1-\dfrac{2}{k} & \text{if} \quad k > 2+\sqrt{2}. \end{cases}$$

Proof. Define $\overline{T}\colon B_2^+(l^\infty) \to B_2^+(c_0)$ as a composition $\overline{T} = T \circ R$, where T, R are maps from previous lemmas. Obviously $T \in L(2k)$ and the minimal displacement of $\overline{T}$ can be estimated as follows:

$$\|x - Tx\| \;\geq\; \max\bigl\{\operatorname{dist}(x, B_2^+(c_0)),\, d_T - \operatorname{dist}(x, B_2^+(c_0))\bigr\} \;>\; 1-\frac{1}{k}$$

for any x. Because $r(B_2^+(l^\infty)) = 1$ (r denotes the Chebyshev radius of the set) we get that

$$\psi_{l^\infty}(k) \geq 1-\frac{2}{k} \qquad \text{for} \quad k \geq 2.$$

This result can be slightly improved by taking a tangent to the graph from 1 because the function ψ is concave with respect to 1 in any space X, see [3, p. 215]. After calculations we get that

$$\psi_{l^\infty}(k) \geq \begin{cases} (3-2\sqrt{2})(k-1) & \text{if} \quad 1 \leq k \leq 2+\sqrt{2}, \\ 1-\dfrac{2}{k} & \text{if} \quad k > 2+\sqrt{2}. \end{cases} \qquad \square$$

It is known that if $l^\infty \subset X$ then there exists a projection $P\colon X \to l^\infty$ with norm 1. So we can formulate a generalization of our theorem.

Theorem 2.4.
If $l^\infty \subset X$ then

$$\psi_X(k) \geq \begin{cases} (3-2\sqrt{2})(k-1) & \text{if} \quad 1 \leq k \leq 2+\sqrt{2}, \\ 1-\dfrac{2}{k} & \text{if} \quad k > 2+\sqrt{2}. \end{cases}$$

Proof. Denote by $B_2(X)$ a closed ball of radius 2 in the space X. Let P be a projection $P\colon X \to l^\infty$ of norm 1. Define $\overline{T}\colon B_2(X) \to B_2^+(c_0)$ as a composition $\overline{T} = T \circ R \circ T_1 \circ P$, where T, R are maps from our lemmas and $T_1 x_i = |x_i|$ for any $i = 1, 2, 3, \dots$ Obviously $T \in L(2k)$, $k \geq 1$, and the minimal displacement of $\overline{T}$ can be estimated as in the proof of the previous theorem. $\square$

The question if the space l^∞ is extremal with respect to the minimal displacement problem, is still open.

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References

[1] Bolibok K., Minimal Displacement and Retraction Problem for Balls in Banach Spaces, PhD thesis, Uniwersytet Marii Curie-Skłodowskiej, Lublin, 1999 (in Polish)

[2] Goebel K., On the minimal displacement of points under lipschitzian mappings, Pacific J. Math., 1973, 45, 151–163

[3] Goebel K., Kirk W.A., Topics in Metric Fixed Point Theory, Cambridge Stud. Adv. Math, 28, Cambridge University Press, Cambridge, 1990

[4] Piasecki Ł., Retracting a ball onto a sphere in some Banach spaces, Nonlinear Anal., 2011, 74(2), 396–399