

Geometry and dynamics of admissible metrics in measure spaces

Research Article

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Abstract: We study a wide class of metrics in a Lebesgue space, namely the class of so-called admissible metrics. We consider the cone of admissible metrics, introduce a special norm in it, prove compactness criteria, define the ε -entropy of a measure space with an admissible metric, etc. These notions and related results are applied to the theory of transformations with invariant measure; namely, we study the asymptotic properties of orbits in the cone of admissible metrics with respect to a given transformation or a group of transformations. The main result of this paper is a new discreteness criterion for the spectrum of an ergodic transformation: we prove that the spectrum is discrete if and only if the ε -entropy of the averages of some (and hence any) admissible metric over its trajectory is uniformly bounded.

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1. Introduction

This paper contains a number of results obtained in the framework of the program outlined by the first author in [10, 15, 17] and concerning the asymptotic dynamics of metrics in measure spaces and its applications to ergodic theory. In the [second section](#), we study the space of so-called admissible metrics on a standard measure space; in the [third section](#), we use the developed machinery to characterize systems with discrete spectrum in terms of scaling entropy. The main idea of our approach is as follows. Consider an action of a countable group G of measurable transformations in a standard

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(Lebesgue) space (X, μ) with a continuous measure and assume that we are given a measurable (regarded as a function of two variables) metric or semimetric ρ such that the corresponding metric space structure on X agrees with the measure space structure (such a metric is called admissible, see below). We iterate the metric using the transformation group G and consider the averages of these iterations over finite subsets of G chosen in a special way (for instance, over Følner sets in amenable groups):

$$\rho_n^{\text{av}}(x, y) = \frac{1}{\#A_n} \sum_{g \in A_n} \rho(gx, gy).$$

For the group \mathbb{Z} with an automorphism T as a generator, we have

$$\rho_n^{\text{av}}(x, y) = \frac{1}{n} \sum_{i=0}^{n-1} \rho(T^i x, T^i y).$$

We suggest to study the asymptotic behavior (as $n \rightarrow \infty$) of this sequence of metrics and its invariants, and to find those invariants that do not depend on the choice of the initial metric. The first example of such an invariant is the ε -entropy of the corresponding metric measure space, more exactly, the scaling entropy. A general principle, which we justify in this paper in the simplest case of a discrete spectrum action of an Abelian group (in particular, \mathbb{Z}), is that these asymptotic characteristics do not depend on the choice of the initial (semi)metric ρ , at least for a wide range of metrics, and thus are ergodic invariants of the action. Most probably, this is also true in many other cases.

Of course, the limit mentioned above does exist almost everywhere by the ergodic theorem (applied to the square of the action on the space $X \times X$) and is an invariant metric. But in most interesting cases, namely, when the orthogonal complement to the constants has no discrete spectrum, this limit metric is constant almost everywhere, so that it determines the discrete topology on X and hence is not admissible in our sense. However, we will be interested not in the limit itself, but in the asymptotic behavior of the average metrics.

The relation between scaling and classical entropies is easy to explain. The Kolmogorov entropy of an automorphism T in Sinai's definition is the limit of the entropy of the product of n rotations of a generating partition for T normalized by n . If this entropy vanishes, no change of the normalization would give a new invariant. In our approach, we suggest to consider the normalized limits of the ε -entropy; and the difference with the classical approach is that we consider not the product of partitions, but the average metric. This allows us to define the asymptotics of the ε -entropy also in the case where the Kolmogorov entropy vanishes. The corresponding growth (in n , for small ε) is determined by the so-called scaling sequence, and if the numerical limit does exist, then it is called the *scaling entropy*, see [11]. As a very special case, this notion includes also topological entropy. Some nontrivial examples for actions of groups of the form $\sum_1^\infty \mathbb{Z}_2$ were studied earlier, see [11, 18]¹.

Similar suggestions, in different contexts and different generality, were studied earlier. Feldman [1], see also the later papers [5] and especially [2], where this problem is considered from the point of view of complexity theory, perceived the role of ε -entropy (without using this term). An important difference of our suggestion from all these papers is that instead of the theory of measurable partitions (i.e., discrete semimetrics) we use the theory of general admissible semimetrics and consider the operation of averaging metrics, which has no simple interpretation in terms of partitions (see the formula above). Averaging is much more natural for ergodic theory than taking the maximum of metrics. In another context, this operation was used in [7]. Considering scaling sequences for the ε -entropy of automorphisms will make it possible to classify the "measure of chaoticity" — from the absence of growth (in the case of discrete spectrum) up to linear growth (in the case of positive Kolmogorov entropy). In between there must be classes of automorphisms with zero Kolmogorov entropy but different scaling entropy. In more traditional (probabilistic) terms, one might say that we suggest to consider the asymptotics of sequences of Hamming-like metrics in the space of realizations of a stationary random process.

Questions about more involved geometric invariants of sequences of metrics apparently were not even posed. What is the difference between the sequence of average metrics on a measure space constructed from a Bernoulli automorphism and that constructed from a non-Bernoulli K -automorphism? The growth of the ε -entropy (the scaling sequence) in

¹ In those papers, the problem arose in connection with the theory of filtrations and the past of Markov processes.

these cases is the same; therefore, to distinguish between them, one needs to consider invariants not of a single metric, but of several consecutive metrics.

To formulate very briefly the idea of the approach to ergodic theory suggested in [10, 15–17], one is to study *random stationary sequences of admissible metrics on a given measure space and their asymptotic invariants*, in contrast to the traditional probabilistic interpretation of this theory as the study of stationary sequences of random variables. It is quite obvious that the information on the shifts contained in metrics is easier to extract than that contained in functions of one variable, and this allows one to hope for a simplification of the whole theory.

The results presented in the [second section](#) of the paper are devoted to preliminary considerations, namely, to the study of admissible metrics on a measure space. On the one hand, Gromov's remarkable work [4] initiated a systematic study of so-called *mm*-spaces (which in [10] were called Gromov triples, or metric triples). The most important fact here is the reconstruction, or classification, theorem of Gromov and Vershik, about a complete system of invariants of nondegenerate *mm*-spaces, see [4, 14] and below, which is a particular case of the classification theorem for measurable functions of several variables [12]. On the other hand, starting from the first author's papers [10, 13], the following point of view on *mm*-spaces is suggested: in contrast to the classical approach, where one fixes a topological space (for instance, a metric compact space) and considers various Borel measures on it, here, on the contrary, one fixes a σ -algebra and a measure and varies *admissible metrics* on this measure space. It is interesting that within this approach, even the notion of a (semi)metric needs to be slightly modified (fortunately, in a harmless way: "an almost metric is a metric"). We consider in detail several equivalent definitions of an admissible metric, which are heavily used in what follows and underlie the whole approach. The admissibility of a metric on a measure space means merely that it is measurable and separable. The original measure is Borel with respect to any admissible metric, and the completion of the original space with respect to an admissible metric is a Polish space with a nondegenerate Borel measure. There are many reformulations of the notion of admissibility, including those involving matrix distributions, projective limits, etc. We consider summable metrics; the space (cone) of admissible metrics lies in $L^1(X \times X, \mu \times \mu)$ and is equipped with a special norm (called the *m*-norm). The convergence in this norm is a "convergence with a regulator," which appears in the theory of partially ordered Banach spaces. We prove a number of properties of this norm and an important compactness criterion for a family of metrics in this norm, which is a generalization of the Kolmogorov–Riesz compactness criterion for L^1 . In one of the subsections we discuss how an admissible metric can be restricted to the elements of a measurable partition. This question is related to a serious problem about the correctness of the restriction of a measurable function of two or more variables to a subset of smaller dimension.

The main result of this paper (the [third section](#)) illustrates this idea; namely, it says that *for an action of \mathbb{Z} (and discrete Abelian groups), the spectrum is discrete if and only if for some (and hence any) admissible metric, the ε -entropy of its averages is bounded*. This criterion does not require explicit calculation of the spectrum or even (as in Kushnirenko's criterion; see the [last subsection](#)) enumeration of the asymptotics of all possible sequences of entropies, etc. It suffices to perform calculations only for one admissible metric. A similar result in a more special situation was obtained by another method in [2, 3].

In the [last subsection](#), we discuss relations of our results with the characterization of discrete spectrum systems in terms of Kirillov–Kushnirenko A-entropy (or sequential entropy) [6] and Kushnirenko's compactness criterion for a set of partitions. The difference between our approaches is that we consider the ε -entropy of the averages of consecutive iterations of a metric rather than the normalized entropy of the supremum over subsequences of partitions, as in [6]. We formulate several open problems and conjectures.

2. The geometry of admissible metrics

2.1. Definitions of admissible metrics on measure spaces

Let (X, μ) be a Lebesgue space. We will be mainly interested in spaces with a normalized (i.e., such that $\mu(X) = 1$) continuous positive measure, but all definitions apply to an arbitrary Lebesgue space, in which the measure may contain atoms.

Definition 2.1.

A metric or semimetric ρ on the space X is called *admissible* if it is measurable, regarded as a function of two variables, on the Lebesgue space $(X \times X, \mu \times \mu)$ and there exists a subset $X_1 \subset X$ of full measure such that the semimetric space (X_1, ρ) is separable.

In other terms, the separability condition is equivalent to the requirement that measure μ is a Radon (or σ -compact) Borel measure w.r.t. (semi)metric ρ .

Since semimetrics play an essential role in our considerations, we use basic notions of the theory of metrics in the case of semimetrics, too. For example, speaking about the Borel σ -algebra of sets in the case of a semimetric space, we mean the σ -algebra generated by the open (in the sense of the semimetric in consideration) sets. Of course, this σ -algebra does not in general separate points. One can easily see that if ρ is an admissible metric (resp. semimetric) in a space (X, μ) , then the measure μ is Borel with respect to ρ , and the completion of appropriate subset $X_1 \subset X$ of full measure with respect to ρ is a complete separable metric (= Polish) space (resp. complete separable semimetric space) in which the measure μ is nondegenerate (nonempty open sets have positive measure).

An important class of admissible metrics is that of *block semimetrics*. Let ξ be a partition of the space (X, μ) into finitely or countably many measurable sets X_i , $i = 1, 2, \dots$; the block semimetric ρ_ξ corresponding to ξ is defined as follows: $\rho_\xi(x, y) = 0$ if x, y lie in the same set X_i for some i , and $\rho_\xi(x, y) = 1$ otherwise. It is called a *cut semimetric* (or just a *cut*) if ξ is a partition into two subsets.

A triple (X, μ, ρ) , where (X, μ) is a Lebesgue space and ρ is an admissible metric, will be called an *admissible metric triple*, or, in short, an *admissible triple*. In what follows, we are mostly interested in the case where the measure μ is continuous (though we do not specify this explicitly), but nevertheless all definitions make sense for an arbitrary (in particular, finite) Lebesgue space. Unless otherwise stated, we assume that an admissible metric ρ is summable:

$$\int_X \int_X \rho(x, y) d\mu(x) d\mu(y) < \infty.$$

In other words, $\rho \in L^1(X \times X)$. However, some results hold without this assumption; moreover, replacing the metric with an equivalent one, we can arrive at the case of a summable metric. Obviously, the (summable) admissible metrics form a cone in the space $L^1(X \times X, \mu \times \mu)$, which will be denoted by $\mathcal{Adm}(X, \mu)$.

The group \mathcal{G} of all automorphisms (i.e., measurable, mod 0 invertible, μ -preserving transformations) of the space (X, μ) acts in $L^1(X \times X, \mu \times \mu)$ in a natural way, and this action preserves the cone $\mathcal{Adm}(X, \mu)$ of admissible metrics. As mentioned in the introduction, in what follows we fix a measure and vary admissible metrics. It is useful to give a definition of an admissible metric which is formally less restrictive, but, however, turns out to be equivalent to the original one.

Definition 2.2.

An *almost metric* on a Lebesgue space (X, μ) is a measurable nonnegative function ρ on $(X \times X, \mu \times \mu)$ such that $\rho(x, y) = \rho(y, x)$ for almost all pairs of points $x, y \in X$ and $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ for almost all triples of points $x, y, z \in X$.

An almost metric ρ is called *essentially separable* if for every $\varepsilon > 0$, the space X can be covered by a countable family of measurable sets with essential diameter (= essential supremum of the distances between points) less than ε .

In [19], the following correction theorem was proved.

Theorem 2.3.

- 1) Let ρ be an almost (semi)metric on X . Then one can modify it on a set of zero measure in X so that the modified function is an almost everywhere finite semimetric on X .
- 2) Besides, if the almost semimetric ρ is essentially separable, then the modified semimetric can be chosen so that the semimetric space (X, ρ) is separable and the corresponding triple is admissible.

Note that the limit in measure (or the almost everywhere limit) of a sequence of (almost) metrics may turn out to be an almost metric, but the correction theorem says that this limit is equivalent to a semimetric. Thus in what follows we always assume that all almost metrics obtained by limit procedures are corrected to semimetrics, that is, the limit of a sequence of semimetrics with respect to almost everywhere convergence is a semimetric or a metric. By the same theorem, the limit of a sequence of semimetrics in the space L^1 can also be assumed to be a semimetric. In what follows, it is convenient to use the following notation.

Definition 2.4.

Let $A \subset X$, and let ρ be a measurable semimetric on X . By $\text{diam}_\rho A$ and $\text{essdiam}_\rho A$ we denote the diameter and the essential diameter of the set A in the semimetric ρ , respectively.

2.2. The entropy of metric measure spaces; equivalent definitions of admissible metrics

Now we introduce the notion of the ε -entropy of a metric on a measure space, which is heavily used in the sequel. The following definition goes back to Kolmogorov.

Definition 2.5.

Let (X, ρ) be a metric space equipped with a Borel probability measure μ . Consider the smallest positive integer k for which X can be represented as the union of sets X_0, X_1, \dots, X_k such that $\mu(X_0) < \varepsilon$ and $\text{diam}_\rho X_j < \varepsilon$ for $j = 1, \dots, k$. The ε -entropy of the admissible triple (X, μ, ρ) is

$$\mathbb{H}_\varepsilon(\rho, \mu) = \log k$$

(the logarithm is binary). If such k does not exist, we set $\mathbb{H}_\varepsilon(\rho, \mu) = \infty$.

However, it turned out that in some situations it is more convenient to use another definition, which was suggested in [15] and involves the Kantorovich metric (or any other natural metric) in the space of measures defined on a metric space.

Definition 2.6.

Let (X, ρ) be a separable metric space. The *Kantorovich* (or *transportation*) metric K_ρ on the simplex of Borel probability measures on X is defined by the formula

$$K_\rho(\mu_1, \mu_2) = \inf_{\Psi} \iint_{X \times X} \rho(x, y) d\Psi(x, y),$$

where Ψ ranges over the set of all Borel probability measures on $X \times X$ whose projections to the factors coincide with the measures μ_1 and μ_2 , respectively. The ε -entropy of an admissible triple (X, μ, ρ) is the following function of ε :

$$\mathbb{H}_\varepsilon^K(\rho, \mu) = \inf \{H(\nu) : K_\rho(\mu, \nu) < \varepsilon\};$$

here ν ranges over all finite atomic measures on X and the entropy of an atomic measure is defined in the usual way: $H(\sum_k c_k \delta_{x_k}) = -\sum_k c_k \log c_k$.

For a compact metric space, estimates relating these two definitions of the ε -entropy are given in [15]. The following theorem contains a series of equivalent definitions of admissible semimetrics, generalizing the results of [15, 19].

Theorem 2.7.

Let ρ be a measurable semimetric on (X, μ) . Then the following conditions are equivalent:

- 1) The triple (X, μ, ρ) is admissible, i.e., the semimetric ρ is admissible for the measure space (X, μ) .
- 2) For every $\varepsilon > 0$, the semimetric ρ has a finite ε -entropy: $\mathbb{H}_\varepsilon(\rho, \mu) < \infty$.
- 3) The measure μ can be approximated in the metric K_ρ by discrete (= finitely supported) measures.
- 4) For μ -almost all $x \in X$ and every $\varepsilon > 0$, the ball of radius ε (in the metric ρ) centered at x has positive measure.
- 5) For every $\varepsilon > 0$, the space X can be represented as the union of sets X_0, X_1, \dots, X_k such that $\mu(X_0) < \varepsilon$ and $\text{essdiam}_\rho X_j < \varepsilon$ for $j = 1, \dots, k$.
- 6) For every measurable set A of positive measure, the essential infimum of the function ρ on $A \times A$ is zero.

Let us comment on some implications.

Proof. In [19] it was proved that conditions 1), 2), and 4) are equivalent. The equivalence of 2) and 5) is obvious, since if $\text{essdiam}_\rho X_j < \varepsilon$, then X_j can be partitioned into two sets one of which has zero measure and the other one has diameter at most 2ε . Indeed, if $\text{essdiam}_\rho X_j < \varepsilon$, then for almost every $x \in X_j$ for almost all $y \in X_j$ the inequality $\rho(x, y) < \varepsilon$ holds. Fix some point $x_0 \in X_j$ such that $\mu(\{y \in X_j : \rho(x_0, y) \geq \varepsilon\}) = 0$. Then, by triangle inequality, $\text{diam}_\rho\{y \in X_j : \rho(x_0, y) < \varepsilon\} < 2\varepsilon$.

Now we prove that 2) implies 3). Since for every $\varepsilon > 0$, the ε -entropy of ρ (in the sense of Definition 2.5) is finite, there exists a partition of X into sets X_0, X_1, \dots, X_k such that $\mu(X_0) < \varepsilon$ and $\text{diam}_\rho X_j < \varepsilon$ for $j \geq 1$. For the set X_0 choose a point $x_0 \in X$, and for each of the sets X_j , $j \geq 1$, choose an arbitrary point $x_j \in X_j$. Consider the atomic measure

$$\nu = \sum_{j=0}^k \mu(X_j) \delta_{x_j}$$

and write the inequality

$$K_\rho(\mu, \nu) \leq \sum_{j=0}^k \int_{X_j} \rho(x_j, y) d\mu(y) \leq \varepsilon + \int_{X_0} \rho(x_0, y) d\mu(y).$$

Choosing x_0 appropriately, we can make the last term not bigger than its mean value

$$\int_X \int_{X_0} \rho(x, y) d\mu(y) d\mu(x).$$

Thus, for an appropriate choice of x_0 , we have

$$K_\rho(\mu, \nu) \leq \varepsilon + \int_{X_0} \int_X \rho(x, y) d\mu(x) d\mu(y).$$

This estimate corresponds to transferring the whole X_i to x_i . The latter expression is small for sufficiently small ε by the absolute continuity of the integral and the summability of the function ρ .

Next we prove that 3) implies 6). If 6) does not hold, then there exist $\varepsilon > 0$ and a set A of positive measure such that $\rho(x, y) \geq \varepsilon$ for almost all pairs $x, y \in A$. But then for every $y \in X$, for almost all $x \in A$, we have $\rho(x, y) \geq \varepsilon/2$, so that $K_\rho(\mu, \nu) \geq \mu(A)\varepsilon/2$ for every atomic measure ν , a contradiction with 3).

Finally, we prove that 6) implies 1), namely, we assume that ρ is not admissible and prove that 6) fails. For every fixed $\varepsilon > 0$, the function $x \mapsto \mu(\{y \in X : \rho(x, y) < \varepsilon\})$ is measurable by Fubini's theorem, so that the set $A_\varepsilon = \{x : \mu(\{y \in X : \rho(x, y) < \varepsilon\}) = 0\}$ is measurable. If ρ is not admissible, then 4) fails, hence for some $\varepsilon > 0$ the set A_ε has positive measure. Taking $A = A_\varepsilon$, we see that the essential infimum of ρ on $A \times A$ is positive. \square

Two more definitions of admissible metrics are given in subsection 2.6, one in terms of averages of distances over sets of positive measure, and the other one in terms of random distance matrices, which are invariants of metric triples.

2.3. The theorem on conditional metrics

In this subsection, we prove a result similar to the well-known theorem on the existence of conditional measures (“Rokhlin’s canonical system of measures”) for measurable partitions: a theorem on the existence of a system of conditional admissible metrics on almost all elements of a partition. Thus we will show that if (X, μ, ρ) is an admissible metric triple, then for every measurable partition ξ of X , almost all elements of ξ can be equipped with a canonical structure of a metric triple with respect to the induced metric. A nontrivial issue is to define metrics on the elements of the partition.

Recall that a measurable partition ξ of a Lebesgue space can be defined as the partition into the inverse images of points under a measurable map from (X, μ) to another Lebesgue space, e.g., under a measurable real-valued function or vector-valued function with values in a separable vector topological space. An intrinsic definition of a measurable partition suggested in [8] relies on the existence of a countable basis of measurable sets determining the partition. For a measurable partition ξ , the quotient space X/ξ (the base of ξ) is a Lebesgue space; the image of μ under the canonical quotient map $\pi: X \rightarrow X/\xi$ is a measure μ_ξ on X/ξ . The main characteristic property of a measurable partition is the existence and uniqueness of a canonical system of conditional measures μ^C on μ_ξ -almost all elements $C \in X/\xi$ of ξ , the spaces (C, μ^C) being Lebesgue spaces. In fact, the theorem on existence of conditional measures is a theorem on an integral representation of the projection in L^2 to the subspace of functions which are constant on the elements of partition ξ , or, in other words, this is an integral representation of the operator of the conditional expectation operator. The crucial fact is that for every μ -measurable map f with values in a space V with a Borel structure (e.g., a measurable real-valued function), and for almost all elements $C \in X/\xi$ of ξ , the restriction $f|_C$ of f to C is measurable with respect to the conditional measure μ^C , and the map $C \mapsto f|_C$ is measurable on the base of ξ . For a summable function f this means that an analog of Fubini’s theorem holds: the integral of f over the whole space is equal to the iterated integral computed first over the elements and then over the base. All these definitions are well-behaved with respect to modifying a measurable partition on a set of zero measure.

Below we will obtain a similar result for measurable partitions of measure spaces equipped with a metric. Consider an admissible triple (X, μ, ρ) and assume that in X we are given a measurable partition ξ . Denote by μ_ξ the quotient measure on the quotient space X/ξ , i.e., on the base of ξ . We will regard elements (fibers) of ξ either as points of the base, denoting them by $C \in X/\xi$, or, if convenient, as subsets of X , writing $C \in \xi$. The conditional measure on an element C will be denoted by μ^C .

Using this notation, we state the theorem on existence of conditional metrics on almost all elements of a measurable partition, and measurability of the dependence of a metric as a function of element C of the partition in appropriate sense. For making this statement rigorous we use a metric invariant of a function of two variables (in particular, of a metric) on a measure space — so-called *matrix distributions* — which was introduced in [12]. This notion gives a simple way to define what does it mean measurability of the family of metrics, which are defined on various spaces (on the elements of a partition) — see item 2) in the theorem.

Theorem 2.8.

- 1) The restriction of the metric ρ to μ_ξ -almost every element $C \in \xi$ of the partition ξ is well defined and determines the structure of an admissible triple (C, μ^C, ρ^C) for almost all $C \in \xi$.
- 2) Let n be a positive integer, let Ω be any open set in the n^2 -dimensional space of $n \times n$ matrices. For almost any element C of ξ one may define by 1) an admissible triple (C, μ^C, ρ^C) . Let $p_\Omega(C)$ denote the probability that a matrix $(\rho(z_i, z_j))_{1 \leq i, j \leq n}$ belongs to Ω , where z_1, \dots, z_n are independent points in C distributed by μ^C . Then p_Ω is a measurable function of C .

It may seem that in order to obtain the required assertions, it suffices to restrict the metric to almost every element of the partition, but this is not so. The problem is that for measurable functions of two (or several) variables, e.g., for an admissible metric, one cannot directly use a Fubini-like theorem on the measurability of restrictions of functions to the elements of the partition. Moreover, in general this is not true for an arbitrary function. Indeed, the set of pairs (x, y) lying in the same fiber of ξ has (in general) zero measure in $X \times X$, hence there is no known canonical way to restrict an arbitrary μ^2 -measurable function $f(x, y)$ to this set.

Hence, in order to prove that the metrics on the elements are admissible and measurable over the base of the partition, one should use special properties of these functions. It turns out that the needed property is admissibility. Note that similar questions, in spite of their importance, have not yet been studied in general setting. We use the separability of an admissible metric, which ensures that this metric can be defined by a vector function of *one variable*. The trick of passing to a sequence for one or both arguments of a function of two arguments, mentioned above and exploited below, was essentially used in [12] for the classification of measurable functions of several variables via a *random choice* of sequences.

Proof. Choose a sequence x_1, x_2, \dots in X , which is dense in some subset X_1 of full measure in X . We use the functions $f_n(\cdot) = \rho(\cdot, x_n)$, $n = 1, 2, \dots$. We also require that those functions are simultaneously measurable on X_1 . Further, note that since the sequence $\{x_n\}$ is dense, we have

$$\rho(x, y) = \inf_n \{f_n(x) + f_n(y)\}.$$

Therefore, for almost all elements C of ξ equipped with the conditional measures μ^C , this formula defines a metric as a measurable function of two variables. The admissibility of the triple (C, μ^C, ρ^C) is straightforward, because a subspace of a separable metric space is separable. The fact that ρ^C is summable with respect to the measure $\mu^C \times \mu^C$ for almost every C easily follows from the triangle inequality and separability.

Now we should explain the measurability statement 2). Without loss of generality, Ω is a cylinder $\{(a_{i,j})_{1 \leq i,j \leq n} : 0 \leq a_{i,j} < p_{i,j}\}$ for fixed positive numbers $p_{i,j}$. The condition $\rho(x, y) = \inf_n \{f_n(x) + f_n(y)\} < p$ is equivalent to the countable number of conditions like $f_n(x) < r_1$, $f_n(y) < r_2$ for some index n and rationals r_1, r_2 with $r_1 + r_2 < p$. So, the probability that a random distance matrix belongs to Ω may be expressed via probabilities that $\rho(z_i, x_n)$ belongs to some interval on a real line. Such events are (at last) independent, and the product of corresponding probabilities is measurable, since each of them is measurable by the Rokhlin theorem. \square

2.4. The space of admissible metrics. The definition and properties of the m-norm

When working with admissible semimetrics, it is convenient to introduce a special norm on the cone of admissible metrics \mathcal{Adm} , which we call the m-norm; it is defined on \mathcal{Adm} and on a wider vector subspace of $L^1(X^2)$.

Definition 2.9.

Given a function $f \in L^1(X^2)$, we define a finite or infinite norm of f as

$$\|f\|_m = \inf \left\{ \|\rho\|_{L^1(X^2)} : \rho \text{ is a semimetric, } \rho(x, y) \geq |f(x, y)| \text{ for almost all } x, y \in X \right\}.$$

Note that $\|\cdot\|_m$ is indeed a norm, in the sense that it is homogeneous and satisfies the triangle inequality. If f is a semimetric, then $\|f\|_m = \|f\|_{L^1(X^2)}$. It follows directly from the definition that for every f we have $\|f\|_m \geq \|f\|_{L^1(X^2)}$. Hence convergence in the m-norm implies convergence in $L^1(X^2)$. In the theory of partially ordered Banach spaces, such convergence is called convergence with a regulator. Note that the operators corresponding to measure-preserving automorphisms preserve also the m-norm.

Consider the set of all functions in $L^1(X^2)$ with finite m-norm:

$$\mathbb{M} = \{f \in L^1(X^2) : \|f\|_m < \infty\}.$$

Clearly, \mathbb{M} is a linear subspace in $L^1(X^2)$.

Lemma 2.10.

The space \mathbb{M} is complete in the m-norm.

Proof. Let f_n be a Cauchy sequence with respect to the m -norm. We will show that it has a limit in the m -norm. Since the L^1 norm is dominated by the m -norm, f_n is also a Cauchy sequence in $L^1(X^2)$, so that it has a limit $f \in L^1(X^2)$. Thinning the sequence, we may assume that f_n converges to f almost everywhere and, besides, $\|f_n - f_{n+1}\|_m < 1/2^n$ for all n . By the definition of the m -norm, this means that there exists a semimetric ρ_n that dominates $|f_n - f_{n+1}|$ almost everywhere and satisfies $\|\rho_n\|_{L^1(X^2)} < 1/2^n$. Note that the semimetric $\sum_{k=n}^{\infty} \rho_k$ dominates the difference $|f_n - f|$ almost everywhere, so that $\|f_n - f\|_m \leq 1/2^{n-1}$. It follows that the sequence f_n converges to f in the m -norm, as required. \square

Now we will study simple properties of convergence of semimetrics.

Lemma 2.11.

If a sequence of semimetrics ρ_n converges to a function ρ in the m -norm, and for every $\varepsilon > 0$ the entropy $\mathbb{H}_\varepsilon(\rho_n, \mu)$ is finite for all sufficiently large n , then ρ is an admissible semimetric.

Corollary 2.12.

If a sequence of admissible semimetrics ρ_n converges to a function ρ in the m -norm, then ρ is also an admissible semimetric.

Proof of Lemma 2.11. Since the sequence ρ_n converges in the m -norm, it also converges in the space $L^1(X^2)$, so that we may assume that the limit function ρ is a semimetric. It remains to prove that ρ is admissible. For this we will show that its ε -entropy is finite for every ε . First we prove an auxiliary proposition.

Proposition 1.

If ρ is a measurable semimetric on (Y, μ) such that $\|\rho\|_{L^1(Y^2)} < \varepsilon^2/2$, then there exist two disjoint sets Y_0, Y_1 with $Y_0 \cup Y_1 = Y$ such that $\mu(Y_0) \leq \varepsilon$ and $\text{diam}_\rho Y_1 \leq \varepsilon$.

Proof. Note that the map $x \mapsto \mu(\{y \in Y : \rho(x, y) \geq \varepsilon/2\})$ is measurable by Fubini's theorem, and its integral over Y is bounded from above by $(\varepsilon^2/2)/(\varepsilon/2) = \varepsilon$ by Chebyshev's inequality. Hence we can choose x_0 such that the measure of the set $Y_0 = \{y \in Y : \rho(x_0, y) \geq \varepsilon/2\}$ does not exceed ε . But for any $x, y \in Y_1 = Y \setminus Y_0$, the triangle inequality implies that $\rho(x, y) \leq \rho(x, x_0) + \rho(y, x_0) \leq \varepsilon$. The proposition follows. \blacksquare

Returning to the proof of the lemma, we fix $\varepsilon > 0$ and prove that $\mathbb{H}_{4\varepsilon}(\rho)$ is finite. For large n , we have $\|\rho_n - \rho\|_m < \varepsilon^2/2$. By the definition of the m -norm, this means that there exists a semimetric p such that $\|p\|_{L^1(X^2)} < \varepsilon^2/2$ and $\rho \leq p + \rho_n$ almost everywhere. As we have just proved, the set X can be partitioned into two sets X_0 and X_1 such that $\mu(X_0) \leq \varepsilon$ and $\rho(x, y) \leq \varepsilon$ for all $x, y \in X_1$. Choosing n large enough, we may assume that the number $\mathbb{H}_\varepsilon(\rho_n)$ is finite, i.e., we can find a partition $X = A_0 \cup A_1 \cup \dots \cup A_k$ such that $\mu(A_0) < \varepsilon$ and $\text{diam}_{\rho_n} A_j < \varepsilon$ for $j \geq 1$.

Now we construct a partition for the semimetric ρ as follows. Put $B_0 = A_0 \cup X_0$ and $B_j = A_j \cap X_1$ for $j = 1, \dots, k$. Clearly, $\mu(B_0) \leq \mu(A_0) + \mu(X_0) < 2\varepsilon$. For every $j > 0$, for almost all $x, y \in B_j$, we have the inequality $\rho(x, y) \leq \rho_n(x, y) + \rho(x, y) \leq \varepsilon + \varepsilon = 2\varepsilon$, which shows that $\text{essdiam}_\rho B_j \leq 2\varepsilon$. Thus we have shown that for every $\varepsilon > 0$ the number $\mathbb{H}_{4\varepsilon}(\rho)$ is finite and, consequently, that the semimetric ρ is admissible. \square

The following simple lemma says that the L_1 -limit of a sequence of admissible semimetrics with uniformly bounded entropies is again an admissible semimetric.

Lemma 2.13.

If M is a set of admissible semimetrics such that the set $\{\mathbb{H}_\varepsilon(\rho, \mu) : \rho \in M\}$ is bounded for every $\varepsilon > 0$, then the closure of M in the space $L^1(X^2)$ consists of admissible semimetrics only.

Proof. Take an arbitrary function ρ from the closure of M in L^1 . We will prove that it is an admissible semimetric. We know that there exists a sequence of semimetrics $\{\rho_n\} \subset M$ that converges to ρ in $L^1(X^2)$. Clearly, ρ is a semimetric, and one should only check that it is admissible.

Assume to the contrary that ρ is not admissible. Then, by Theorem 2.7, there exist $\varepsilon > 0$ and a set $A \subset X$ of positive measure such that $\rho(x, y) \geq \varepsilon$ for almost all $x, y \in A$. Decreasing ε if necessary, we may assume that $\mu(A) \geq \varepsilon$. Using the boundedness of entropies for $\varepsilon/2$, for each of the semimetrics ρ_n we find a partition $X = X_0 \cup X_1 \cup \dots \cup X_k$ such that $\text{diam}_{\rho_n} X_i \leq \varepsilon/2$ for all $i = 1, \dots, k$ and $\mu(X_0) \leq \varepsilon/2$. Of course, this partition may depend on n , but the number k can be chosen to be universal, since the entropies are bounded. Note that at least one of the sets $X_i \cap A$, $i \geq 1$, has measure not less than $\varepsilon/2k$. Moreover, for almost all $x, y \in X_i \cap A$, we have

$$\rho(x, y) - \rho_n(x, y) \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}, \quad \text{whence} \quad \|\rho - \rho_n\|_{L^1(X^2)} \geq \left(\frac{\varepsilon}{2k}\right)^2 \frac{\varepsilon}{2} > 0.$$

The latter inequality contradicts the convergence of ρ_n to ρ in $L^1(X^2)$, and the lemma follows. \square

In conclusion of this subsection, we prove a lemma on pointwise convergence of admissible semimetrics.

Lemma 2.14.

Assume that a sequence of admissible semimetrics ρ_n converges to an admissible semimetric ρ_{lim} almost everywhere with respect to the measure $\mu \times \mu$. Then there exists a set $X' \subset X$ of full measure such that for any $x, y \in X'$,

$$\limsup_n \rho_n(x, y) = \rho_{\text{lim}}(x, y).$$

Besides, if $x, y \in X'$ and $\rho_{\text{lim}}(x, y) = 0$, then $\lim_n \rho_n(x, y) = 0$.

Proof. Consider the function $\bar{\rho}(x, y) = \limsup_n \rho_n(x, y)$. The functions $\bar{\rho}$ and ρ_{lim} coincide on a set of full measure in X^2 . We must prove that they coincide on the square of a set X' of full measure in X . Note that the function $\bar{\rho}$ satisfies the triangle inequality everywhere (as upper limit of semimetrics); also it is finite almost everywhere with respect to the measure μ^2 , because the function ρ_{lim} is finite a.e. Put $X'' = \{x \in X : \mu(\{y : \bar{\rho}(x, y) = +\infty\}) = 0\}$. Note that $\mu(X'') = 1$. We will prove that $\bar{\rho}(x, y) < +\infty$ for any $x, y \in X''$. Indeed, if $\bar{\rho}(x, y) = +\infty$, then for every $z \in X$ we have either $\bar{\rho}(x, z) = +\infty$ or $\bar{\rho}(y, z) = +\infty$, contradicting the choice of X'' . Thus on X'' the semimetric $\bar{\rho}$ is finite and coincides almost everywhere with ρ_{lim} . Using the characterization of admissibility in terms of the measures of balls from Theorem 2.7 for the semimetrics ρ_{lim} and $\bar{\rho}$, we see that $\bar{\rho}$ is also admissible. Then, by [19, Theorem 3], there exists a set $X' \subset X''$ of full measure such that $\rho_{\text{lim}} = \bar{\rho}$ on the square of X' . The last claim is obvious. \square

2.5. Convergence of admissible metrics. A precompactness criterion

Lemma 2.15.

Assume that a sequence of uniformly bounded semimetrics ρ_n converges to an admissible semimetric ρ in L^1 . Then this sequence converges in the m -norm to the same limit.

Proof. Let R be a constant bounding all semimetrics ρ_n, ρ . Fix $\varepsilon > 0$ and, using the admissibility of ρ , find a partition of the space X into sets A_0, A_1, \dots, A_k such that $\mu(A_0) < \varepsilon$ and $\text{diam}_{\rho} A_j \leq \varepsilon^2$ for $j > 0$. We may assume that

$$\delta = \min \{\mu(A_j) : j = 1, \dots, k\} > 0.$$

Note that for every $j > 0$ the sequence of restricted semimetrics $\rho_n \upharpoonright_{A_j^2}$ converges to the semimetric $\rho \upharpoonright_{A_j^2}$ in the space $L^1(A_j^2)$. By construction, the limit semimetric does not exceed ε^2 everywhere on A_j , hence for sufficiently large n we have

$$\|\rho_n \upharpoonright_{A_j^2}\|_{L^1(A_j^2)} \leq 2\varepsilon^2 \mu(A_j)^2.$$

Now consider the set A_j equipped with the normalized measure $\mu/\mu(A_j)$ and apply Proposition 1 to the restriction of the semimetric ρ_n to A_j . We see that A_j can be partitioned into two sets $B_j(n), C_j(n)$ such that $\mu(B_j(n)) \leq 2\varepsilon\mu(A_j)$ and $\text{diam}_{\rho_n} C_j(n) \leq 2\varepsilon$. This immediately implies that $\mu(C_j(n)) \geq (1-2\varepsilon)\mu(A_j)$.

Choose n so large that these inequalities hold for all $j = 1, \dots, k$. We put $C(n) = \bigcup_{j=1}^k C_j(n)$ and prove that if n is sufficiently large, then $|\rho_n(u, v) - \rho(u, v)| \leq 10\varepsilon$ for any $u, v \in C(n)$. If $u, v \in C_j(n)$ for some j , then $|\rho_n(u, v) - \rho(u, v)| \leq \rho_n(u, v) + \rho(u, v) \leq 3\varepsilon$ by construction. Now let $u_0 \in C_i(n), v_0 \in C_j(n)$, and $i \neq j$. If $|\rho_n(u_0, v_0) - \rho(u_0, v_0)| > 10\varepsilon$, then for all $u \in C_i(n), v \in C_j(n)$ we have

$$|\rho_n(u, v) - \rho(u, v)| \geq |\rho_n(u_0, v_0) - \rho(u_0, v_0)| - (\rho_n(u_0, u) + \rho_n(v_0, v) + \rho(u_0, u) + \rho(v_0, v)) > 4\varepsilon.$$

But then $\|\rho_n - \rho\|_{L^1(X^2)} \geq 4\varepsilon\mu(C_i)\mu(C_j) \geq 4\varepsilon(1-2\varepsilon)^2\delta^2$, which cannot be true for large n . Thus for all sufficiently large n , for any two points $u, v \in C(n)$ we have $|\rho_n(u, v) - \rho(u, v)| \leq 10\varepsilon$. It follows from the construction that the measure of $C(n)$ is large, more exactly, $\mu(C(n)) \geq (1-2\varepsilon)(1-\varepsilon)$.

Define a metric p_n as follows. On the set $C(n) \times C(n)$ it is identically equal to 10ε , and on the remaining set it is equal to $2R + 10\varepsilon$. We have just proved that on $C(n)$ this metric dominates the difference $|\rho_n - \rho|$. On the remaining set, it also dominates the distance, because all original semimetrics are bounded by R . Since R is fixed, for sufficiently small ε the metric p_n has an arbitrarily small L^1 norm. Thus the sequence p_n converges to ρ in the m -norm, and the lemma follows. \square

In what follows, we need a lemma on truncations of semimetrics. Given an arbitrary function f and a real number R , denote by f^R the truncation of f at level R , that is, $f^R(\cdot) = \min(f(\cdot), R)$.

Lemma 2.16.

For a summable semimetric p on the space (X, μ) and every $R > 0$,

$$\|p - p^{2R}\|_m \leq 2 \int_{p > R} p \, d\mu^2.$$

Proof. Choosing an arbitrary point $x \in X$, consider the ball $B = \{y \in X : p(x, y) \leq R\}$ and its complement $A = X \setminus B$. Now we define a semimetric q as follows:

$$q(u, v) = \begin{cases} 0 & \text{if } u, v \in B, \\ p(u, v) & \text{if } u, v \in A, \\ p(u, x) & \text{if } u \in A, v \in B, \\ p(v, x) & \text{if } u \in B, v \in A. \end{cases}$$

One can easily check that q is indeed a semimetric and, besides, for any $u, v \in X$ we have $p(u, v) - p^{2R}(u, v) \leq q(u, v)$. Thus, by the definition of the m -norm,

$$\begin{aligned} \|p - p^{2R}\|_m &\leq \|q\|_{L^1(X^2)} = \int_{X \times X} q \, d\mu^2 = \left(\int_{A \times A} + \int_{A \times B} + \int_{B \times A} + \int_{B \times B} \right) q \, d\mu^2 = \int_{A \times A} p(u, v) \, d\mu(u) d\mu(v) + 2 \int_{A \times B} p(u, x) \, d\mu(u) d\mu(v) \\ &\leq \int_{A \times A} (p(u, x) + p(x, v)) \, d\mu(u) d\mu(v) + 2\mu(B) \int_A p(u, x) \, d\mu(u) \\ &= 2(\mu(A) + \mu(B)) \int_A p(u, x) \, d\mu(u) = 2 \int_A p(u, x) \, d\mu(u). \end{aligned}$$

Now we can optimize this bound by choosing x . Note that the average of the right-hand side over $x \in X$ coincides with $2 \int_{p > R} p \, d\mu^2$; hence, choosing x appropriately, we obtain the desired bound. \square

We use this lemma to deduce a more general theorem.

Theorem 2.17.

Assume that a sequence of semimetrics ρ_n converges to an admissible semimetric ρ in the space L^1 . Then this sequence converges in the m -norm to the same limit.

Proof. We just use the two lemmas already proved. Fix $\delta > 0$ and, using the absolute continuity of the integral of ρ , choose $R > 0$ so large that

$$\int_{\rho > R/2} \rho d\mu^2 < \delta.$$

Since the sequence ρ_n converges to ρ in $L^1(X^2)$, for sufficiently large n we have

$$\int_{\rho_n > R} \rho_n d\mu^2 < 2\delta.$$

The truncations ρ_n^{2R} converge to ρ^{2R} in the space $L^1(X^2)$, since for any functions f, g we have

$$\|f^{2R} - g^{2R}\|_{L^1(X^2)} \leq \|f - g\|_{L^1(X^2)}.$$

Applying Lemma 2.15 to the truncations, we see that for sufficiently large n , $\|\rho_n^{2R} - \rho^{2R}\|_m \leq \delta$. Using Lemma 2.16 twice, we can write the inequality

$$\|\rho_n - \rho\|_m \leq \|\rho_n - \rho_n^{2R}\|_m + \|\rho_n^{2R} - \rho^{2R}\|_m + \|\rho - \rho^{2R}\|_m \leq 4\delta.$$

Thus the sequence ρ_n converges to ρ in the m -norm, as required. \square

This theorem easily implies the following corollary.

Corollary 2.18.

A set of admissible semimetrics is compact in the m -norm if and only if it is compact in L^1 .

In the remaining part of this subsection we prove a precompactness criterion for the m -norm.

Theorem 2.19.

Let M be a set of admissible semimetrics on (X, μ) . Then M is precompact in the m -norm if and only if the following two conditions hold:

- 1) (uniform integrability) the set M is uniformly integrable on X^2 ;
- 2) (uniform admissibility) for every $\varepsilon > 0$ there exists a partition of X into finitely many sets X_1, \dots, X_k such that for every semimetric $\rho \in M$ there exists a set $A \subset X$ of measure less than ε such that $\text{diam}_\rho(X_j \setminus A) < \varepsilon$.

Note that condition 2) in the statement of the theorem can be replaced with the equivalent condition in which diam is replaced by essdiam . Moreover, each of these conditions implies that the set $\{\mathbb{H}_\varepsilon(\rho) : \rho \in M\}$ is bounded for every $\varepsilon > 0$.

It is worth mentioning that we will use not only the definition of uniform integrability, but also its reformulation. We will say that a family of functions $K \subset L^1(\Omega, \nu)$ is uniformly integrable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every set $A \subset \Omega$ with $\nu(A) < \delta$, for every function $f \in K$,

$$\int_A |f| d\nu < \varepsilon.$$

Now we proceed to the proof of the theorem.

Proof. First we will prove that if M is precompact in the m -norm, then conditions 1) and 2) are satisfied. Note that since the m -norm dominates the L^1 norm, the set M is precompact in the space L^1 and hence uniformly integrable.

Consider an arbitrary finite partition ξ of the space X into sets X_1, \dots, X_k . Assume that for some semimetric ρ the partition ξ is an ε -partition, i.e., there exists an exceptional set A such that $\mu(A) < \varepsilon$ and $\text{diam}_\rho(X_j \setminus A) < \varepsilon$, $j = 1, \dots, k$. Using Proposition 1, one can easily see that there exists $\delta > 0$ such that if $\|\rho - \rho_1\|_m < \delta$, then ξ is an ε -partition for ρ_1 , too. That is, the set of semimetrics for which a given partition is an ε -partition is open in the m -norm. We will refer to this set as corresponding to ξ . By Corollary 2.12, the closure of the set M in the m -norm consists only of admissible semimetrics, each having a finite ε -partition. Let us cover the closure of M (which is a compact set) by the open sets corresponding to finite partitions. This open cover has a finite subcover. Clearly, the product of the corresponding partitions is a universal ε -partition for all semimetrics in M , i.e., condition 2) is satisfied.

Now we will prove that conditions 1) and 2) are sufficient for M to be precompact. First we prove that the set M is precompact in the space $L^1(X^2)$. It suffices to find, for every $\varepsilon > 0$, a finite 4ε -net in the L^1 norm. The uniform integrability of the family M means that

$$\lim_{R \rightarrow +\infty} \sup_{\rho \in M} \iint_{\rho > R} \rho d(\mu \times \mu) = 0.$$

Hence for sufficiently large R , all truncations of the functions are close in the L^1 norm (and even in the m -norm) to the corresponding semimetrics from M . Therefore, it suffices to search for an ε -net in the set of truncated semimetrics. For sufficiently large R , we have

$$\|\rho^R - \rho\|_m < \varepsilon$$

for every $\rho \in M$. Note that the universal partition from condition 2) remains universal also for all truncations ρ^R . The set of truncated semimetrics will be denoted by M^R .

Fix a small number $\delta > 0$ which will be specified later, and, using condition 2), find a universal δ -partition $X = X_1 \cup \dots \cup X_k$. For every function $\rho \in M^R$, find an exceptional set A of measure at most δ such that $\text{diam}_\rho(X_j \setminus A) < \delta$ for $j = 1, \dots, k$. Put $Y_j = X_j \setminus A$ and define a function $\bar{\rho} \in L^1(X^2)$ on each of the sets $X_i \times X_j$ as the average of ρ over the set $Y_i \times Y_j$. In the case where one of the sets Y_j has zero measure, we set the value of $\bar{\rho}$ on this set equal to zero. We will prove that $\bar{\rho}$ is close to ρ in the space $L^1(X^2)$. First, both functions are bounded by R . Second, for any $u_1, u_2 \in Y_i$, $v_1, v_2 \in Y_j$, we have the obvious inequality

$$|\rho(u_1, v_1) - \rho(u_2, v_2)| \leq \rho(u_1, u_2) + \rho(v_1, v_2) < 2\delta.$$

Hence

$$\int \int_{Y_i \times Y_j} |\rho - \bar{\rho}| d\mu^2 < 2\delta \mu(Y_i) \mu(Y_j).$$

The union of all sets of the form $Y_i \times Y_j$ is exactly $(X \setminus A)^2$, whence

$$\int \int_X |\rho - \bar{\rho}| d\mu^2 < 2\delta \mu^2(X \setminus A) + 2R\mu(A) < 2\delta(1 + R),$$

which is small for sufficiently small δ . Thus we can approximate every function from M^R by the corresponding function $\bar{\rho}$ with accuracy $\varepsilon/2$. But the set of all such functions $\bar{\rho}$ is bounded in $L^1(X^2)$ and is contained in a finite-dimensional subspace, so that it has a finite $\varepsilon/2$ -net. It follows that in M we can find a finite 4ε -net with respect to the norm of the space $L^1(X^2)$.

Thus M is precompact in $L^1(X^2)$. Consider its closure \bar{M} in $L^1(X^2)$. By Lemma 2.13 (the condition of this lemma holds because of the uniform admissibility), all functions from \bar{M} are admissible semimetrics. Thus the set \bar{M} , which is compact in L^1 , consists of admissible semimetrics only, so that, by Corollary 2.18 of Theorem 2.17, it is compact in the m -norm. Hence the set M is precompact in the m -norm. \square

Theorems 2.17, 2.19 and Lemma 2.13 easily imply the following corollary.

Corollary 2.20.

If M is a precompact set in the m -norm that consists of admissible semimetrics, then its closures in $L^1(X^2)$ and in the m -norm coincide and consist of admissible semimetrics only. Also, ε -entropies of semimetrics in M are uniformly bounded for any fixed $\varepsilon > 0$. In particular, this holds for a sequence of admissible semimetrics, converging in m -norm (and hence by Theorem 2.17 for a sequence of admissible semimetrics, converging in L^1 to admissible semimetric).

The following criterion of precompactness deals with convex sets of metrics. It is suggested by applications in ergodic theory.

Theorem 2.21.

Let M be a uniformly integrable convex family of admissible semimetrics in the space \mathbb{M} . Then M is precompact in the m -norm if and only if the ε -entropies of semimetrics in M are uniformly (with respect to semimetric) bounded for every fixed $\varepsilon > 0$.

Proof. The precompactness of M implies the uniform boundedness of the ε -entropies, e.g., by item 2) of Theorem 2.19. Now we prove that if the ε -entropies are uniformly bounded for every $\varepsilon > 0$, then M is precompact in $L^1(X^2)$. This will imply that M is precompact in the m -norm. Indeed, a set is precompact if and only if every sequence of elements of this set has a Cauchy subsequence. Thus if M is precompact in L^1 , then every sequence of elements of M has a Cauchy subsequence, which converges to a semimetric ρ in L^1 ; since the ε -entropies are uniformly bounded, it follows from Lemma 2.11 that this semimetric is admissible. Then, by Theorem 2.17, the sequence converges to ρ also in the space \mathbb{M} .

Assume that M is not precompact in L^1 . Then, for some $c > 0$, we can choose a sequence of semimetrics ρ_1, ρ_2, \dots in M such that $\|\rho_i - \rho_j\|_{L^1} > c$ for all indices $1 \leq i < j < \infty$. For the moment, fix $\varepsilon > 0$ whose value will be specified later. Find a positive integer k such that for every metric $\rho \in M$ there exists a partition of X into sets X_0, X_1, \dots, X_k such that $\mu(X_0) < \varepsilon$ and $|\rho(x, y)| < \varepsilon$ for all $x, y \in X_i$, $i = 1, 2, \dots, k$.

Consider the semimetric $\rho = (\rho_1 + \dots + \rho_n)/n$; by convexity, $\rho \in M$. The value of n will also be specified later. Consider the corresponding partition of X into sets X_0, X_1, \dots, X_k . Choose points p_i in X_i arbitrarily for $i = 1, \dots, k$. For $s = 1, 2, \dots, n$, consider the function d_s on $X \times X$ defined as

$$d_s(x, y) = \begin{cases} 0 & \text{if } x \in X_0 \text{ or } y \in X_0, \\ \rho_s(p_i, p_j) & \text{if } x \in X_i, y \in X_j, \quad 1 \leq i, j \leq k. \end{cases}$$

We will estimate the sum of the L^1 -distances between the pairs of functions d_s, ρ_s on $X \times X$. The measure of the set $(X_0 \times X) \cup (X \times X_0)$ is less than 2ε ; the integral over this set of each of the functions ρ_s does not exceed some value $\delta(\varepsilon)$ which is small provided that ε is small (this is the uniform integrability of M). On $X_i \times X_j$ we have

$$|\rho_s(x, y) - d_s(x, y)| = |\rho_s(x, y) - \rho_s(p_i, p_j)| \leq \rho_s(x, p_i) + \rho_s(y, p_j).$$

We sum these inequalities over $s = 1, \dots, n$. In the right-hand side, the sums $\sum_s \rho_s(x, p_i) = n\rho(x, p_i)$, $\sum_s \rho_s(y, p_j) = n\rho(y, p_j)$ appear, each not exceeding εn . Integrating over $X_i \times X_j$ and summing over all $i, j = 1, 2, \dots, k$ yields

$$\sum_s \iint_{X \times X} |\rho_s - d_s| \leq \delta(\varepsilon)n + 2\varepsilon n.$$

Now assume that $\delta(\varepsilon) + 2\varepsilon < c/10$. Then the estimate $\|\rho_s - d_s\| < c/5$ holds at least for $n/2$ indices s .

Note that all metrics d_s lie in the same space L of piecewise constant functions, which has dimension $k^2 + 1$. Besides, their norms are bounded by a constant depending only on the uniform bound on the norms of semimetrics in M . It follows

that if n is sufficiently large, then among any $n/2$ of these metrics there are two, say d_s, d_t , with distance at most $c/5$ from each other (indeed, otherwise the balls in L of radius $c/10$ centered at these functions would be disjoint and would lie in a ball of a bounded radius, which is impossible for large n from volume considerations; note that the bound on n here depends only on the dimension of the space, but not on its structure). But if $\|\rho_s - d_s\| < c/5$, $\|\rho_t - d_t\| < c/5$, $\|d_s - d_t\| < c/5$, then $\|\rho_s - \rho_t\| < c$, contradicting the assumption. \square

Note that the criterion may be rephrased for not necessarily convex family of semimetrics: ε -entropies of all finite convex combinations must be uniformly bounded, and if it is the case, then the family is precompact. It immediately follows from Theorem 2.21 and the fact that a set in Banach space is precompact if and only if its convex hull is precompact. In the following special case we see that even not all convex combinations are necessary for assuring in precompactness.

Theorem 2.22.

Let (X, ρ) be admissible semimetric triple, T be measure-preserving transform on X (not necessarily invertible). Denote $T^k \rho(x, y) = \rho(T^k x, T^k y)$ and $\rho_n^{\text{av}} = n^{-1} \sum_{k=1}^n T^k \rho$. Assume that for any $\varepsilon > 0$, ε -entropies of semimetrics ρ_n^{av} are uniformly bounded. Then the orbit $\{\rho, T\rho, T^2\rho, \dots\}$ of ρ under action of T is precompact (say, in m -norm).

Proof. Assume the contrary, then for some $\varepsilon > 0$ and some positive integers $n_1 < n_2 < \dots$ the mutual distances between metrics $T^{n_i} \rho$ are not less than ε . We know from the proof of Theorem 2.21 that there exists dimension D depending on ε and, if n is large enough, there exists a subspace L_D of dimension D such that not less than, say, $n/2$ metrics $T^i \rho$, $i = 1, 2, \dots, n$, are $\varepsilon/9$ -close to L_D . Also, a ball of radius, say, $2 \int \rho + 2\varepsilon$ in L_D has $(\varepsilon/9)$ -net of cardinality at most $C = C(\varepsilon, \rho)$. Hence we may find at least $n/2C$ indices $i_1 < i_2 < \dots < i_k \leq n$, $k \geq n/2C$, such that mutual distances between metrics $T^{i_s} \rho$ do not exceed $\varepsilon/3$.

Consider pairs of integers (a, p) , where $1 \leq a \leq k$, $1 \leq p \leq M$, $M = 5C + 1$. Then all sums $i_a + n_p$ are less than $2n$ (if n is large enough), while there are more than $2n$ such sums. Then by pigeonhole principle there exist $i_a < i_b$ and $n_p < n_q$ such that $i_a + n_p = i_b + n_q$. Hence the distance between metrics $T^{i_a} \rho$ and $T^{i_b} \rho$ coincides with the distance between $T^{n_p} \rho$ and $T^{n_q} \rho$, while the latter is not less than ε and the former is not greater than $\varepsilon/3$. A contradiction. \square

2.6. Matrix definitions of admissible metrics

Using Lemma 2.16, one can characterize the admissibility of (summable) metrics in terms of the behavior of the traces of the matrices of block averages of metrics.

Theorem 2.23.

Let ρ be a measurable summable metric defined on a Lebesgue space (X, μ) . Consider a partition λ of X into n sets of equal measure, $X = \bigsqcup_{i=1}^n \Delta_i$, and construct the matrix $A_{\rho, \lambda}$ of averages of ρ over λ :

$$A_{\rho, \lambda}(i, j) = n^2 \int_{\Delta_i \times \Delta_j} \rho d\mu^2.$$

- 1) If $\inf (\text{tr } A_{\rho, \lambda})/n = 0$, where the infimum is taken over all n and over all partitions of X into n parts of equal measure, then the metric ρ is admissible.
- 2) Assume that the metric ρ is admissible and a sequence of partitions $\lambda_1, \lambda_2, \dots$ satisfies the Lebesgue density theorem (i.e., for every measurable subset $Y \subset X$, for almost every point $y \in Y$, the density of Y in the element $\lambda_k(y)$ of the partition λ_k that contains y tends to 1 as $k \rightarrow +\infty$). Then

$$\lim_{k \rightarrow +\infty} \frac{1}{n_k} \text{tr } A_{\rho, \lambda_k} = 0,$$

where n_k is the number of parts in λ_k . This property is satisfied, for example, for a sequence of dyadic partitions, for partitions of an interval into equal subintervals, partitions of a square into equal rectangles, etc.

Proof. 1) If ρ is not admissible, then, by Theorem 2.7, there exist $c > 0$ and a measurable set Y of measure $\mu(Y) \geq c$ such that $\rho(x, y) \geq c$ for almost all pairs $x, y \in Y$. Put $m_k = \mu(\Delta_k \cap Y)$. Then

$$n^2 \int_{\Delta_k^2} \rho d\mu^2 \geq cn^2 m_k^2;$$

summing over k yields

$$\text{tr } A_{\rho, \lambda} \geq cn^2 \sum_{k=1}^n m_k^2 \geq cn \left(\sum_{k=1}^n m_k \right)^2 \geq c^3 n,$$

so that the infimum in question is not less than c^3 , a contradiction.

2) First consider arbitrary ρ and λ . Averaging the triangle inequality $\rho(x, y) \leq \rho(x, z) + \rho(y, z)$ over $x, y \in \Delta_k, z \in \Delta_m$ yields $A_{\rho, \lambda}(k, k) \leq 2A_{\rho, \lambda}(k, m)$. Now, averaging over the pairs k, m , we see that

$$\frac{1}{n} \text{tr } A_{\rho, \lambda} \leq 2\|\rho\|_{L_1}.$$

This immediately implies that

$$\left| \frac{1}{n} \text{tr } A_{\rho, \lambda} - \frac{1}{n} \text{tr } A_{\rho', \lambda} \right| \leq 2\|\rho - \rho'\|_{\text{m}}.$$

Since every summable admissible semimetric can be approximated in the m-norm by its truncations, Lemma 2.16, it suffices to prove the required assertion under the assumption that the semimetric ρ is bounded.

Fix $\varepsilon > 0$ and find a partition $X = \bigsqcup_{i=0}^N X_i$ of X into a set X_0 of measure less than ε and sets X_1, \dots, X_N of ρ -diameter less than ε . That of the sets X_i which contains $y \in X$ will be denoted by $X(y)$, by analogy with $\lambda(y)$. The Lebesgue density theorem (more exactly, its assumption) implies the following: the measure of the set of points y for which

$$\mu(X(y) \cap \lambda_k(y)) \leq \frac{1}{2n_k}$$

tends to zero as k tends to infinity. Take the union of the set of such exceptional y 's with X_0 and call the obtained set Y_0 (here Y_0 depends on k and has measure $< \varepsilon$ for large k). Put $n = n_k$ and denote the elements of the partition λ_k by $\Delta_1, \dots, \Delta_n$. In $\bigcup_{j=1}^n \Delta_j^2 \subset X^2$ consider the set E of points (x, y) such that $x \in Y_0$ or $y \in Y_0$. Obviously, $\mu^2(E) \leq 2\mu(Y_0)/n$. Put $E_1 = \bigcup \Delta_j^2 \setminus E$. Note that on E_1 the semimetric ρ does not exceed ε pointwise. Indeed, let $x, y \in \Delta_j, x, y \notin Y_0, x \in X_i, y \in X_l$. Then necessarily $i = l$, since otherwise summing up the inequalities $\mu(\Delta_j \cap X_s) > 1/(2n)$ for $s = i, l$ leads to a contradiction. Thus

$$\int_{E_1} \rho d\mu^2 \leq \varepsilon \mu^2(E_1) \leq \frac{\varepsilon}{n} \quad \text{and} \quad \int_E \rho d\mu^2 \leq \mu^2(E) \cdot \text{diam}_\rho X \leq \frac{2\varepsilon}{n} \text{diam}_\rho X.$$

Adding these two inequalities and recalling that ε is arbitrary yields

$$\int_{\bigcup \Delta_j^2} \rho d\mu^2 = o\left(\frac{1}{n}\right),$$

as required. □

Let x_1, \dots, x_n be points chosen in X independently at random. The classification theorem [4, 14] says that a metric triple is determined up to isomorphism by the corresponding distribution of the distance matrices $\rho(x_i, x_j)_{1 \leq i, j \leq n}$ for all n . Therefore, the admissibility of a metric must also be expressible in terms of this distribution. Among various ways to give such a description, we confine ourselves to the following one.

Theorem 2.24.

1) If a metric ρ is not admissible, then there exists $c > 0$ such that the probability of the following event tends to one as n tends to infinity:

$$\begin{aligned} & \text{there is a set of indices } I \subset \{1, 2, \dots, n\} \text{ of cardinality} \\ & \text{at least } cn \text{ such that } \rho(x_i, x_j) \geq c \text{ for all distinct } i, j \in I. \end{aligned} \quad (P_c)$$

2) If a metric ρ is admissible, then for every $c > 0$ the probability of (P_c) tends to zero.

In both cases, the rate of convergence to 1 or 0 is at least exponential in n .

Proof. 1) Find $c > 0$ and a measurable set $Y \subset X$ of measure $2c$ such that $\rho(x, y) \geq c$ for almost all pairs $x, y \in Y$. Then on the average, Y contains $2cn$ points among x_1, \dots, x_n , and the probability that the number of such points is at most cn tends to 0 exponentially in n (by standard large deviations estimates in the Law of Large Numbers for Bernoulli independent summands). The probability that a pair of such points is at distance at most ε is zero. Therefore, with probability tending to one exponentially, a required set of indices does exist.

2) Let ρ be an admissible metric. Partition X into a set X_0 of measure $< c/2$ and sets X_1, \dots, X_N of ρ -diameter $\leq c/2$. Note that if a required set of indices I is found, then for every $i = 1, \dots, N$ the point x_k lies in X_i for at most one index $k \in I$. Therefore, for $n > 10N/c$ this implies that at least $2cn/3$ points among x_1, \dots, x_n fall into X_0 . But, again, this happens with probability exponentially small in n . \square

Remark.

In conclusion of this section, we mention an important problem from the theory of metric measure spaces. We define an integral averaging operator as follows. Let $\rho \in L^2_{\mu \times \mu}(X \times X)$. Consider the following linear operator $I_\rho \equiv I$:

$$I(f)(y) \equiv \int_X \rho(x, y) f(x) d\mu(x),$$

where $f \in L^2_\mu(X)$. Roughly speaking, this operator measures the weighted average distance between the points of the space.

Obviously, I is a self-adjoint Hilbert–Schmidt operator in $L^2_\mu(X)$. It is of great interest to study its spectrum and, in particular, the leading eigenvalues. It may happen that some metric invariants of an action of a group G on X can be expressed in terms of joint characteristics of the operator I and the unitary operators U_g , $g \in G$. Since the spectrum of the random distance matrix is a complete invariant of an admissible triple, it is of interest to study this spectrum and compare it with the spectrum of the averaging operator I .

3. The dynamics and ε -entropy of admissible metrics; discreteness of the spectrum

3.1. Scaling entropy and the statement of the discreteness criterion

The theory of admissible metrics and semimetrics which we considered in the [second section](#), being of interest in itself, also leads to new applications to ergodic theory. These applications rely on replacing the dynamics of measure-preserving transformations in the original measure spaces by the dynamics of the associated transformations in the spaces of admissible metrics. This should be compared with the transition $T \mapsto U_T$ from measure-preserving transformations to unitary operators in L^2 in the early 1930s. Let T be a transformation of a Lebesgue space (X, μ) preserving the measure μ ; then we can consider the transformation R_T of the cone of admissible metrics $\mathcal{Adm}(X, \mu)$ defined by the

formula $R_T(\rho)(x, y) = \rho(Tx, Ty)$. The set of the admissible metrics of type $R_T^n(\rho)(x, y) \equiv \rho_n(x, y) = \rho(T^n x, T^n y)$, $n \in \mathbb{Z}$, we called T -orbit of ρ . Introduce the averaging operator $M_n = (1/n) \sum_{k=0}^{n-1} R_T^k$:

$$(M_n \rho)(x, y) = \frac{1}{n} \sum_{k=0}^{n-1} \rho(T^k x, T^k y).$$

It is clear that M_n sends every semimetric ρ to a new semimetric $\rho_n^{\text{av}} = M_n \rho$, and we are interested in the study of its properties as n tends to infinity.

In fact, we study the action of the unitary operator $U_T \otimes U_T$ and averages of its powers. However, the crucial point is that we consider this action on the cone of admissible metrics rather than simply in L^1 . Recall the definition of the scaling entropy of an automorphism introduced in [11, 15], see also [18].

Definition 3.1.

Let T be an automorphism of a Lebesgue space (X, μ) . For an arbitrary $\varepsilon > 0$ and an arbitrary semimetric ρ , we define the *class of scaling sequences* for the automorphism T and the semimetric ρ as the family of all nondecreasing sequences $\{c_n\}$ such that

$$0 < \liminf_{n \rightarrow \infty} \frac{\mathbb{H}_\varepsilon(\rho_n^{\text{av}})}{c_n} \leq \limsup_{n \rightarrow \infty} \frac{\mathbb{H}_\varepsilon(\rho_n^{\text{av}})}{c_n} < \infty.$$

All sequences in the same class are equivalent. If the limit exists, it is called the *scaling ε -entropy* of T with respect to the semimetric ρ and scaling sequence $\{c_n\}$. Finally, if the limit of these ε -entropies as $\varepsilon \rightarrow 0$ exists with some normalization in ε , then it is called the *scaling entropy* of T (with respect to the semimetric ρ , scaling sequence and normalization).

In the calculations performed so far in concrete examples, the latter limit does exist and does not depend on the choice of an admissible metric. A special role is played by the class of bounded nondecreasing scaling sequences. The main result of this paper is the following theorem.

Theorem 3.2.

Let T be a measure-preserving automorphism of a Lebesgue space (X, μ) . Then the following conditions are equivalent:

- 1) T has a purely discrete spectrum.
- 2) For every admissible semimetric $\rho \in L^1(X^2)$ and every $\varepsilon > 0$, the scaling sequences are bounded.
- 3) For some admissible metric $\rho \in L^1(X^2)$ and every $\varepsilon > 0$, the scaling sequences are bounded.

Remark 3.3.

By individual ergodic theorem, the limiting average semimetric

$$\rho^{\text{av}} = \lim_{n \rightarrow \infty} \rho_n^{\text{av}}(x, y)$$

does exist almost everywhere. The results of Section 2 show that it is admissible if and only if for any $\varepsilon > 0$ the scaling sequences of ρ_n^{av} are uniformly bounded by n (the “if” part follows from Lemma 2.13, “only if” part from Corollary 2.20). It allows to reformulate Theorem 3.2, replacing condition 2) to: *For every admissible semimetric $\rho \in L^1(X^2)$ the limiting average metric ρ^{av} is admissible*, and analogously for condition 3).

The implication $2) \Rightarrow 3)$ is trivial, and the proof of the other two ones is given below; the proof relies on the obtained results on admissible metrics.

3.2. Proof of the main theorem; the implication 1) \Rightarrow 2)

Here we use the result obtained in the [second section](#) on the precompactness of a family of admissible metrics in the m -norm. Since the automorphism T has purely discrete spectrum, its tensor square $T^{\otimes 2}$ (acting on $X \times X$) also has purely discrete spectrum. It implies that the $T^{\otimes 2}$ -orbit of any function $f \in L^2(X \times X)$ is precompact. Take any admissible semimetric ρ on X . Our nearest goal is to prove that the T -orbit of ρ is precompact in $L^1(X \times X)$.

Assume the contrary, then for some $c > 0$ and some infinite subset $\mathcal{N} \subset \mathbb{N}$ we have $\|\rho_n - \rho_k\| \geq c$ for all distinct $n, k \in \mathcal{N}$. Choose large $M > 0$ so that $\|\rho - \rho^M\| < c/3$, where ρ^M is the truncation of ρ at level M . Since taking truncation commutes with action of T , we get

$$\|\rho_n^M - \rho_k^M\| \geq \|\rho_n - \rho_k\| - \|\rho_n - \rho_n^M\| - \|\rho_k - \rho_k^M\| \geq c - c/3 - c/3 = c/3$$

for all $n, k \in \mathcal{N}$. Hence for a bounded metric ρ^M its T -orbit also has a separated infinite subset. But it belongs to $L^2(X \times X)$, hence its orbit is precompact even in L^2 , and so in L^1 . A contradiction.

So we see that the T -orbit of ρ is precompact in L^1 , hence its closure in L^1 is compact. But ρ is admissible, hence by [Lemma 2.13](#) this closure contains only admissible metrics. Then it is compact also in m -norm by [Corollary 2.18](#). So, its convex hull is precompact in m -norm. Hence ε -entropies of the metrics from this convex hull are uniformly bounded by [Corollary 2.20](#), as desired.

Remark 3.4.

Actually, the following more general fact is proved. Discreteness of spectrum of T implies that ε -entropies of all convex combinations of semimetrics in the T -orbit of a given admissible semimetric ρ are uniformly bounded (but not only for averages over initial segments).

A typical and, by von Neumann's classical theorem, general example of a transformation with discrete spectrum is a rotation on a compact abelian group. By [Remark 3.3](#) and already proved part of [Theorem 3.2](#) we see that averaged (over orbit of the rotation) metric is then admissible. It is clear that instead of averaging over orbit of a rotation we can consider the averaging over the closure of the orbit, which coincides with the whole group in the ergodic case. Below we prove the analog of this fact for general (not necessary Abelian) compact group. The proof is very similar to the above proof of part of [Theorem 3.2](#). Also, we prove that an admissible rotation-invariant metric must be continuous.

Proposition 2.

For an arbitrary admissible metric ρ on a compact group G endowed with Haar measure, the average of the metric ρ with respect to the compact subgroup of the group of translations is admissible. The average over whole group is, moreover, invariant, and hence continuous.

Proof. Note that the map $G \rightarrow L^1(G^2)$, $g \mapsto \rho_g(x, y) = \rho(gx, gy)$, is continuous (by continuity of rotation in mean). Hence its image I is compact in L^1 . Then it is compact also in m -norm by [Corollary 2.18](#). Then its convex hull is precompact in m -norm and so ε -entropies of its elements are uniformly bounded by [Corollary 2.20](#). The averaged metric $\int_H \rho_g d\mu_H$ (where H is a compact subgroup of G , μ_H is a Haar measure on H) lies in the closed (say, in L^1) convex hull of I and hence is admissible by [Lemma 2.13](#). Now we will show that the averaged metric over whole G is continuous. Since this metric is translation-invariant, it suffices to prove that it is continuous at unity. The admissibility criterion, [Theorem 2.7](#), says that for almost all $x \in G$, the ball $B = \{y \in G : \rho(x, y) \leq r\}$ of radius $r > 0$ centered at x has positive measure. But then by Steinhaus theorem, see, for example [\[9\]](#), the set $B \cdot B^{-1}$ contains a neighborhood of unity, and for every $z \in B \cdot B^{-1}$, by the triangle inequality and the invariance of ρ , we have $\rho(1, z) \leq 2r$, which proves that the metric is continuous at unity. \square

3.3. Proof of the implication 3) \Rightarrow 1)

Now we will prove the implication 3) \Rightarrow 1): if there exists an admissible metric ρ such that the corresponding class of scaling sequences consists of bounded sequences for every $\varepsilon > 0$, then the automorphism T has a purely discrete spectrum. Clearly, one may assume that ρ is bounded by replacing it with a truncation if necessary.

We use the following known criterion of discreteness of spectrum for a unitary operator U in Hilbert space: *U-orbit of any element is precompact*. This is a corollary of the spectral theorem for unitary operator. Recall that a slightly more general fact is true: *U-orbit of x is precompact if and only if x lies in the closed span of eigenvectors of U* . Finally, for the unitary operator corresponding to the automorphism T on the Lebesgue space (X, \mathfrak{A}, μ) , this closed span is a space of functions in L^2 , measurable w.r.t. some σ -subalgebra $\mathfrak{B} \subset \mathfrak{A}$ (or, in other words, the space of functions, constant on almost all parts of some measurable partition ξ). For the square-summable function of two variables $f(x, y)$ on $X \times X$, the precompactness of its T -orbit $\{f(T^n x, T^n y) : n = 1, 2, \dots\}$ therefore implies that f is measurable with respect to the sub-algebra \mathfrak{B}^2 . In particular, for almost all x functions $f(x, \cdot)$ are \mathfrak{B} -measurable and for almost all parts of the corresponding partition ξ the functions $f(x, \cdot)$ coincide a.e. for a.e. x from this part. Assume that it holds for the bounded (or just square summable) admissible metric $f = \rho$. But then for any two points u, v the functions $f(u, \cdot)$ and $f(v, \cdot)$ are different on the ball $B(u, \rho(u, v)/3)$, which has positive measure for almost all u by Theorem 2.7. In other words, for almost all u there is no v such that functions $f(u, \cdot)$ and $f(v, \cdot)$ coincide a.e. (Such functions are called in [12] “pure functions of two variables”, this property is important in the classification theorem.) It implies that partition ξ is trivial and so the spectrum of T is purely discrete.

Now, in order to finish the proof of implication 1) \Rightarrow 3) it suffices to combine the above general techniques and Theorem 2.22.

3.4. Further remarks

3.4.1. Relation to A-entropy

In [6], another discreteness criterion for the spectrum of an automorphism was proved; it is also based on the notion of entropy (in that case, sequential, or A-, or Kirillov–Kushnirenko entropy). According to this criterion, the spectrum of an automorphism T is discrete if and only if

$$\limsup_{n \rightarrow \infty} \frac{1}{n} H \left(\prod_{k=1}^n T^{i_k} \xi \right) = 0 \quad (1)$$

for every finite partition ξ and an arbitrary sequence $i_1 < i_2 < \dots$ of positive integers. Here $H(\cdot)$ is the entropy of a finite partition. One can easily check that the entropy $H(\cdot)$ in criterion (1) can be replaced with the ε -entropy (when $\varepsilon > 0$ takes all positive values). Kushnirenko's proof is based on the following two reductions.

- (I) The spectrum of T is discrete if and only if the set of partitions $\{T^n \xi : n = 1, 2, \dots\}$ is precompact with respect to some natural metric on partitions. Since the number of parts in the partition $T^n \xi$ is fixed, various natural metrics turn out to be equivalent. For our purposes, it is convenient to consider the distance in L^1 or in the m-norm between the block semimetrics corresponding to partitions.
- (II) Such a family is precompact if and only if the normalized entropies (1) tend to zero.

The product of partitions appearing in (1) corresponds to the maximum of the associated block metrics. However, our main Theorem 3.2 involves averages of semimetrics. So, the precompactness criterion (II) is to be compared with our Theorem 2.21. It is not a complete analog of Kushnirenko's criterion: first, it applies to general admissible semimetrics; second, deals with averages rather than maxima; third, uses the entropy of metrics rather than partitions. In the particular case, where one deals with the convex hull of a family of cut semimetrics corresponding to partitions into two parts of equal measure, Kushnirenko's criterion follows from the condition of criterion in Theorem 2.21. In the general situation, the relation between two criteria is not quite clear; for instance, we do not know any exact generalization of Kushnirenko's criterion to the case of general semimetrics.

3.4.2. Conjectures

The asymptotics of the scaling entropy for an arbitrary automorphism is not known. Most probably, in the other extreme case, i.e., for actions with positive Kolmogorov entropy, the answer can be obtained in the same way as in the discrete spectrum case. Namely, we state the following conjecture.

Conjecture 3.5.

For any automorphism T with positive entropy, the scaling sequence has order n . In other words, for every admissible metric ρ ,

$$\lim_{\epsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{\mathbb{H}_\epsilon(\rho_n^T)}{\phi(\epsilon)n} = h(T),$$

where $\phi(\epsilon)$ is a function, possibly depending on ρ , and $h(T)$ is the classical entropy of T .

In [15] we formulate a weaker conjecture that the equality is true for generic admissible metric. But it seems that using Shannon–McMillan–Breiman theorem it is possible to prove the above conjecture.

As to zero entropy — it is not yet known what intermediate — between bounded and linear — growth the scaling sequences for automorphisms can have. Most probably, logarithmic growth with different bases can be achieved (for oricycles, adic transformations, etc.). For arbitrary groups, the growth of scaling sequences lies between bounded growth and the growth of the number of words of given length in the group. For the groups $\sum_1^\infty \mathbb{Z}_p$, examples are already found in [11, 18] where the scaling entropy grows as an arbitrary integer power of the logarithm of the number of words of given length. It is still plausible that the growth does not depend on the choice of admissible metric.

However, recall that entropy characteristics are just the simplest (“unary,” or “dimensional”) invariants of the dynamics of metrics. There are other asymptotic invariants of the sequence of average metrics with respect to automorphism.

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