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Milnor fibration at infinity for mixed polynomials

Research Article

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Abstract: We study the existence of Milnor fibration on a big enough sphere at infinity for a mixed polynomial $f\colon \mathbb{R}^{2n} \to \mathbb{R}^2$. By using strongly non-degenerate condition, we prove a counterpart of Némethi and Zaharia's fibration theorem. In particular, we obtain a global version of Oka's fibration theorem for strongly non-degenerate and convenient mixed polynomials.

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1. Introduction

In the local case of germs of holomorphic polynomial functions with isolated singularities, it is well known that there exists a locally trivial fibration $\varphi = f/|f|\colon S_r \setminus f^{-1}(0) \to S^1$ in a sufficiently small sphere which is called a Milnor fibration [9]. Unfortunately, in the global case of holomorphic polynomials, the Milnor fibration $f/|f|$ at infinity does not exist in general. In some special cases, $f\colon \mathbb{C}^n \to \mathbb{C}$ has no atypical values at infinity, for instance: "convenient polynomials with non-degenerate Newton principal part at infinity" (Kouchnirenko [8]), polynomials which are "tame" (Broughton [3, 4]), "quasi-tame" (Némethi [10, 11]). In these cases, the Milnor fibration $f/|f|$ at infinity exists in a sufficiently large sphere which is equivalent to the fibration $f\colon f^{-1}(S^1_R) \to S^1_r$ for R sufficiently large. In [13], Némethi and Zaharia considered a special class of holomorphic polynomials called "semitame" whose atypical values are contained in $\{0\}$. It was shown that for semitame polynomials, the Milnor fibration at infinity exists. When $n = 2$, Bodin in [2] proved that the Milnor fibration at infinity exists if and only if f is semitame. Recently, Oka introduced the notion of "mixed polynomials" which is a polynomial function $\mathbb{C}^n \to \mathbb{C}$ with variables $\mathbf{z}$ and $\bar{\mathbf{z}}$, i.e. a real polynomial application $\mathbb{R}^{2n} \to \mathbb{R}^2$. By defining non-degeneracy conditions for mixed function germs, Oka showed in [15, Theorem 29, 33, 36] that for a strongly

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non-degenerate convenient mixed function germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, there exist positive numbers r_0, δ_0 and $\delta \ll \delta_0$, such that for any $r \leq r_0$,

$$f\colon f^{-1}(D^*_\delta) \cap B^{2n}_r \to D^*_\delta$$

is locally trivial fibration and the topological isomorphism class does not depend on the choice of r and δ_0. Moreover, $\varphi = f/|f|\colon S^{2n-1}_r \setminus K_r \to S^1$ also is a locally trivial fibration which is equivalent to the above fibration. For mixed polynomials, one can also ask under which condition does the Milnor fibration $f/|f|$ at infinity exist? In this paper, we present an approach to this problem by using the strong non-degeneracy condition at infinity defined in [6]. Consider a mixed polynomial $f\colon \mathbb{C}^n \to \mathbb{C}$. Inspired by Oka's construction in the local case, we prove a similar result in the global setting.

Theorem 1.1.
If f is a Newton strongly non-degenerate mixed polynomial, then there exist $\delta_0 > 0$ and $R_0 > 0$ sufficiently large such that for any $\delta \geq \delta_0$ and $R > R_0$,

$$\frac{f}{|f|}\Big|\colon S^{2n-1}_R \setminus f^{-1}(D_\delta) \to S^1$$

is a locally trivial fibration for $R \geq R_0$ and it is equivalent to the global fibration $f_|\colon f^{-1}(S^1_\delta) \to S^1_\delta$.

In the above theorem, we do not put any other conditions on atypical values of f: unlike in the semitame setting of holomorphic case, we take a big enough disk D_δ in order to bound all the atypical values of f. As a consequence of the above theorem, we get the following global version of [15, Theorems 29, 33, 36].

Corollary 1.2.
If f is a Newton strongly non-degenerate convenient mixed polynomial, then there exists $R_0 > 0$ sufficiently large such that for all $R \geq R_0$ the Milnor fibration at infinity

$$\frac{f}{|f|}\Big|\colon S^{2n-1}_R \setminus K \to S^1$$

exists and it is equivalent to the global fibration $f_|\colon f^{-1}(S^1_\delta) \to S^1_\delta$, where $\delta > 0$ is sufficiently large.

We will review some basic definitions and properties of mixed polynomials in Section 2. In order to get an effective estimation of atypical values of $f/|f|$, we define the ρ-regularity for $f/|f|$ in Section 3, which allows us to get a type of formulation like [6, Theorem 1.1]. The proofs of Theorem 1.1 and Corollary 1.2 will be given in Section 4. Our Example 4.3 shows that the semitame condition is not sufficient to insure the existence of the Milnor fibration $f/|f|$ at infinity in the mixed setting.

2. Preliminaries

2.1. Mixed singularity and homogeneous polynomials

Let $f = (g, h)\colon \mathbb{R}^{2n} \to \mathbb{R}^2$ be a polynomial application, where $g(x_1, \dots, y_n)$ and $h(x_1, \dots, y_n)$ are real polynomials. By writing $\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathbb{C}^n$, where $z_k = x_k + iy_k$ for $k = 1, 2 \dots n$, we get a polynomial function $f\colon \mathbb{C}^n \to \mathbb{C}$ in variables $\mathbf{z}$ and $\bar{\mathbf{z}}$, namely,

$$f(\mathbf{z}, \bar{\mathbf{z}}) = g\left(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2i}\right) + ih\left(\frac{\mathbf{z}+\bar{\mathbf{z}}}{2}, \frac{\mathbf{z}-\bar{\mathbf{z}}}{2i}\right),$$

and reciprocally for a polynomial function $f\colon \mathbb{C}^n \to \mathbb{C}$ in variables $\mathbf{z}$ and $\bar{\mathbf{z}}$, we can consider it as a polynomial application $(\operatorname{Re} f, \operatorname{Im} f)$. Then f is called a *mixed polynomial*, after [15]. We write f as follows:

$$f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu} \mathbf{z}^{\nu} \bar{\mathbf{z}}^{\mu} \tag{1}$$

where $c_{\nu,\mu} \neq 0$, $\mathbf{z}^{\nu} = z_1^{\nu_1} \cdots z_n^{\nu_n}$ and $\bar{\mathbf{z}}^{\mu} = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n}$ for n-tuples $\nu = (\nu_1, \ldots, \nu_n)$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n$. In the sequel, given a mixed polynomial f, we consider f as in the form of (1). For a mixed polynomial f, we shall often use derivation with respect to $\mathbf{z}$ and $\bar{\mathbf{z}}$ and use the following notation:

$$\mathrm{d}f = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right), \qquad \bar{\mathrm{d}}f = \left(\frac{\partial f}{\partial \bar{z}_1}, \ldots, \frac{\partial f}{\partial \bar{z}_n} \right).$$

Definition 2.1.

We call w a mixed singularity of $f\colon \mathbb{C}^n \to \mathbb{C}$, if w is a critical point of the mapping $f = (g, h)\colon \mathbb{R}^{2n} \to \mathbb{R}^2$.

By abuse of notation, we continue to denote the set of mixed singularities for a mixed polynomial f by $\operatorname{Sing} f$. The next proposition gives us a straight way to calculate the locus of mixed singularities.

Proposition 2.2 ([15, Proposition 1]).

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a mixed polynomial. Then $w \in \mathbb{C}^n$ is a mixed singularity of f if and only if there exists a complex number λ with $|\lambda| = 1$ such that $\overline{\mathrm{d}f(\mathbf{w}, \bar{\mathbf{w}})} = \lambda \bar{\mathrm{d}} f(\mathbf{w}, \bar{\mathbf{w}})$.

For mixed polynomials, we have two notions of homogeneous polynomials introduced in [7, 14].

Definition 2.3.

A mixed polynomial $f\colon \mathbb{C}^n \to \mathbb{C}$ is called *radial weighted homogeneous* if there exist n integers $q_1, \ldots, q_n$ with $\gcd(q_1, \ldots, q_n) = 1$ and a positive integer m_r such that $\sum_{j=1}^{n} q_j(\nu_j + \mu_j) = m_r$ for every n-tuples ν and μ with $c_{\nu,\mu} \neq 0$. We call $(q_1, \ldots, q_n)$ the radial weight of f and m_r the radial degree of f. More precisely, f is radial weighted homogeneous of type $(q_1, \ldots, q_n; m_r)$ if and only if it verifies the following equation for all $t \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$:

$$f(t \circ \mathbf{z}) = f\left(t^{q_1} z_1, \ldots, t^{q_n} z_n, t^{q_1} \bar{z}_1, \ldots, t^{q_n} \bar{z}_n\right) = t^{m_r} f(\mathbf{z}, \bar{\mathbf{z}}).$$

From Definition 2.3, we see that if $f = (g, h)\colon \mathbb{R}^{2n} \to \mathbb{R}^2$ is a radially weighted homogeneous mixed polynomial, then g and h are real weighted homogeneous polynomials with the same weights and degrees as f, *i.e.*, $\operatorname{weight} x_j = \operatorname{weight} x_j = q_j$.

Definition 2.4.

A mixed polynomial $f\colon \mathbb{C}^n \to \mathbb{C}$ is called *polar weighted homogeneous* if there exist n integers $p_1, \ldots, p_n$ with $\gcd(p_1, \ldots, p_n) = 1$ and a non-negative integer m_p such that $\sum_{j=1}^{n} p_j(\nu_j - \mu_j) = m_p$ for every n-tuples ν and μ. We call $(p_1, \ldots, p_n)$ the polar weight of f and m_r the polar degree of f. More precisely, f is polar weighted homogeneous of type $(p_1, \ldots, p_n; m_p)$ if and only if it verifies the following equation for all $\lambda \in S^1$:

$$f(\lambda \circ \mathbf{z}) = f\left(\lambda^{p_1} z_1, \ldots, \lambda^{p_n} z_n, \lambda^{-p_1} \bar{z}_1, \ldots, \lambda^{-p_n} \bar{z}_n\right) = \lambda^{m_p} f(\mathbf{z}, \bar{\mathbf{z}}).$$

Example 2.5.

Let $f, g\colon \mathbb{C}^2 \to \mathbb{C}$, $f(x, y) = |x|^2 + |y|^2$ and $g(x, y) = x^2 + x^4 \bar{y}^2 + y^2$. We see that f is a radial weighted homogeneous polynomial of radial weight $(1, 1)$ and degree 2, but f is not polar weighted homogeneous; and g is a polar weighted homogeneous polynomial of polar weight $(1, 1)$ and degree 2, but g is not radial weighted homogeneous.

2.2. Newton non-degeneracy at infinity

In this section, we review the definitions of Newton polyhedron and non-degeneracy conditions introduced in [6]. Let f be a mixed polynomial.

Definition 2.6.
We call $\operatorname{supp} f = \{\nu + \mu \in \mathbb{N}^n : c_{\nu,\mu} \neq 0\}$ the *support* of f. We denote by $\overline{\operatorname{supp} f}$ the convex hull of the set $\operatorname{supp} f \setminus \{0\}$. We say that f is *convenient* if the intersection of $\operatorname{supp} f$ with each coordinate axis is non-empty. The *Newton polyhedron* of a mixed polynomial f, denoted by $\Gamma_0(f)$, is the convex hull of the set $\{0\} \cup \operatorname{supp} f$. The *Newton boundary at infinity*, denoted by $\Gamma^+(f)$, is the union of the faces of the polyhedron $\Gamma_0(f)$ which do not contain the origin. By the "face" we mean a face of any dimension.

Definition 2.7.
For any face Δ of $\overline{\operatorname{supp} f}$, we denote the restriction of f to $\Delta \cap \operatorname{supp} f$ by $f_\Delta = \sum_{\nu+\mu\in\Delta\cap\operatorname{supp} f} c_{\nu,\mu}\mathbf{z}^\nu\bar{\mathbf{z}}^\mu$. The mixed polynomial f is called *non-degenerate* if $\operatorname{Sing} f_\Delta \cap f_\Delta^{-1}(0) \cap \mathbb{C}^{*n} = \emptyset$, for each face Δ of $\Gamma^+(f)$. We say that f is *Newton strongly non-degenerate* if $\operatorname{Sing} f_\Delta \cap \mathbb{C}^{*n} = \emptyset$ for any face Δ of $\Gamma^+(f)$.

It is easily seen that these two non-degeneracy conditions are not equivalent, but they coincide in the holomorphic setting. Let us recall the definition of bad faces for mixed polynomials. (See also [6, 12].)

Definition 2.8.
A face $\Delta \subseteq \overline{\operatorname{supp} f}$ is called *bad* if

(i) there exists a hyperplane $H \subset \mathbb{R}^n$ with equation $a_1x_1 + \cdots + a_nx_n = 0$ (where $x_1, \ldots, x_n$ are the coordinates of $\mathbb{R}^n$) such that

- (i_1) there exist i and j with $a_i < 0$ and $a_j > 0$,
- (i_2) $H \cap \overline{\operatorname{supp} f} = \Delta$.

A bad face Δ is called *strictly bad* if it satisfies in addition the following:

(ii) the affine subspace of the same dimension spanned by Δ contains the origin.

The sets of bad and strictly bad faces of $\overline{\operatorname{supp} f}$ will be denoted respectively by $\mathfrak{B}$ and $\mathfrak{S}\mathfrak{B}$

Let us review here the notions of Milnor set and asymptotic ρ-non-regular values.

Definition 2.9.
The *Milnor set* of a mixed polynomial f is

$$M(f) = \left\{\mathbf{z} \in \mathbb{C}^n : \text{ there exist } \lambda \in \mathbb{R} \text{ and } \mu \in \mathbb{C}^* \text{ such that } \lambda\mathbf{z} = \mu\overline{\mathrm{d}f}(\mathbf{z},\bar{\mathbf{z}}) + \bar{\mu}\bar{\mathrm{d}}f(\mathbf{z},\bar{\mathbf{z}})\right\}.$$

Definition 2.10.
The set of *asymptotic ρ-nonregular values* of a mixed polynomial f is

$$S(f) = \left\{c \in \mathbb{C} : \text{ there exists } \{\mathbf{z}_k\}_{k\in\mathbb{N}} \subset M(f), \text{ such that } \lim_{k\to\infty} \|\mathbf{z}_k\| = \infty \text{ and } \lim_{k\to\infty} f(\mathbf{z}_k,\bar{\mathbf{z}}_k) = c\right\}.$$

3. Approximation of atypical values of $f/|f|$

Let us denote by φ the function $f/|f|\colon \mathbb{C}^n \setminus V(f) \to S^1$ where $V(f) = f^{-1}(0)$.

Lemma 3.1.

For $\mathbf{z} \in \mathbb{C}^n \setminus V(f)$, the fibre $\varphi^{-1}(\varphi(\mathbf{z}, \bar{\mathbf{z}}))$ does not intersect transversely the sphere $S^{2n-1}_{\|\mathbf{z}\|}$ at $\mathbf{z} \in \mathbb{C}^n$ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$\lambda \mathbf{z} = i\bar{f}\,\bar{\mathrm{d}}f(\mathbf{z}, \bar{\mathbf{z}}) - if\,\overline{\mathrm{d}f}(\mathbf{z}, \bar{\mathbf{z}}). \tag{2}$$

In particular, $\operatorname{Sing}\varphi = \big\{\mathbf{z} \in \mathbb{C}^n \setminus V(f) : \bar{f}\,\bar{\mathrm{d}}f(\mathbf{z}, \bar{\mathbf{z}}) = f\,\overline{\mathrm{d}f}(\mathbf{z}, \bar{\mathbf{z}})\big\}$.

Proof. Observe first the normal vector of φ is linearly dependent to the normal vector of $-\mathrm{Re}\,(i\log f)$ over $\mathbb{R}$. Therefore instead of φ, we consider the non-transversality for the function $-\mathrm{Re}\,(i\log f)$. By [6, Lemma 2.1], the non-transversality of the fiber $\varphi^{-1}(\varphi(\mathbf{z}, \bar{\mathbf{z}}))$ and the sphere $S^{2n-1}_{\|\mathbf{z}\|}$ implies

$$\gamma \mathbf{z} = -\mu\,\overline{\mathrm{d}\mathrm{Re}\,(i\log f)}(\mathbf{z}, \bar{\mathbf{z}}) - \bar{\mu}\,\bar{\mathrm{d}}\mathrm{Re}\,(i\log f)(\mathbf{z}, \bar{\mathbf{z}}) \tag{3}$$

for some $\gamma \in \mathbb{R}$ and $\mu \in \mathbb{R}^*$. By definition of $\bar{\mathrm{d}}\mathrm{Re}\,(i\log f)$ and $\overline{\mathrm{d}\mathrm{Re}\,(i\log f)}$, we have

$$-\bar{\mathrm{d}}\mathrm{Re}\,(i\log f) = i\,\frac{\bar{\mathrm{d}}f(\mathbf{z}, \bar{\mathbf{z}})}{f}, \qquad -\overline{\mathrm{d}\mathrm{Re}\,(i\log f)} = -i\,\frac{\overline{\mathrm{d}f}(\mathbf{z}, \bar{\mathbf{z}})}{\bar{f}}.$$

Multiplying the two sides of (3) by $|f|^2$, we conclude (2), where $\lambda = (\gamma/\mu)|f|^2 \in \mathbb{R}$. In particular, taking $\lambda = 0$ in (2), we obtain $\operatorname{Sing}\varphi$. □

Combining with the above lemma, we are led to definition of ρ-regularity for φ. (In general case, this regularity condition is defined in [1]).

Definition 3.2.

We call by the *ρ-non-regular locus* of φ the semi-algebraic set

$$M(\varphi) = \big\{\mathbf{z} \in \mathbb{C}^n \setminus V(f) : \text{there exists } \lambda \in \mathbb{R} \text{ such that } \lambda\mathbf{z} = i\bar{f}\,\bar{\mathrm{d}}f(\mathbf{z}, \bar{\mathbf{z}}) - if\,\overline{\mathrm{d}f}(\mathbf{z}, \bar{\mathbf{z}})\big\}.$$

and we call by *asymptotic ρ-non-regular values* of $f/|f|$ the set

$$S(\varphi) = \Big\{c \in S^1 : \text{there exists } \{\mathbf{z}_k\}_{k\in\mathbb{N}} \subset M(\varphi) \text{ such that } \lim_{k\to\infty} \|\mathbf{z}_k\| = \infty \text{ and } \lim_{k\to\infty} \varphi(\mathbf{z}_k, \bar{\mathbf{z}}_k) = c\Big\}.$$

Recall the notation of Milnor set $M(f)$ and the asymptotic ρ-non-regular set $S(f)$. The above definition enables us to obtain the following structure result for $S(\varphi)$.

Lemma 3.3.

$S(\varphi)$ is semi-algebraic and $M(\varphi) \subset M(f) \setminus V(f)$.

Proof. The inclusion $M(\varphi) \subset M(f) \setminus V(f)$ follows from Definitions 2.9 and 3.2. Since $M(\varphi)$ is a semi-algebraic set, we now proceed analogously to the proof of [6, Proposition 2.1] and we see that $S(\varphi)$ is semi-algebraic. □

Our next proposition shows that under some homogeneous condition, $\operatorname{Sing}\varphi$ can be equal to $M(\varphi)$.

Proposition 3.4.

If f is a mixed radial weighted homogeneous polynomial and is not constant, then $\operatorname{Sing}\varphi = \operatorname{Sing} f \setminus V(f) = M(\varphi)$.

Proof. Let us denote the radial weights of f by $q_1, \dots, q_n$ and the radial degree of f by m_r, where $q_1, \dots, q_n \in \mathbb{Z}$ and $m_r \neq 0$. First, we have $\operatorname{Sing}\varphi \subset \operatorname{Sing} f \setminus V(f)$ and $\operatorname{Sing}\varphi \subset M(\varphi)$. To prove the equality, let $\mathbf{a} \in \operatorname{Sing} f$ and $f(\mathbf{a}, \bar{\mathbf{a}}) \neq 0$. Therefore there exists $\lambda \in S^1$ such that for $1 \leq i \leq n$,

$$\overline{\frac{\partial f}{\partial z_i}}(\mathbf{a}, \bar{\mathbf{a}}) = \lambda \frac{\partial f}{\partial \bar{z}_i}(\mathbf{a}, \bar{\mathbf{a}}). \tag{4}$$

Since f is radial weighted homogeneous, by Euler's lemma, we have

$$\sum_{i=1}^{n} q_i a_i \frac{\partial f}{\partial z_i}(\mathbf{a}, \bar{\mathbf{a}}) + \sum_{i=1}^{n} q_i \bar{a}_i \frac{\partial f}{\partial \bar{z}_i}(\mathbf{a}, \bar{\mathbf{a}}) = m_r f(\mathbf{a}, \bar{\mathbf{a}}). \tag{5}$$

Let

$$A = \sum_{i=1}^{n} q_i a_i \frac{\partial f}{\partial z_i}(\mathbf{a}, \bar{\mathbf{a}}), \qquad B = \sum_{i=1}^{n} q_i \bar{a}_i \frac{\partial f}{\partial \bar{z}_i}(\mathbf{a}, \bar{\mathbf{a}}).$$

Multiplying (4) by $q_i \bar{a}_i$, we obtain

$$\bar{A} = \lambda B \tag{6}$$

which implies $A\bar{A} = B\bar{B}$ since $\lambda \in S^1$. From (5), (6) and $f(\mathbf{a}, \bar{\mathbf{a}}) \neq 0$, we therefore get $AB \neq 0$. Consequently,

$$\frac{\overline{f(\mathbf{a}, \bar{\mathbf{a}})}}{f(\mathbf{a}, \bar{\mathbf{a}})} = \frac{\bar{A} + \bar{B}}{A + B} = \frac{B\bar{A} + B\bar{B}}{B(A+B)} = \frac{B\bar{A} + A\bar{A}}{B(A+B)} = \lambda$$

which proves that $\mathbf{a} \in \operatorname{Sing}\varphi$ from (4). Thus, we have $\operatorname{Sing}\varphi = \operatorname{Sing} f \setminus V(f)$. Using Euler vector field as in the proof of [1, Proposition 3.2], we have $M(\varphi) \subset \operatorname{Sing} f \setminus V(f)$. □

For simplicity of notation, we write $\varphi_\triangle = f_\triangle / |f_\triangle|$ for the restriction of $f/|f|$, where $\triangle$ is a face of $\overline{\operatorname{supp} f}$.

Theorem 3.5.

If f is Newton strongly non-degenerate at infinity for any face of $\overline{\operatorname{supp} f}$, then $M(\varphi)$ is bounded and $S(\varphi) = \emptyset$.

Proof. Assume that $M(\varphi)$ is not bounded, then by Curve selection lemma at infinity [13, Lemma 2], [6, Lemma 2.3], there exists $\mathbf{z}(t)$ of $M(\varphi)$ a real analytic path defined on a small enough interval $]0, \varepsilon[$ such that $\|\mathbf{z}(t)\| \to \infty$ as $t \to 0$. Since $\mathbf{z}(t) \subset M(\varphi)$, there exists a real analytic curve $\lambda(t)$, such that for $t \in]0, \varepsilon[$ we have

$$\lambda(t)\mathbf{z}(t) = i\bar{f}\,\overline{\mathrm{d}}f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) - if\,\overline{\mathrm{d}f}(\mathbf{z}(t), \bar{\mathbf{z}}(t)). \tag{7}$$

Suppose here $\lambda(t) \not\equiv 0$ and let $I = \{i : z_i(t) \not\equiv 0\}$. Then $I \neq \emptyset$ since $\|\mathbf{z}(t)\| \to \infty$, $t \to 0$. Assuming that $I = \{1, \dots, m\}$, we write the expansions of $f(\mathbf{z}(t), \bar{\mathbf{z}}(t))$, $\mathbf{z}(t)$ and $\lambda(t)$ explicitly as follows:

$$z_i(t) = a_i t^{p_i} + \text{h.o.t.}, \qquad \text{where} \quad a_i \neq 0, \quad p_i \in \mathbb{Z}, \quad 1 \leq i \leq m, \tag{8}$$

$$f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = \begin{cases} bt^\delta + \text{h.o.t.} & \text{if} \quad \lim_{t\to 0} f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = 0 \quad \text{or} \quad \infty; \qquad \text{where} \quad b \in \mathbb{C}^*, \quad \delta \neq 0, \\ c + bt^\delta + \text{h.o.t.} & \text{if} \quad \lim_{t\to 0} f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = c; \qquad \text{where} \quad c, b \in \mathbb{C}^*, \quad \delta > 0. \end{cases} \tag{9}$$

$$\lambda(t) = \lambda_0 t^\gamma + \text{h.o.t.}, \qquad \text{where} \quad \lambda_0 \in \mathbb{R}^*, \quad \gamma \in \mathbb{Z}, \quad \lambda(t) \in \mathbb{R}. \tag{10}$$

Set $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{C}^{*I}$, $\mathbf{P} = (p_1, \dots, p_m) \in \mathbb{R}^m$ and consider the linear function $l_{\mathbf{P}} = \sum_{i=1}^{m} p_i x_i$ defined on $\overline{\operatorname{supp} f^I}$. Let $\triangle$ be the *maximal face* of $\overline{\operatorname{supp} f^I}$ where $l_{\mathbf{P}}$ takes its minimal value, say this value is $d_{\mathbf{P}}$. We have

$$f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) = f^I_\triangle(\mathbf{a}, \bar{\mathbf{a}})\, t^{d_{\mathbf{P}}} + \text{h.o.t.} \tag{11}$$

Let us discuss the following two cases:

(I) If $f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) \to 0$ or ∞ as $t \to 0$, we get $d_{\mathrm{P}} \leq \delta$. Since $\|\mathbf{z}(t)\| \to \infty$, this implies $p = \min_{j \in I} p_j < 0$. Now using (8)–(11) in (7), we get

$$i\bar{b}\, \frac{\partial f^I_\triangle}{\partial \bar{z}_i}(\mathbf{a}, \bar{\mathbf{a}}) - ib\, \overline{\frac{\partial f^I_\triangle}{\partial z_i}}(\mathbf{a}, \bar{\mathbf{a}}) = \begin{cases} \lambda_0 a_i & \text{if} \quad d_{\mathrm{P}} - p_i + \delta = p_i + \gamma, \\ 0 & \text{if} \quad d_{\mathrm{P}} - p_i + \delta < p_i + \gamma. \end{cases} \tag{12}$$

Let $J = \{j : d_{\mathrm{P}} - p_j + \delta = p_j + \gamma\}$. We suppose $J \neq \emptyset$ which gives $J = \{j : p_j = p = \min_{1 \leq j \leq m} p_j < 0\}$. Consider the derivative of $f(\mathbf{z}(t), \bar{\mathbf{z}}(t))$ with respect to t. On one hand, we have

$$\frac{\mathrm{d} f(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{\mathrm{d} t} = b \delta t^{\delta - 1} + \text{h.o.t.} \tag{13}$$

On the other hand,

$$\frac{\mathrm{d} f(\mathbf{z}(t), \bar{\mathbf{z}}(t))}{\mathrm{d} t} = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial z_i} \cdot \frac{\partial z_i}{\partial t} + \frac{\partial f}{\partial \bar{z}_i} \cdot \frac{\partial \bar{z}_i}{\partial t} \right) = \left[\left\langle \mathbf{Pa}, \overline{\mathrm{d} f^I_\triangle}(\mathbf{a}, \bar{\mathbf{a}}) \right\rangle + \left\langle \mathbf{P}\bar{\mathbf{a}}, \overline{\bar{\mathrm{d}} f^I_\triangle(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle \right] t^{d_{\mathrm{P}} - 1} + \text{h.o.t.} \tag{14}$$

where $\mathbf{Pa} = (p_1 a_1, \ldots, p_m a_m)$. From (12), we obtain

$$\operatorname{Re} \left\langle \mathbf{Pa}, i\bar{b}\, \bar{\mathrm{d}} f^I_\triangle(\mathbf{a}, \bar{\mathbf{a}}) - ib\, \overline{\mathrm{d} f^I_\triangle}(\mathbf{a}, \bar{\mathbf{a}}) \right\rangle = \sum_{i \in J} \lambda_0 p \|a_j\|^2 \neq 0. \tag{15}$$

If $d_{\mathrm{P}} < \delta$, then comparing the orders of the expansions (13) and (14) with respect to t, we have $\left\langle \mathbf{Pa}, \overline{\mathrm{d} f^I_\triangle}(\mathbf{a}, \bar{\mathbf{a}}) \right\rangle + \left\langle \mathbf{P}\bar{\mathbf{a}}, \overline{\bar{\mathrm{d}} f^I_\triangle(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle = 0$ and $f^I_\triangle(\mathbf{a}, \bar{\mathbf{a}}) = 0$. Multiplying (14) by $i\bar{b}$ and comparing the real parts of the equality, we obtain a contradiction with (15). If $d_{\mathrm{P}} = \delta$, then by (13), we have $\operatorname{Re} \left\langle \mathbf{Pa}, i\bar{b}\, \bar{\mathrm{d}} f^I_\triangle(\mathbf{a}, \bar{\mathbf{a}}) - ib\, \overline{\mathrm{d} f^I_\triangle}(\mathbf{a}, \bar{\mathbf{a}}) \right\rangle = \operatorname{Re}(i|b|^2 \delta) = 0$, which contradicts (15). It follows that $J = \emptyset$. Hence $\mathbf{a} \in \mathbb{C}^{*I}$ is a singularity of $f^I_\triangle$ and $f^I_\triangle(\mathbf{a}, \bar{\mathbf{a}}) = b$. By [6, Remark 3.3], this is contrary to the strong non-degeneracy of f^I.

(II) If $f(\mathbf{z}(t), \bar{\mathbf{z}}(t)) \to c \in \mathbb{C}^*$, $t \to 0$, comparing the orders of expansions (10) and (12) with respect to t, we have $d_{\mathrm{P}} < \delta$. Now using (9)–(12) in (8), we get

$$i\bar{c}\, \frac{\partial f^I_\triangle}{\partial \bar{z}_i}(\mathbf{a}, \bar{\mathbf{a}}) - ic\, \overline{\frac{\partial f^I_\triangle}{\partial z_i}}(\mathbf{a}, \bar{\mathbf{a}}) = \begin{cases} \lambda_0 a_i & \text{if} \quad d_{\mathrm{P}} - p_i = p_i + \gamma, \\ 0 & \text{if} \quad d_{\mathrm{P}} - p_i < p_i + \gamma. \end{cases} \tag{16}$$

Let $J = \{j : d_{\mathrm{P}} - p_j = p_j + \gamma\}$. We suppose $J \neq \emptyset$ which implies $J = \left\{j : p_j = p = \min_{1 \leq j \leq m} p_j < 0\right\}$. We derive $f(\mathbf{z}(t), \bar{\mathbf{z}}(t))$ with respect to t. On one hand, we get (13). On the other hand, we have (14). From (16), we obtain

$$\operatorname{Re} \left\langle \mathbf{Pa}, i\bar{c}\, \bar{\mathrm{d}} f^I_\triangle(\mathbf{a}, \bar{\mathbf{a}}) - ic\, \overline{\mathrm{d} f^I_\triangle}(\mathbf{a}, \bar{\mathbf{a}}) \right\rangle = \sum_{i \in J} \lambda_0 p \|a_j\|^2 \neq 0. \tag{17}$$

Since $d_{\mathrm{P}} < \delta$, comparing the orders of the expansions (13) and (14), we have $\left\langle \mathbf{Pa}, \overline{\mathrm{d} f^I_\triangle}(\mathbf{a}, \bar{\mathbf{a}}) \right\rangle + \left\langle \mathbf{P}\bar{\mathbf{a}}, \overline{\bar{\mathrm{d}} f^I_\triangle(\mathbf{a}, \bar{\mathbf{a}})} \right\rangle = 0$. Multiplying (14) by $i\bar{c}$ and comparing the real parts, we obtain a contradiction with (17). It follows that $J = \emptyset$. Hence $\mathbf{a} \in \mathbb{C}^{*I}$ is a singularity of $f^I_\triangle$ and $f^I_\triangle(\mathbf{a}, \bar{\mathbf{a}}) = 0$. By [6, Remark 3.3], this is contrary to the non-degeneracy of f^I.

In general, if we do not assume the strong non-degeneracy of f and let $\mathbf{A} = (\mathbf{a}, 1, 1, \ldots, 1)$ with the i^{th} coordinate $z_i = 1$ for $i \notin I$, then we have the following conclusion:

(a) If $d_{\mathsf{P}} < \delta$, then A is a singularity of $V(f_\triangle)$. (b) If $d_{\mathsf{P}} = \delta$, then $\mathsf{A} \in \operatorname{Sing}\varphi = \operatorname{Sing} f_\triangle \setminus V(f_\triangle)$ by Proposition 3.4. When $\lambda(t) \equiv 0$, by comparing the orders with respect to t in (7), we have

$$\begin{cases} i\bar{b}\,\dfrac{\partial f^I_\triangle}{\partial \bar{z}_i}(\mathbf{a},\bar{\mathbf{a}}) - ib\,\overline{\dfrac{\partial f^I_\triangle}{\partial z_i}}(\mathbf{a},\bar{\mathbf{a}}) = 0 & \text{if} \quad \lim\limits_{t\to 0} f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = 0 \quad \text{or} \quad \infty, \\ i\bar{c}\,\dfrac{\partial f^I_\triangle}{\partial \bar{z}_i}(\mathbf{a},\bar{\mathbf{a}}) - ic\,\overline{\dfrac{\partial f^I_\triangle}{\partial z_i}}(\mathbf{a},\bar{\mathbf{a}}) = 0 & \text{if} \quad \lim\limits_{t\to 0} f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = c \in \mathbb{C}^*. \end{cases} \tag{18}$$

It follows that $\mathbf{a} \in \mathbb{C}^{*I}$ is a singularity of $f^I_\triangle$. By [6, Remark 3.3], this is contrary to the non-degeneracy of f^I. Hence $M(\varphi)$ is bounded and $S(\varphi) = \emptyset$. □

We now proceed to formulate the analogue of [6, Theorem 1.1]. Recall the notation $\mathfrak{SB}$ for the union of strictly bad faces of $\overline{\operatorname{supp} f}$.

Theorem 3.6.

Let $f\colon \mathbb{C}^n \to \mathbb{C}$ be a mixed polynomial. Suppose that f is Newton strongly non-degenerate polynomial which depends effectively on all the variables. Let $f(0) = 0$ and $0 \notin S(f)$. Then

$$S(\varphi) \subset \bigcup_{\triangle \in \mathfrak{SB}} \varphi_\triangle\big(\operatorname{Sing}\varphi_\triangle \cap \mathbb{C}^{*n}\big).$$

Proof. We use the same notations as that in the proof of Theorem 3.5. For any $c_0 \in S(\varphi)$, by the Curve Selection Lemma at infinity, there exists $\mathbf{z}(t)$ of $M(\varphi)$ a real analytic path defined on a small enough interval $]0,\varepsilon[$ such that

$$\lim_{t\to 0} \|\mathbf{z}(t)\| = \infty \qquad \text{and} \qquad \lim_{t\to 0} \varphi(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = c_0,$$

where either $f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = bt^\delta + \text{h.o.t.}$ and $d_{\mathsf{P}} \le \delta < 0$, $c_0 = b/|b|$, or $f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = c + bt^\delta + \text{h.o.t.}$ and $d_{\mathsf{P}} \le 0, c \in \mathbb{C}^*, c_0 = c/|c|$. Consider $\lambda(t) \not\equiv 0$, if $\operatorname{ord}_t f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) < 0$, we are in the case (I) of the proof of Theorem 3.5, then we get that $\mathbf{a} \in \mathbb{C}^{*I}$ is a singularity of $f^I_\triangle$. If $\operatorname{ord}_t f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = 0$, we are in case (II) of the same proof, then $\mathbf{a} \in \mathbb{C}^{*I}$ is a singularity of $f^I_\triangle$.

Set $A = (\mathbf{a}, 1, 1, \dots, 1)$ with the i^{th} coordinate $z_i = 1$ for $i \notin I$. By recalling the definition of Newton boundary at infinity for mixed polynomial, we have the following two cases:

(I) If $d_{\mathsf{P}} < 0$, then, from [6, Lemma 3.1], we conclude that $\triangle$ is a face of $\Gamma^+(f^I)$. On the other hand since $\mathbf{a}$ is a singularity of $f^I_\triangle$, by [6, Remark 3.3] this contradicts the Newton strong non-degeneracy of f^I.

(II) If $d_{\mathsf{P}} = \delta = 0$, then, from [6, Lemma 3.1], it follows that either $\triangle$ is a face of $\Gamma^+(f^I)$ or $\triangle$ satisfies condition (ii) of Definition 2.8. Assume first $\triangle$ is a face of $\Gamma^+(f^I)$, then we get the same contradiction as that in (I). Thus $\triangle$ verifies condition (ii) of Definition 2.8. We proceed to show that $\triangle$ is strictly bad face of $\overline{\operatorname{supp} f}$. Let us denote by d the minimal value of the restriction of l_{P} to $\overline{\operatorname{supp} f}$. Since $\overline{\operatorname{supp} f^I} = \overline{\operatorname{supp} f} \cap \mathbb{R}^I_+$, we have $d \le d_{\mathsf{P}} = 0$. Let H be the hyperplane of the equation $\sum_{i=1}^m p_i x_i + q\sum_{i=m+1}^n x_i = 0$, where $q > -d+1 > 0$. Hence for any $\mathbf{x} = (x_1, \dots, x_n) \in \overline{\operatorname{supp} f} \setminus \overline{\operatorname{supp} f^I}$, the value of $\sum_{i=1}^m p_i x_i + q\sum_{i=m+1}^n x_i$ is positive. We therefore get $\triangle = \overline{\operatorname{supp} f^I} \cap H = \overline{\operatorname{supp} f} \cap H$. On the other hand, note that $p_1 = p = \min_{1\le i\le m} p_i < 0$ and $q > 0$. If $\triangle$ does not satisfy condition (i_1) of Definition 2.8, then we have $m = n$ and $p_i \le 0$ for all $1 \le i \le n$. It follows that f cannot depend on z_1 otherwise d_{P} will be negative. This contradicts the effectiveness of f. Hence we conclude that $\triangle$ is a strictly bad face of $\overline{\operatorname{supp} f}$. Since $d_{\mathsf{P}} = 0$, we obtain $c = f^I_\triangle(\mathbf{a},\bar{\mathbf{a}}) = f_\triangle(\mathsf{A},\overline{\mathsf{A}}) \neq 0$. By $\mathsf{A} \in \operatorname{Sing}\varphi_\triangle$ and Proposition 3.4, we get $c_0 \in \varphi_\triangle(\operatorname{Sing}\varphi_\triangle)$. When $\lambda(t) \equiv 0$, it follows that $\mathbf{a} \in \mathbb{C}^{*I}$ is a singularity of $f^I_\triangle$ from (18). In the same manner as above reasoning, we get the desired conclusion. □

Remark 3.7.

In particular, if a mixed polynomial f is Newton strongly non-degenerate at infinity and convenient, then by [6, Corollary 4.1], we have $S(f) = \emptyset$. Combining this conclusion with the above theorem, we get $S(\varphi) = \emptyset$ since $\mathfrak{SB} = \emptyset$.

4. Fibration at infinity

Recall that for a strongly non-degenerate polynomial f, we have the monodromy fibration $f_|: f^{-1}(S^1_\delta) \to S^1_\delta$ over some circle S^1_δ of radius δ which is sufficiently large. We define two vectors on $\mathbb{C}^n \setminus V(f)$:

$$v_1(\mathbf{z},\bar{\mathbf{z}}) = \overline{\mathrm{d}\log f}(\mathbf{z},\bar{\mathbf{z}}) + \bar{\mathrm{d}}\log f(\mathbf{z},\bar{\mathbf{z}}), \qquad v_2(\mathbf{z},\bar{\mathbf{z}}) = i\big(\overline{\mathrm{d}\log f}(\mathbf{z},\bar{\mathbf{z}}) - \bar{\mathrm{d}}\log f(\mathbf{z},\bar{\mathbf{z}})\big),$$

which have the following geometrical meanings: $v_1(\mathbf{z},\bar{\mathbf{z}})$ is the normal vector of $\log|f|$ and $v_2(\mathbf{z},\bar{\mathbf{z}})$ is the normal vector of $-i\log f/|f|$. In order to prove Theorem 1.1, we shall first prove the following proposition.

Proposition 4.1.
Under the same assumption as that in Theorem 1.1, there exists $\delta_2 > 0$ sufficiently large, such that for any $\mathbf{z}$ from $\{\mathbf{z} \in \mathbb{C}^n : |f(\mathbf{z},\bar{\mathbf{z}})| \geq \delta_2\}$ the three vectors $\mathbf{z}$, $v_1(\mathbf{z},\bar{\mathbf{z}})$, $v_2(\mathbf{z},\bar{\mathbf{z}})$ are either linearly independent over $\mathbb{R}$ or they are linearly dependent over $\mathbb{R}$ with the following relation:

$$\mathbf{z} = a v_1(\mathbf{z},\bar{\mathbf{z}}) + b v_2(\mathbf{z},\bar{\mathbf{z}}), \qquad a > 0.$$

Proof. Since f is strongly non-degenerate at infinity, by [6, Theorem 1.1], $f(\operatorname{Sing} f) \cup S(f)$ is bounded. Let us suppose that $f(\operatorname{Sing} f) \cup S(f) \subset D_{\delta_1}$. For $|f(\mathbf{z},\bar{\mathbf{z}})|$ sufficiently large we shall prove either $\mathbf{z}, v_1(\mathbf{z},\bar{\mathbf{z}}), v_2(\mathbf{z},\bar{\mathbf{z}})$ are linearly independent over $\mathbb{R}$ or $\mathbf{z} = a v_1(\mathbf{z},\bar{\mathbf{z}}) + b v_2(\mathbf{z},\bar{\mathbf{z}})$ where $|a| + |b| \neq 0$. If $v_1(\mathbf{z},\bar{\mathbf{z}})$ and $v_2(\mathbf{z},\bar{\mathbf{z}})$ are linearly dependent over $\mathbb{R}$, we have $\mathbf{z} \in \operatorname{Sing} f \setminus V(f)$. Hence $f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) \subset f(\operatorname{Sing} f)$. This contradicts the boundedness of $f(\operatorname{Sing} f)$. It follows that $v_1(\mathbf{z},\bar{\mathbf{z}})$ and $v_2(\mathbf{z},\bar{\mathbf{z}})$ are linearly independent over $\mathbb{R}$ for $|f(\mathbf{z},\bar{\mathbf{z}})|$ sufficiently large. We are reduced to proving the proposition for $a > 0$. In the remainder of the proof, we assume $a < 0$. Note also that for $a = 0$, the proof still works. By the Curve Selection Lemma at infinity, there exist analytic curves $\mathbf{z}(t) \in \mathbb{C}^n$, $a(t) < 0$ and $b(t) \in \mathbb{R}$ defined on a small enough interval $]0,\varepsilon[$ such that

$$\begin{gathered} \lim_{t\to 0} \|\mathbf{z}(t)\| = \infty, \qquad \lim_{t\to 0} f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = \infty. \\ \mathbf{z}(t) = a(t) v_1(\mathbf{z},\bar{\mathbf{z}})(t) + b(t) v_2(\mathbf{z},\bar{\mathbf{z}})(t). \end{gathered} \tag{19}$$

Let $I = \{i : z_i(t) \not\equiv 0\}$. Without loss of generality we can assume $I = \{1,\dots,m\}$, then we have

$$\begin{aligned} z_i(t) &= a_i t^{p_i} + \text{h.o.t.}, && \text{where} \quad a_j \neq 0, \quad p_i \in \mathbb{Z}, \quad i \in I, \\ f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) &= b t^q + \text{h.o.t.}, && \text{where} \quad b \in \mathbb{C}^*, \quad q \in \mathbb{Z}, \quad q < 0, \\ a(t) &= \lambda_0 t^{\nu_0} + \text{h.o.t.}, && \text{where} \quad \lambda_0 \in \mathbb{R}, \quad \nu_0 \in \mathbb{Z}, \\ b(t) &= \beta_0 t^{\nu_0} + \text{h.o.t.}, && \text{where} \quad \beta_0 \in \mathbb{R}, \quad \nu_0 \in \mathbb{Z}, \end{aligned}$$

where $|\lambda_0| + |\beta_0| \neq 0$. If $\lambda_0 \in \mathbb{R}^*$, then, by our assumption $a(t) < 0$, we have $\lambda_0 < 0$. To shorten notation, we write $\mathbf{a} = (a_1,\dots,a_m) \in \mathbb{C}^{*I}$, $\mathbf{P} = (p_1,\dots,p_m) \in \mathbb{R}^m$ and consider the linear function $l_{\mathbf{P}} = \sum_{i=1}^m p_i x_i$ defined on $\overline{\operatorname{supp} f^I}$. Let $\triangle$ be the *maximal face* of $\overline{\operatorname{supp} f^I}$ where $l_{\mathbf{P}}$ takes its minimal value, say this value is $d_{\mathbf{P}}$. We have $d_{\mathbf{P}} \leq \operatorname{ord}_t f(\mathbf{z}(t),\bar{\mathbf{z}}(t)) = q < 0$. By the above expansions, we get from (19):

$$\begin{aligned} \lambda_0 \left(\frac{(\overline{\partial f^I_\triangle})/(\partial z_i)(\mathbf{a},\bar{\mathbf{a}})}{\bar{b}} + \frac{(\partial f^I_\triangle)/(\partial \bar{z}_i)(\mathbf{a},\bar{\mathbf{a}})}{b} \right) &+ i\beta_0 \left(\frac{(\overline{\partial f^I_\triangle})/(\partial z_i)(\mathbf{a},\bar{\mathbf{a}})}{\bar{b}} - \frac{(\partial f^I_\triangle)/(\partial \bar{z}_i)(\mathbf{a},\bar{\mathbf{a}})}{b} \right) \\ &= \begin{cases} a_i & \text{if} \quad d_{\mathbf{P}} - p_i - q + \nu_0 = p_i, \\ 0 & \text{if} \quad d_{\mathbf{P}} - p_i - q + \nu_0 < p_i. \end{cases} \end{aligned} \tag{20}$$

Let $J = \{j \in I : d_{\mathbf{P}} - p_j - q + v_0 = p_j\}$. We observe that $J = \{j \in I : p_j = p = \min_{j \in I} p_j < 0\}$. If $J = \emptyset$, then from (20), we have $\mathbf{a} \in \operatorname{Sing} f^I_\triangle$. Since $d_{\mathbf{P}} < 0$, by [6, Lemma 3.1], we conclude that $\triangle$ is a face of $\Gamma^+(f^I)$. This contradicts the Newton strongly non degeneracy of f^I. Hence $J \neq \emptyset$. To deduce a contradiction, consider the following expansion:

$$\frac{\lambda_0 + i\beta_0}{\overline{b}} \frac{\mathrm{d}\overline{f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d}t} + \frac{\lambda_0 - i\beta_0}{b} \frac{\mathrm{d}f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d}t} = 2\lambda_0 q t^{q-1} + \text{h.o.t.}$$

We also have

$$\frac{\mathrm{d}f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d}t} = \left[\left\langle \mathbf{Pa}, \overline{\mathrm{d}f^I_\triangle}(\mathbf{a}, \overline{\mathbf{a}})\right\rangle + \left\langle \mathbf{P}\overline{\mathbf{a}}, \overline{\overline{\mathrm{d}}f^I_\triangle(\mathbf{a}, \overline{\mathbf{a}})}\right\rangle\right] t^{d_{\mathbf{P}}-1} + \text{h.o.t.}$$

By (20), we obtain

$$\frac{\lambda_0 + i\beta_0}{\overline{b}} \frac{\mathrm{d}\overline{f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d}t} + \frac{\lambda_0 - i\beta_0}{b} \frac{\mathrm{d}f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d}t} = \left(2 \sum_{j \in J} p \|a_j\|^2\right) t^{d_{\mathbf{P}}-1} + \text{h.o.t.}$$

Since $d_{\mathbf{P}} \leq q$, comparing the two expansions of

$$\frac{\lambda_0 + i\beta_0}{\overline{b}} \frac{\mathrm{d}\overline{f}(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d}t} + \frac{\lambda_0 - i\beta_0}{b} \frac{\mathrm{d}f(\mathbf{z}(t), \overline{\mathbf{z}}(t))}{\mathrm{d}t},$$

it follows that $d_{\mathbf{P}} = q$ and $\lambda_0 = \sum_{j \in J} p\|a_j\|^2/q > 0$ from $p < 0$ and $q < 0$. This contradicts $\lambda_0 < 0$. □

Remark 4.2.
In the holomorphic setting, the parallel results to this proposition are [9, Lemma 4.4] and [13, Lemmas 4 and 5]; in the mixed setting, this is a global analogue of [15, Lemma 34].

Proof of Theorem 1.1. The proof is done as that in the case of a holomorphic polynomial. The strong non-degeneracy of f yields a global fibration $f_| : f^{-1}(S^1_\delta) \to S^1_\delta$, where $\delta > 0$ is sufficiently large. Since S^1_δ is compact and $f(\operatorname{Sing} f) \cup S(f)$ is bounded, there exists $R_0 > 0$ sufficiently large such that all the fibers intersect S_R transversely for any $R \geq R_0$. We therefore get the restriction

$$f_| : f^{-1}(S^1_\delta) \cap B_R \to S^1_\delta,$$

which is equivalent to the global fibration. By Proposition 4.1, there exists a non-zero vector field ω on $N = \{\mathbf{z} \in B_R : |f(\mathbf{z}, \overline{\mathbf{z}})| \geq \delta\}$ such that

$$\operatorname{Re}\langle w(\mathbf{z}), v_2(\mathbf{z}, \overline{\mathbf{z}})\rangle = 0, \qquad \operatorname{Re}\langle w(\mathbf{z}), v_1(\mathbf{z}, \overline{\mathbf{z}})\rangle > 0, \qquad \operatorname{Re}\langle w(\mathbf{z}), \mathbf{z}\rangle > 0.$$

Along the integral curve $\gamma(t, \mathbf{z}_0)$ of w with $\gamma(0, \mathbf{z}_0) = \mathbf{z}_0 \in N$, the argument of $f\big(\gamma(t, \mathbf{z}_0), \overline{\gamma(t, \mathbf{z}_0)}\big)$ is constant and $\big|f\big(\gamma(t, \mathbf{z}_0), \overline{\gamma(t, \mathbf{z}_0)}\big)\big|$, $\|\gamma(t, \mathbf{z}_0)\|$ are monotone increasing. Thus for every $\mathbf{z}_0 \in N$, there exists a unique $h(\mathbf{z}_0) \in S^{2n-1}_R \setminus f^{-1}(D_\delta)$ and $t_0 \in \mathbb{R}_+$ such that $\|\gamma(t_0, h(\mathbf{z}_0))\| = R$. Consequently, there is an isomorphism $\phi : f^{-1}(S^1_\delta) \cap B_R \to S^{2n-1}_R \setminus f^{-1}(D_\delta)$. We therefore get $f/|f|_| : S^{2n-1}_R \setminus f^{-1}(D_\delta) \to S^1$ a locally trivial fibration which is equivalent to the fibration $f_| : f^{-1}(S^1_\delta) \cap B_R \to S^1_\delta$. So $f/|f|_| : S^{2n-1}_R \setminus f^{-1}(D_\delta) \to S^1$ is also equivalent to the global one. This completes our proof. □

Proof of Corollary 1.2. From Remark 3.7, it follows that $S(\varphi) = \emptyset$ and $M(\varphi)$ is bounded. On the other hand from the proof of [6, Corollary 4.1], we have $M(f) \cap V(f)$ is bounded. Thus we can construct vector fields like in the local case (see [9, 15]) and we have $f/|f|_| : S^{2n-1}_R \setminus K \to S^1$ is a locally trivial fibration. Note that the proof of Theorem 1.1 yields that this fibration is equivalent to the global fibration $f_| : f^{-1}(S^1_\delta) \to S^1_\delta$, where $\delta > 0$ is sufficiently large. □

Example 4.3 ([15, Example 5 IV]).
Consider a mixed polynomial

$$f(\mathbf{z}, \bar{\mathbf{z}}) = \frac{1}{4} z_1^2 - \frac{1}{4} \bar{z}_1^2 + z_1 \bar{z}_1 - (1+i)(z_1 + z_2)(\bar{z}_1 + \bar{z}_2).$$

Then we have

(a) f is not Newton strongly non-degenerate at infinity and $S(f) = \emptyset$.

(b) $\operatorname{Sing} f = \{\mathbf{z} \in \mathbb{C}^2 : z_1 = 0,\ z_2 \in \mathbb{C}\} \cup \{\mathbf{z} \in \mathbb{C}^2 : z_1 + z_2 = 0,\ z_1 - i\bar{z}_1 = 0\} \cup \{\mathbf{z} \in \mathbb{C}^2 : z_1 + z_2 = 0,\ z_1 + i\bar{z}_1 = 0\}$.

(c) $M(\varphi)$ is not bounded and $S(\varphi) = \{-(1+i)/\sqrt{2}, (2 \pm i)/\sqrt{5}\}$.

Remark 4.4.
The above example is due to Oka. In the holomorphic case, Némethi and Zaharia proved the existence of the Milnor fibration at infinity for semitame polynomials in [13]. The definition of semitame is equivalent to $S(f) \subset \{0\}$. But this example shows that in the mixed case, the condition $S(f) \subset \{0\}$ fails to insure the existence of the Milnor fibration $f/|f|$ at infinity. We also observe that the Newton strong non-degeneracy condition at infinity of Theorem 1.1 can not be replaced by Newton non-degeneracy condition at infinity.

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