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Numerical solution of inverse spectral problems for Sturm–Liouville operators with discontinuous potentials

Research Article

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Abstract: We consider Sturm–Liouville differential operators on a finite interval with discontinuous potentials having one jump. As the main result we obtain a procedure of recovering the location of the discontinuity and the height of the jump. Using our result, we apply a generalized Rundell–Sacks algorithm of Rafler and Böckmann for a more effective reconstruction of the potential and present some numerical examples.

MSC: 31A25, 34A55, 34L05, 47E05, 65D18

Keywords: Sturm–Liouville differential operators • Discontinuous potentials • Inverse spectral problems • Numerical solution

1. Introduction

This paper is devoted to the numerical solution of the inverse spectral problem for Sturm–Liouville differential operators with discontinuous potentials on a finite interval. Direct and inverse spectral problems with discontinuous potentials appear in various problems of applied mathematics, mechanics, physics, geophysics and other branches of natural sciences. Usually such problems are connected with discontinuous material properties. The inverse problem of reconstructing the material properties of a medium from data collected outside of the medium is in such situations of central importance (see e.g. [5, 7, 10, 19, 21]).

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We consider the Sturm–Liouville differential equation

$$-y'' + q(x)y = \lambda y \tag{1}$$

on $[0,\pi]$ under different boundary conditions. As the inverse problem, the task is to determine the potential function $q(x)$ from eigenvalues of corresponding spectral problems.

We concentrate below on the Dirichlet spectrum and on the Dirichlet–Neumann spectrum. So, let us add two pairs of boundary conditions:

$$y(0) = y(\pi) = 0, \tag{2}$$

$$y(0) = y'(\pi) = 0. \tag{3}$$

Denote by $(\mu_n)_{n\geqslant 1}$ the sequence of eigenvalues of the boundary problem (1) & (2) and by $(\nu_n)_{n\geqslant 0}$ the sequence of eigenvalues of the boundary problem (1) & (3). The task is to recover $q(x)$ from the given $(\mu_n)_{n\geqslant 1}$ and $(\nu_n)_{n\geqslant 0}$.

The theory of Sturm–Liouville inverse problems has been developed fairly completely in many papers and textbooks [6, 8, 11–14]. In particular, there is a well-known classical result of Borg from which we can conclude that $(\mu_n)_{n\geqslant 1}$ and $(\nu_n)_{n\geqslant 0}$ determine $q(x)$ uniquely. Numerically these problems were investigated in [1, 2, 9, 18], see also references therein. Numerical methods for inverse Sturm–Liouville problems often make use of the fact that good numerical methods are available for matrix inverse eigenvalue problems [3] and direct Sturm–Liouville problems [15]. A shortcoming of many numerical techniques developed for such inverse problems is that they work satisfactory only provided the potentials to be reconstructed are smooth. Otherwise these algorithms work ineffectively: we can observe oscillations throughout all the interval. If the properties of the medium do suffer jump discontinuities we need some additional effort in order to get a satisfactory numerical reconstruction method. Rafler and Böckmann [16] recently proposed a generalized version of the Rundell–Sacks algorithm [17]; their method allows to deal with a general reference potential which can be adapted to estimations of the jump-discontinuity points and jump heights of the unknown potential.

The main goal of this work is to present a procedure that is able to recover both the point of discontinuity as well as the height of the jump. Following which we may apply a suitable numerical method for solving the inverse problem. In our paper we use the generalized Rundell–Sacks algorithm [16] with a special form of the reference potential. The use of constant reference potentials is a feature of many numerical methods, thus they also can be adapted to use appropriate non-constant reference potentials.

We work with unknown potentials that have the special following form:

$$q(x) = \begin{cases} q_1(x) + b, & 0 \leqslant x \leqslant a; \\ q_1(x), & a < x \leqslant \pi, \end{cases} \tag{4}$$

where $q_1(x) \in \mathrm{AC}[0,\pi]$, i.e., $q_1(x)$ is an absolutely continuous function on the segment $[0,\pi]$.

The paper is structured as follows. In Section 2 we prove two theorems and draw a conclusion how to recover the point of discontinuity and height of the jump. Section 3 describes the numerical algorithm. Section 4 contains some numerical examples. In Section 5 we give a brief conclusion.

2. Recovering the point of discontinuity and the height of the jump

Let us define an auxiliary eigenvalue sequence $(\lambda_n)_{n\geqslant 1}$ as follows:

$$\lambda_{2n+1} = \nu_n, \qquad n \geqslant 0, \qquad\qquad \lambda_{2n} = \mu_n, \qquad n \geqslant 1.$$

If we extend $q(x)$ from $[0,\pi]$ to $[0,2\pi]$ by $q(2\pi - x) = q(x)$, $x \in [0,\pi]$, then, it is clear that $(\lambda_n)_{n\geqslant 1}$ is the Dirichlet spectrum for (1) on $[0,2\pi]$. Later in this paper we will work with this spectrum for convenience.

Recall the asymptotics for $(\lambda_n)_{n\geqslant 1}$ [4]:

$$\lambda_n = \left(\frac{n}{2}\right)^2 + A + c_n,$$

where $A = 1/(2\pi)\int_0^{2\pi} q(x)\,dx$, $c_n = o(1)$, $n\to\infty$. Let $(\lambda_n)_{n\geqslant 1}$ be given. Then A and c_n can be recovered by the formula

$$A = \lim_{n\to\infty} b_n = \lim_{N\to\infty}\sum_{n=N}^{2N}\frac{b_n}{N+1},$$

where $b_n = \lambda_n - (n/2)^2$ is known for all n. Let us define the following function:

$$p_N(x) = \frac{2\pi i}{N+1}\sum_{n=N}^{2N} c_n\, n\, e^{inx}, \qquad x\in[0,\pi], \tag{5}$$

with $c_n = \lambda_n - (n/2)^2 - A$.

Theorem 2.1.

The following relation holds:

$$p_N(x) = p_N^*(x) + o(1), \qquad N\to\infty, \tag{6}$$

where

$$p_N^*(x) = \frac{b}{N+1}\cdot\frac{e^{i(2N+1)(x-a)} - e^{iN(x-a)}}{e^{i(x-a)}-1}. \tag{7}$$

Proof. It is known [20] that

$$\lambda_n = \left(\frac{n}{2}\right)^2 + \frac{1}{2\pi}\int_0^{2\pi} q(x)[1-\cos nx]\,dx + o\left(\frac{1}{n}\right), \qquad n\to\infty.$$

Using representation (4) we infer

$$\begin{aligned}\lambda_n &= \left(\frac{n}{2}\right)^2 + \frac{1}{\pi}\int_0^a b[1-\cos nx]\,dx + \frac{1}{\pi}\int_0^\pi q_1(x)[1-\cos nx]\,dx + o\left(\frac{1}{n}\right)\\ &= \left(\frac{n}{2}\right)^2 + \frac{ab}{\pi} + \frac{1}{\pi}\int_0^\pi q_1(x)\,dx - \frac{b\cdot\sin na}{\pi n} - \frac{1}{\pi}\int_0^\pi q_1(x)\cos nx\,dx + o\left(\frac{1}{n}\right).\end{aligned}$$

Therefore

$$c_n = -\frac{b\sin na}{\pi n} - \frac{1}{\pi}\int_0^\pi q_1(x)\cos nx\,dx + o\left(\frac{1}{n}\right).$$

Notice that

$$-\frac{2bi}{N+1}\sum_{n=N}^{2N} e^{inx}\sin na = -\frac{b}{N+1}\left[\sum_{n=N}^{2N} e^{in(x+a)} - \sum_{n=N}^{2N} e^{in(x-a)}\right] = \frac{b}{N+1}\left[\frac{e^{i(2N+1)(x-a)} - e^{iN(x-a)}}{e^{i(x-a)}-1} + \frac{e^{iN(x+a)} - e^{i(2N+1)(x+a)}}{e^{i(x+a)}-1}\right].$$

Then $p_N(x)$ has the following form:

$$p_N(x) = p_N^*(x) + h_{1N}(x) + h_{2N}(x) + h_{3N}(x), \tag{8}$$

where

$$p_N^*(x) = \frac{b}{N+1}\left[\frac{e^{i(2N+1)(x-a)} - e^{iN(x-a)}}{e^{i(x-a)}-1}\right], \qquad h_{1N}(x) = \frac{b}{N+1}\left[\frac{e^{iN(x+a)} - e^{i(2N+1)(x+a)}}{e^{i(x+a)}-1}\right],$$
$$h_{2N}(x) = -\frac{2i}{N+1}\sum_{n=N}^{2N} n\, e^{inx}\int_0^\pi q_1(y)\cos ny\, dy, \qquad h_{3N}(x) = \frac{2\pi i}{N+1}\sum_{n=N}^{2N}\varepsilon_n n\, e^{inx},$$

and ε_n satisfies $\lim\limits_{n\to\infty} n\varepsilon_n = 0$. Let us consider

$$|h_{1N}(x)| = \left|\frac{b}{N+1}\cdot\frac{\sin((N+1)(x+a)/2)}{\sin((x+a)/2)}\right| \leqslant \left|\frac{b}{N+1}\right|\cdot\frac{1}{|\sin((x+a)/2)|}.$$

As $x+a\in(0,2\pi)$, hence $|\sin((x+a)/2)|>0$. The function $\sin((x+a)/2)$ as a continuous function takes its minimum value on a closed interval:

$$\left|\sin\frac{x+a}{2}\right| \geqslant C, \qquad x\in[0,\pi],$$

where C is a positive constant. So, we get $h_{1N}(x) = \mathrm{O}(1/N)$. Now let us consider $h_{2N}(x)$. Since $q_1(x)\in \mathrm{AC}[0,\pi]$, using the Riemann–Lebesgue theorem, we get

$$\begin{aligned}|h_{2N}(x)| \leqslant \frac{2}{N+1}\sum_{n=N}^{2N}\left|n\int_0^\pi q_1(y)\cos ny\, dy\right| &= \frac{2}{N+1}\sum_{n=N}^{2N}\left|n\,\frac{\sin ny}{n}\,q_1(y)\Big|_0^\pi - n\,\frac{1}{n}\int_0^\pi q_1'(y)\sin ny\, dy\right| \\ &= \frac{2}{N+1}\sum_{n=N}^{2N}\left|\int_0^\pi q_1'(y)\sin ny\, dy\right| = \mathrm{o}(1).\end{aligned}$$

Thus, $h_{2N}(x) = \mathrm{o}(1)$. Further,

$$|h_{3N}(x)| = \left|\frac{2\pi i}{N+1}\sum_{n=N}^{2N}\varepsilon_n n\, e^{inx}\right| \leqslant \frac{2\pi}{N+1}\sum_{n=N}^{2N}|n\varepsilon_n| \leqslant C\,\frac{2N-N}{N+1}\,N\max_{N\leqslant n\leqslant 2N}|\varepsilon_n|,$$

where $C = 4\pi$,

$$\lim_{N\to\infty} C\,\frac{N}{N+1}\,N\max_{N\leqslant n\leqslant 2N}|\varepsilon_n| = 0.$$

Hence, $h_{3N}(x) = \mathrm{o}(1)$. Thus, using representation (8), we obtain the assertion of the theorem. □

The following theorem shows how we can determine a and b.

Theorem 2.2.

Let $p_N^*(x)$ be the function defined in (7). Then

$$\lim_{N\to\infty} p_N^*(a) = b, \tag{9}$$
$$\lim_{N\to\infty} p_N^*(x) = 0, \qquad x\neq a, \tag{10}$$

where the convergence is uniform on any set $[0,\pi]\setminus(a-\delta, a+\delta)$, $\delta>0$.

Proof. Using the L'Hôpital's rule we have

$$\lim_{x\to a}\frac{b}{N+1}\cdot\frac{e^{i(2N+1)(x-a)}-e^{iN(x-a)}}{e^{i(x-a)}-1}=\lim_{x\to a}\frac{b}{N+1}\cdot\frac{i(2N+1)\,e^{i(2N+1)(x-a)}-iNe^{iN(x-a)}}{i\,e^{i(x-a)}}=b.$$

Thus, (9) is proved. Consider now

$$|p^*_N(x)|=\left|\frac{b}{N+1}\cdot\frac{\sin((N+1)(x-a)/2)}{\sin((x-a)/2)}\right|\cdot\left|\frac{\sin((N+1)(x-a)/2)}{\sin((x-a)/2)}\right|\leqslant\frac{1}{|\sin((x-a)/2)|}.$$

Let us notice, that $|\sin((x-a)/2)|>0$, since we examine the case when $x\neq a$. Then, taking the limit we get

$$\lim_{N\to\infty}p^*_N(x)=0,\qquad x\neq a.$$

We got (10) pointwise. Now let us show the uniformness. The function $\sin((x-a)/2)$ as a continuous function takes its minimum value on a closed interval:

$$\left|\sin\frac{x-a}{2}\right|\geqslant C,\qquad x\in[0,\pi]\setminus(a-\delta,a+\delta),$$

where C is a positive constant, $\delta>0$. Therefore

$$|p^*_N(x)|\leqslant\left|\frac{b}{C(N+1)}\right|,\qquad x\in[0,\pi]\setminus(a-\delta,a+\delta).\tag{11}$$

The assertion of (10) now follows directly from (11). Theorem 2.2 is proved. □

Corollary 2.3.
For all $\delta>0$ there exists $N(\delta)=N_\delta$ such that if $N>N_\delta$ and x^* is point of the global maximum of $p_N(x)$, then $x^*\in(a-\delta,a+\delta)$.

From Corollary 2.3 the numerical algorithm of finding characteristics of discontinuity follows. Algorithm efficiency depends on how rapidly the remainder term in (6) decreases. If the function $q_1(x)$ is smoother than it is regarded in this paper, we can get more precise estimates. This question is certainly important and will be examined in future works.

3. Numerical algorithm

Let $(\mu_n)_{n\geqslant1}$ and $(\nu_n)_{n\geqslant0}$ be given.

Algorithm 1

1. Define $(\lambda_n)_{n\geqslant1}$ as follows: $\lambda_{2n+1}=\nu_n$, $n\geqslant0$, $\lambda_{2n}=\mu_n$, $n\geqslant1$.
2. Calculate A using the following formula:

$$A=\lim_{N\to\infty}\sum_{n=N}^{2N}\frac{b_n}{N+1},\qquad b_n=\lambda_n-\left(\frac{n}{2}\right)^2.$$

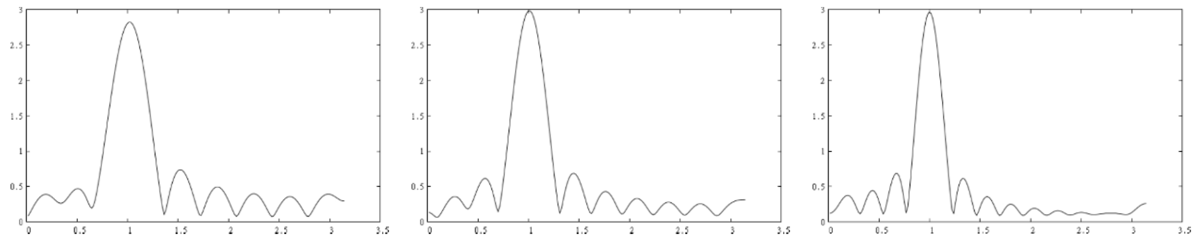
3. Calculate $c_n=\lambda_n-(n/2)^2-A$.
4. Construct the function $p_N(x)$ from eigenvalues, using (5).
5. Recover the point of discontinuity a as a global maximum of $|p_N(x)|$.
6. Compute the approximation of the height of the jump b by $b=p_N(a)$.
7. Apply a numerical method from [16] using a reference potential with jump discontinuity b at a for reconstruction the unknown potential.

4. Numerical examples

Let us consider

$$q(x)=\begin{cases} x(\pi-x)+3, & 0\leqslant x\leqslant 1;\\ x(\pi-x), & 1<x\leqslant\pi. \end{cases} \tag{12}$$

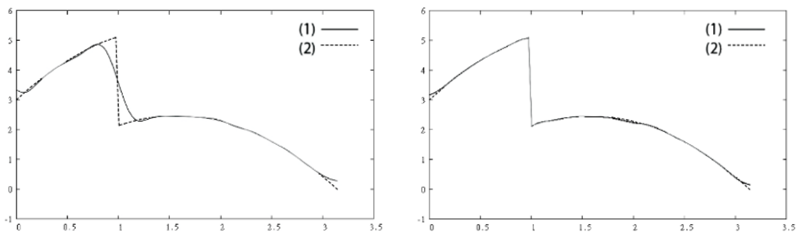
Figure 1. The modulus of the function $p_N(x)$ for the case when $q(x)$ has the form (12): a) corresponds to $N=17$, b) corresponds to $N=19$, c) corresponds to $N=26$.



a) b) c)

Notice, that using rather small values of N, we can observe the accurately allocated global maximum. This allows to use this method even if large eigenvalues are known inexactly, because of inaccuracy of measurements.

Figure 2. Line (1): reconstruction of potential (12): a) using the reference potential $p(x)=(1/\pi)\int_0^\pi q(x)\,dx$, b) using the adapted reference potential $p(x)$, where we recover $a=1.0122$, $b=2.9849$ from $p_N(x)$, $N=19$. Line (2) in both cases is an exact function (12).

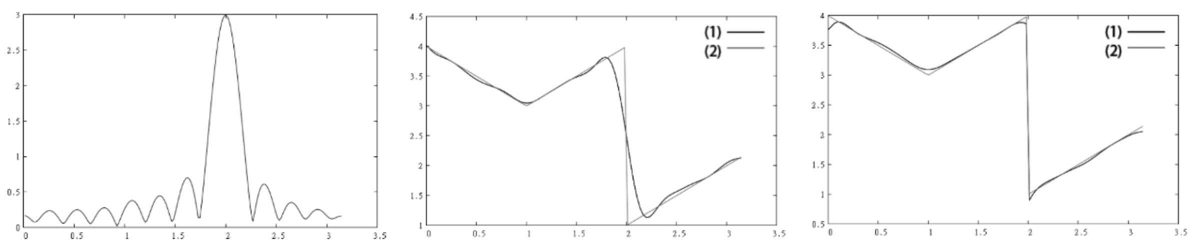


a) b)

Also let us present some examples for

$$q(x)=\begin{cases} |x-1|+3, & 0\leqslant x\leqslant 2;\\ |x-1|, & 2<x\leqslant\pi. \end{cases} \tag{13}$$

Figure 3. a) $|p_N(x)|$, where $N=22$. Line (1) on b) and c): reconstruction of potential (13), b) using the reference potential $p(x)=(1/\pi)\int_0^\pi q(x)\,dx$, c) using the adapted reference potential $p(x)$, where we recover $a=2.007$, $b=2.989$. Line (2) in both cases is an exact function (13).



a) b) c)

Table 1. Absolute and relative errors of the reconstructed potentials

Line	L^2-error Absolute	L^2-error Relative	L^∞-error Absolute	L^∞-error Relative
Figure 2 (a), (1)	0.08927	0.45866	0.26923	1.37606
Figure 2 (b), (1)	0.01014	0.05212	0.03203	0.16371
Figure 3 (b), (1)	0.08541	0.44819	0.33916	1.35667
Figure 3 (c), (1)	0.01391	0.07299	0.05910	0.23642

These values of N are also used as the finite numbers of eigenvalues sequences to produce the retrieved potentials.

5. Conclusion

In this paper for the first time theoretical justification of finding characteristics of discontinuity of the required potential was given. Also the algorithm of finding these characteristics was described. Using our procedure we can recover a and b with sufficiently good approximation and then reconstruct the potential, using the generalized Rundell–Sacks method. We have practically the same accuracy of this method as in cases where we know a and b a priori.

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