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The combinatorial derivation and its inverse mapping

Research Article

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Abstract: Let G be a group and $\mathcal{P}_G$ be the Boolean algebra of all subsets of G. A mapping $\Delta\colon \mathcal{P}_G \to \mathcal{P}_G$ defined by $\Delta(A) = \{g \in G : gA \cap A \text{ is infinite}\}$ is called the combinatorial derivation. The mapping Δ can be considered as an analogue of the topological derivation $d\colon \mathcal{P}_X \to \mathcal{P}_X$, $A \mapsto A^d$, where X is a topological space and A^d is the set of all limit points of A. We study the behaviour of subsets of G under action of Δ and its inverse mapping ∇. For example, we show that if G is infinite and $\mathcal{I}$ is an ideal in $\mathcal{P}_G$ such that $\Delta(A) \in \mathcal{I}$ and $\nabla(A) \subseteq \mathcal{I}$ for each $A \in \mathcal{I}$ then $\mathcal{I} = \mathcal{P}_G$.

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1. Introduction

Let G be a group with the identity e. We denote by $\mathcal{P}_G$ and $\mathcal{F}_G$ the families of all and all finite subsets of G. A subset A of a group G is called

- *large* if $G = FA$ for some $F \in \mathcal{F}_G$;
- *small* if $L \setminus A$ is large for each large subset L;
- *thick* if, for each $F \in \mathcal{F}_G$, there exists $a \in A$ such that $Fa \subseteq A$;
- *prethick* if FA is thick for some $F \in \mathcal{F}_G$;
- *thin* if $gA \cap A$ is finite for each $g \in G$, $g \neq e$;
- *sparse* if, for each infinite subset $S \subseteq G$, there exists a finite subset $F \subset S$ such that $\bigcap_{g\in F} gA$ is finite.

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We use the following elementary observations. A subset A is small if and only if A is not prethick. A subset A is thick if and only if $G\setminus A$ is not large. Each thin subset of a group G is sparse, and each sparse subset of an infinite group G is small. All above defined families of subsets are left and right translation invariant. In the dynamical terminology [4, p. 85], large and prethick subsets are called syndetic and piecewise syndetic. We note also that all above subsets can be defined in the much more general context of balleans [10, 12, 13].

The families Sm_G and Sp_G of all small and sparse subsets of G are ideals in the Boolean algebra $\mathcal{P}_G$ (see [2] and [7] respectively). Recall that a subset $\mathcal{I}\subseteq\mathcal{P}_G$ is an ideal if $\mathcal{I}$ is closed under taking subsets and finite unions (we do not suppose that $G\notin\mathcal{I}$). So the smallest ideal is $\mathcal{I}_\varnothing=\{\varnothing\}$ and the largest ideal is $\mathcal{P}_G$.

Following [11], we define a mapping $\Delta\colon\mathcal{P}_G\to\mathcal{P}_G$ by

$$\Delta(A)=\{g\in G: gA\cap A \text{ is infinite}\},$$

and say that Δ is the *combinatorial derivation*. The mapping Δ can be considered as an analogue of the topological derivation $d\colon\mathcal{P}_X\to\mathcal{P}_X$, $A\mapsto A^d$, where X is a topological space and A^d is the set of all limit points of A (see [11]). Clearly, $\Delta(A)=\Delta^{-1}(A)$ and $e\in\Delta(A)$ if and only if A is infinite. We denote

$$\mathrm{Sym}_G=\{\varnothing\}\cup\big\{A\subseteq G: A=A^{-1},\, e\in A\big\}.$$

Evidently, $\Delta(A)\subseteq AA^{-1}$ but each infinite group G contains a subset A such that $G=AA^{-1}$ and $\Delta(A)=\{e\}$, see [5, Theorem 3]. We use also the inverse to Δ, multivalued mapping ∇ defined by

$$\nabla(A)=\{B\subseteq G:\Delta(B)=A\}.$$

By Lemma 2.6, if G is infinite, the range of Δ is Sym_G.

In this paper, we explore the behaviour of the subsets defined above and some other families of subsets of a group under the action of the operations Δ and ∇. If G is finite then $\Delta(G)=\varnothing$ so in what follows all groups under consideration are supposed to be infinite. "Countable" means "countably infinite". Thus, we have the following strict inclusions [6]:

$$\mathcal{I}_\varnothing\subset\mathcal{F}_G\subset\mathcal{T}_G\subset\langle\mathcal{T}_G\rangle\subset\mathrm{Sp}_G\subset\mathrm{Sm}_G,$$

where $\mathcal{T}_G$ is the family of all thin subsets of G, $\langle\mathcal{T}_G\rangle$ is the ideal generated by $\mathcal{T}_G$.

2. Lemmata

Lemma 2.1.

If $A\in\mathcal{P}_G$ and $F\in\mathcal{F}_G$ then $\Delta(FA)=F\Delta(A)F^{-1}$.

Proof. Since F is finite, $g\in\Delta(FA)$ if and only if there are $a,b\in F$ such that $gaA\cap bA$ is infinite, equivalently, $b^{-1}gaA\cap A$ is infinite. □

Lemma 2.2.

If A is a thick subset of G then $\Delta(A)=G$.

Proof. We fix $g\in G$ and, for each natural number n, pick a finite subset F_n such that $|F_n|=n$. Since A is thick, there is $a_n\in A$ such that $\big(g^{-1}F_n\cup F_n\big)a_n\subset A$. It follows that $F_na_n\subset gA\cap A$. Hence, $gA\cap A$ is infinite and $g\in\Delta(A)$. □

Lemma 2.3 ([3, Corollary 9]).

If A is prethick then $\Delta(A)$ is large.

Lemma 2.4.
If $\Delta(A)$ is sparse then A is sparse.

Proof. We use the following observation. Let S be a subset of G such that $XY \subseteq S$ for some infinite subsets X, Y of G. Then $Y \subseteq \bigcap_{g\in F} gS$ for each finite subset F of X^{-1}. Hence S is not sparse.

If A is not sparse, there is an infinite subset Z such that $\bigcap_{g\in F} gA$ is infinite for each finite subset F of Z. It follows that $gA \cap hA$ is infinite for all $g, h \in Z$ so $Z^{-1}Z \subseteq \Delta(A)$ and, by the above paragraph, $\Delta(A)$ is not sparse. □

Lemma 2.5.
Let G be a group, $X = \{x_\alpha : \alpha < |G|\}$ be a subset of G, and for each $\alpha < |G|$, let $X_\alpha = \{x_\beta : \beta < \alpha\}$. If $x_\alpha \notin X_\alpha X_\alpha^{-1} X_\alpha$ for every $\alpha < |G|$ then $|gX \cap X| \leqslant 2$ for every $g \in G \setminus \{e\}$.

Proof. Assume the contrary and pick distinct $\alpha_1, \alpha_2, \alpha_3$ and distinct $\beta_1, \beta_2, \beta_3$ such that $gx_{\alpha_i} = x_{\beta_i}$, $i \in \{1,2,3\}$. Changing the numeration, we may suppose that either $\alpha_1 < \beta_1$, $\alpha_2 < \beta_2$ or $\alpha_1 > \beta_1$, $\alpha_2 > \beta_2$. In the first case, we may suppose that $\beta_2 > \beta_1$. Then $x_{\beta_1}x_{\alpha_1}^{-1} = x_{\beta_2}x_{\alpha_2}^{-1}$ implies $x_{\beta_2} \in X_{\beta_2}X_{\beta_2}^{-1}X_{\beta_2}$. The second case is analogous. □

Lemma 2.6.
For every subset $A \in \mathrm{Sym}_G$, there exist two thin subsets X, Y such that $\Delta(X \cup Y) = A$.

Proof. If $A = \varnothing$, we can take any finite subsets X, Y. Suppose that $A \neq \varnothing$ and $|G| = \aleph_0$. We write the elements of A in a sequence $(a_n)_{n\in\omega}$, $a_0 = e$. If A is finite, all but finitely many a_n are equal to e. Also, we represent G as a union $G = \bigcup_{n\in\omega} F_n$ of an increasing chain $\{F_n : n \in \omega\}$ of finite subsets such that $e \in F_n$, $F_n = F_n^{-1}$.

Then we choose inductively a sequence $(Z_n)_{n\in\omega}$ of finite subsets of G of the form

$$Z_n = \left\{x_{n0}, \dots, x_{nn}, a_0x_{n0}, \dots, a_nx_{nn}\right\}$$

such that, for all $m \leqslant n < \omega$, the following conditions are satisfied:

(a) $Z_mZ_n^{-1} \cap F_n \subseteq \left\{a_0, \dots, a_n, a_0^{-1}, \dots, a_n^{-1}\right\}$,

(b) $F_n\{x_{n0}, \dots, x_{nn}\} \cap \bigcup_{i<n}\{x_{i0}, \dots, x_{in}\} = \varnothing$, $F_n\{a_0x_{n0}, \dots, a_nx_{nn}\} \cap \bigcup_{i<n}\{a_0x_{i0}, \dots, a_nx_{in}\} = \varnothing$,

(c) $F_nx_{ni} \cap F_nx_{nj} = \varnothing$, $F_na_ix_{ni} \cap F_na_jx_{nj} = \varnothing$ for all distinct $i, j \in \{0, \dots, n\}$.

After ω steps, we put

$$X = \bigcup_{n\in\omega}\{x_{n0}, \dots, x_{nn}\}, \qquad Y = \bigcup_{n\in\omega}\{a_0x_{n0}, \dots, a_nx_{nn}\},$$

and note that $A \subseteq \Delta(X \cup Y)$. If $g \in G \setminus A$, by (a), the set $g(X \cup Y) \cap (X \cup Y)$ is finite. Hence $\Delta(X \cup Y) = A$. By (b) and (c), the subsets X and Y are thin.

If $|A| \leqslant \aleph_0$ but $|G| > \aleph_0$, we take a countable subgroup H of G such that $A \subseteq H$, forget about G and find two thin subsets X, Y of H such that $\Delta(X \cup Y) = A$ in H. Since $gA \cap A = \varnothing$ for each $g \in G \setminus H$, we have $\Delta(X \cup Y) = A$ in G.

At last, let $|A| > \aleph_0$. Repeating the arguments from the above paragraph, we may suppose that $|A| = |G|$. We use the lexicographic ordering $\prec$ on $|G| \times \omega$ ($(\gamma, m) \prec (\alpha, n)$ if $\gamma < \alpha$ or $\gamma = \alpha$ and $m < n$), denote by $\lambda(G)$ the set of all limit ordinals $< |G|$, and put $|\alpha| = |\{\gamma : \gamma < \alpha\}|$.

We enumerate $A = \{a_\alpha : \alpha < |G|\}$ and construct inductively a sequence $(Z_\alpha)_{\alpha<|G|}$ of finite subsets of G and an increasing sequence $(H_\alpha)_{\alpha<|G|}$ of subgroups of G such that if $\alpha \in \lambda(G)$ and $n \in \omega$ then $|H_{\alpha+n}| = \max\{\aleph_0, |\alpha|\}$,

$$Z_{\alpha+n} = \left\{x_{\alpha+n,0}, \dots, x_{\alpha+n,n}, a_\alpha x_{\alpha+n,0}, \dots, a_{\alpha+n}x_{\alpha+n,n}\right\}$$

and the following conditions are satisfied:

(d) $Z_{\alpha+n} \subset H_{\alpha+n+1} \setminus H_{\alpha+n}$,

(e) $Z_{\alpha+n}Z_{\alpha+n}^{-1} \subset \{a_\alpha^{\pm 1}, \dots a_{\alpha+n}^{\pm 1}\} \cup (H_{\alpha+n+1} \setminus H_{\alpha+n}) \cup \{e\}$,

(f) $x_{\alpha+n,m} \notin X_{\alpha+n,m}X_{\alpha+n,m}^{-1}X_{\alpha+n,m}$, where

$$X_{\alpha+n,m} = \{x_{\gamma+i,j} : 0 \leqslant j \leqslant i,\, (\gamma+i,j) \prec (\alpha+n,m)\},$$

(g) $a_{\alpha+m}x_{\alpha+n,m} \notin Y_{\alpha+n,m}Y_{\alpha+n,m}^{-1}Y_{\alpha+n,m}$, where

$$Y_{\alpha+n,m} = \{a_{\gamma+j}x_{j+i,j} : 0 \leqslant j \leqslant i,\, (\gamma+i,j) \prec (\alpha+n,m)\}.$$

We describe a strategy of the choice. We put $H_0 = \{e\}$ and suppose that, for some $\alpha \in \lambda(G)$ and $n \in \omega$, $H_{\alpha+n}$ and $\{Z_{\gamma+i} : \gamma < \lambda(G),\, i \in \omega, (\gamma, i) \prec (\alpha, n)\}$ have been chosen. We choose $x_{\alpha+n,0}, \dots, x_{\alpha+n,n}$ to satisfy (f), (g) and

$$Z_{\alpha+n} \cap H_{\alpha+n} = \varnothing, \qquad Z_{\alpha+n}Z_{\alpha+n}^{-1} \subset \{a_\alpha^{\pm 1}, \dots a_{\alpha+n}^{\pm 1}\} \cup (G \setminus H_{\alpha+n}) \cup \{e\}.$$

After that we denote by $H_{\alpha+n+1}$ the subgroup generated by $H_{\alpha+n} \cup Z_{\alpha+n}$. If $\alpha \in \Lambda(G)$, we put $H_\alpha = \bigcup_{\gamma<\alpha} H_\gamma$. After $|G|$ steps, we put

$$X = \{x_{\gamma+n,m} : m \leqslant n < \omega,\, \gamma < \lambda(G)\}, \qquad Y = \{a_{\gamma+m}x_{\gamma+n,m} : m \leqslant n < \omega,\, \gamma < \lambda(G)\},$$

and $Z = X \cup Y$. By the construction, $A \subseteq \Delta(Z)$. If $g \in \Delta(Z) \setminus A$, then (d), (e) and finiteness of each Z_n imply $|gZ \cap Z| < \aleph_0$. Hence $\Delta(Z) = A$. By (f), (g) and Lemma 2.5, X and Y are thin. □

3. Results

For a family $\mathcal{F}$ of subsets of a group G, we put $\Delta(\mathcal{F}) = \{\Delta(A) : A \in \mathcal{F}\}$, and say that $\mathcal{F}$ is Δ-*complete* if $\Delta(\mathcal{F}) \subseteq \mathcal{F}$. If $\nabla(A) \subseteq \mathcal{F}$ for each $A \in \mathcal{F}$, we say that $\mathcal{F}$ is ∇-*complete*.

Clearly, $\Delta(\mathcal{F}_G) = \mathcal{I}_\varnothing$ and $\Delta(\mathcal{T}_G) = \{\{e\}, \varnothing\}$, so $\mathcal{F}_G$ and $\mathcal{T}_G$ are Δ-complete. By Lemma 2.6, $\mathcal{F}_G$ and $\mathcal{T}_G$ are not ∇-complete. By [6, Theorem 3.1], each subset $A \in \mathcal{P}_G$ with finite $\Delta(A)$ can be partitioned into finitely many thin subsets.

Theorem 3.1.

For every group G, the following statements hold:

(i) $\Delta(\langle \mathcal{T}_G \rangle) = \Delta(\mathrm{Sp}_G) = \Delta(\mathrm{Sm}_G) = \mathrm{Sym}_G$;

(ii) Sp_G *and* Sm_G *are* ∇*-complete;*

(iii) $\langle \mathcal{T}_G \rangle$ *is not* ∇*-complete.*

Proof. For (i), we note that $\langle \mathcal{T}_G \rangle \subset \mathrm{Sp}_G \subset \mathrm{Sm}_G$ and apply Lemma 2.6. For (ii) apply Lemmas 2.4 and 2.3, respectively.

(iii) Passing to subgroups, we may suppose that G is countable. We choose inductively a sequence $(F_n)_{n\in\omega}$ of finite subsets of G such that $e \in F_n$, $|F_n| = n+1$ and the set $T = \bigcup_{n\in\omega} F_nF_n^{-1}$ is thin. Modifying the arguments proving Lemma 2.6, we can choose a doubling sequence $(g_{mn})_{m,n\in\omega}$ in G such that the family $\{F_ng_{mn} : m, n \in \omega\}$ is disjoint and $\Delta\big(\bigcup_{m,n\in\omega} F_ng_{mn}\big) = T$. We put $A = \bigcup_{m,n\in\omega} F_ng_{mn}$ so $\Delta(A) \in \mathcal{T}_G$.

Since $e \in F_n$, $|F_n| = n+1$ and the subsets $\{F_ng_{mn} : m \in \omega\}$ are pairwise disjoint for each $n \in \omega$, we conclude that $A \notin \langle \mathcal{T}_G \rangle$ using the following characterization of $\langle \mathcal{T}_G \rangle$ from [7]. A subset $S \in \langle \mathcal{T}_G \rangle$ if and only if there exists a natural number n such that, for each $F \in \mathcal{F}_G$, there exists $K \in \mathcal{F}_G$ such that $|Fg \cap S| \leqslant m$ for every $g \in G \setminus K$. □

Theorem 3.2.

If an ideal $\mathcal{I}$ in $\mathcal{P}_G$ is Δ-complete and ∇-complete then $\mathcal{I} = \mathcal{P}_G$.

Proof. Since $\varnothing \in \mathcal{I}$ and $\mathcal{I}$ is ∇-complete, $\mathcal{F}_G \subseteq \mathcal{I}$, in particular, $\{e\} \in \mathcal{I}$. Then ∇-completeness gives $\mathcal{T}_G \subseteq \mathcal{I}$ and so $\langle \mathcal{T}_G \rangle \subseteq \mathcal{I}$. By Theorem 3.1, $\mathrm{Sym}_G \subseteq \mathcal{I}$. Since $\mathcal{I}$ is an ideal, $\mathcal{I} = \mathcal{P}_G$. □

An ideal $\mathcal{I}$ in $\mathcal{P}_G$ is called a *group ideal* if $\mathcal{F}_G \subseteq \mathcal{I}$ and $AB^{-1} \in \mathcal{I}$ for all $A, B \in \mathcal{I}$. By [9], there are $2^{\mathfrak{c}}$ distinct group ideals in $\mathcal{P}_G$ for a countable group G.

Theorem 3.3.

Let $\mathcal{I}$ be a group ideal in $\mathcal{P}_G$ such that $\mathcal{I} \neq \mathcal{P}_G$. Then $\mathcal{I} \subset \mathrm{Sm}_G$ and $\mathcal{I}$ is Δ-complete.

Proof. Suppose that some member $A \in \mathcal{I}$ is not small. Then A is prethick so $F - A$ is thick for some $F \in \mathcal{F}_G$. By Lemma 2.2, $\Delta(FA) = G$. Clearly, $\Delta(FA) \subseteq FA(FA)^{-1}$. Since $\mathcal{I}$ is a group ideal, $FA(FA)^{-1} \in \mathcal{I}$ so $\mathcal{I} = \mathcal{P}_G$. Hence, $\mathcal{I} \subseteq \mathrm{Sm}_G$. By [5, Theorem 3], Sm_G is not a group ideal so $\mathcal{I} \subset \mathrm{Sm}_G$. If $A \in \mathcal{I}$ then $AA^{-1} \in \mathcal{I}$. Since $\Delta(A) \subseteq AA^{-1}$ and $\mathcal{I}$ is an ideal, we have $\Delta(A) \in \mathcal{I}$ so $\mathcal{I}$ is Δ-complete. □

Let G be an amenable group and let μ be a Banach measure on G (i.e., a finitely additive, probability, left invariant measure $\mu\colon \mathcal{P}_G \to [0,1]$). We consider the ideal $\mathcal{N}_\mu = \{A \in \mathcal{P}_G : \mu(A) = 0\}$ and the ideal $\mathcal{N}_G = \bigcap\{\mathcal{N}_\mu : \mu \text{ is a Banach measure on } G\}$ of the absolute null sets on G. By [6, Theorems 5.1 and 5.2], $\mathrm{Sp}_G \subset \mathcal{N}_G \subset \mathrm{Sm}_G$.

Theorem 3.4.

For an amenable group G, the following statements hold:

(i) *$\mathcal{N}_G$ and $\mathcal{N}_\mu$ are not Δ-complete;*

(ii) *$\mathcal{N}_G$ and $\mathcal{N}_\mu$ are ∇-complete.*

Proof. (i) We use Lemma 2.6 to choose two thin subsets X, Y such that $\Delta(X \cup Y) = G$. Since $X \cup Y$ is sparse, we conclude that $\mathcal{N}_G$ and $\mathcal{N}_\mu$ are not Δ-complete.

(ii) It suffices to show that if $\mu(A) > 0$ then $\mu(\Delta(A)) > 0$. We pick a natural number n such that $\mu(A) > 1/n$, and choose $g_1, \dots, g_k \in G$, $k < n$, such that $\mu(gA_i \cap gA_j) = 0$ for all distinct $i, j \in \{1, \dots, k\}$, and, for each $g \in G$, there is $i \in \{1, \dots, k\}$ such that $\mu(gA \cap g_iA) > 0$. We fix $g \in G$ and pick $i \in \{1, \dots, k\}$ such that $\mu(gA \cap g_iA) > 0$. Since $gA \cap g_iA$ is infinite, we have $g_i^{-1}g \in \Delta(A)$. Hence $G = \{g_1, \dots, g_k\}\Delta(A)$ and $\mu(\Delta(A)) > 0$. □

For a family $\mathcal{F}$ of subsets of a group G, we denote by $\nabla^*(\mathcal{F})$ the smallest by inclusion ∇-complete family of subsets of G containing $\mathcal{F}$. It is easy to see that

$$\nabla^*(\mathcal{F}) = \bigcup_{n \in \omega} \nabla^n(\mathcal{F}),$$

where $\nabla^0(\mathcal{F}) = \mathcal{F}$ and $\nabla^{n+1}(\mathcal{F}) = \{A \subseteq G : \Delta(A) \in \nabla^n(\mathcal{F})\}$. If $\mathcal{F} \subseteq \nabla(\mathcal{F})$ then $\nabla^n(\mathcal{F}) \subseteq \nabla^{n+1}(\mathcal{F})$ for each $n \in \omega$. In particular, $\nabla^n(\mathcal{I}_\varnothing) = \nabla^{n+1}(\mathcal{I}_\varnothing)$. We note that $A \in \nabla^*(\mathcal{I}_\varnothing)$ if and only if $\nabla^n(A) = \{\varnothing\}$ for some $n \in \omega$.

By Theorem 3.1 (ii), $\nabla^*(\mathcal{I}_\varnothing) \subseteq \mathrm{Sp}_G$. We show (Theorem 3.5) that $\nabla^*(\mathcal{I}_\varnothing)$ is much smaller than Sp_G. Following [6], we say that a subset $A \subseteq G$ is *k-sparse*, $k \in \mathbb{N}$, if every infinite subset X of G contains a subset F such that $|F| = k$ and $\bigcap_{g \in F} gA$ is finite. We denote by k-Sp_G the family of all k-sparse subsets of G. By [6, Lemma 3.2], $A \in 2$-Sp_G if and only if $X^{-1}X \nsubseteq \Delta(A)$ for every infinite subset X of G.

Following [8], we define a family $\tau(\mathcal{F})$ by the rule: $A \in \tau(\mathcal{F})$ if and only if $gA \cap A \in \mathcal{F}$ for each $g \in G$, $g \neq e$. A family $\mathcal{F}$ is τ-complete if $\tau(\mathcal{F}) \subseteq \mathcal{F}$. We denote by $\tau^*(\mathcal{F})$ the smallest τ-complete family containing $\mathcal{F}$. If $\mathcal{F}$ is a left translation invariant ideal of $\mathcal{P}_G$ and $\{e\} \in \mathcal{F}$, a characterization of $\tau^*(\mathcal{F})$ is given in [1]: $A \in \tau^*(\mathcal{F})$ if and only if for any sequence $(g_n)_{n \in \omega}$ in $G \setminus \{e\}$ there is a number $m \in \omega$ such that

$$\bigcup_{k_0, \dots, k_n \in \{0,1\}} g_0^{k_0} \dots g_n^{k_n} A \in \mathcal{F}.$$

Theorem 3.5.
For a group G, the following statements hold:

(i) $\nabla^*(\mathcal{I}_\varnothing) \subseteq 2\text{-}\mathrm{Sp}_G$;

(ii) $\nabla^*(\mathcal{I}_\varnothing)$ *and* $2\text{-}\mathrm{Sp}_G$ *are not ideals;*

(iii) *if G is torsion free then* $2\text{-}\mathrm{Sp}_G \subset \tau^*(\mathcal{F}_G)$.

Proof. (i) It suffices to verify that $2\text{-}\mathrm{Sp}_G$ is ∇-complete. If not, then there is $X \notin 2\text{-}\mathrm{Sp}_G$ such that $\Delta(X) \in 2\text{-}\mathrm{Sp}_G$. We choose an infinite subset Y of G such that $gX \cap hX$ is infinite for all distinct $g, h \in Y$. Then $Y^{-1}Y \subseteq \Delta(X)$ so $\Delta(X)$ is not 2-sparse.

(ii) By [6, Theorem 3.3], there are two thin subsets A, B of G such that $A \cup B$ is not 2-sparse. Since $A, B \in \nabla^*(\mathcal{I}_\varnothing)$, we can apply (i).

(iii) We fix $A \in \mathrm{Sp}_G$ and an arbitrary sequence $(g_n)_{n\in\omega}$ in $G \setminus \{e\}$. Since G is torsion free, we can choose $\{0,1\}$-sequence $(m_n)_{n\in\omega}$ in ω such that the sequence $\left(g_0^{m_0}\dots g_n^{m_n}\right)_{n\in\omega}$ is injective. Since A is 2-sparse, there are $p, q \in \omega$, $p < q$, such that

$$g_0^{m_0}\dots g_p^{m_p} A \cap g_0^{m_0}\dots g_q^{m_q} A \in \mathcal{F}_G,$$

so $g_p^{m_p}\dots g_q^{m_q} A \cap A \in \mathcal{F}_G$. Hence, $A \in \tau^*(\mathcal{F}_G)$ and $2\text{-}\mathrm{Sp}_G \subseteq \mathcal{F}_G$.

By [1, Theorem 1.2], $\tau^*(\mathcal{F}_G)$ is an ideal, so, by (ii), the inclusion $2\text{-}\mathrm{Sp}_G \subset \tau^*(\mathcal{F}_G)$ is strict. □

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