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On stable conjugacy of finite subgroups of the plane Cremona group, I

Research Article

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Abstract: We discuss the problem of stable conjugacy of finite subgroups of Cremona groups. We compute the stable birational invariant $H^1(G, \mathrm{Pic}(X))$ for cyclic groups of prime order.

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1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic 0. The *Cremona group* $\mathrm{Cr}_n(\Bbbk)$ over $\Bbbk$ is the group of birational automorphisms of the projective space $\mathbb{P}^n$, or, equivalently, the group of $\Bbbk$-automorphisms of the field $\Bbbk(x_1, x_2, \dots, x_n)$ of rational functions in n independent variables. Adding new variables one gets a tower $\mathrm{Cr}_n(\Bbbk) \subset \mathrm{Cr}_{n+1}(\Bbbk) \subset \dots$ Subgroups $G_1, G_2 \subset \mathrm{Cr}_n(\Bbbk)$ are said to be *stably conjugate* if they are conjugate in some $\mathrm{Cr}_m(\Bbbk) \supset \mathrm{Cr}_n(\Bbbk)$. Stable conjugacy of Cremona groups is an analog of stable birational equivalence [5, 15] of varieties over non-closed fields. A subgroup $G \subset \mathrm{Cr}_n(\Bbbk)$ is said to be *linearizable* if the embedding is induced by a linear action of G on the projective space $\mathbb{P}^n$. A subgroup $G \subset \mathrm{Cr}_n(\Bbbk)$ is said to be *stably linearizable* if it is stably conjugate to linearizable one.

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We are interested in stable conjugacy of subgroups of $\mathrm{Cr}_n(\Bbbk)$ (cf. [13]). An example of a subgroup $G \subset \mathrm{Cr}_n(\Bbbk)$ (for $n = 3$) that is not stably conjugate to any subgroup induced by a linear action was constructed by Popov [13]. The construction is based on the example of Artin and Mumford [2] and uses non-triviality of torsions of $H^3(Y, \mathbb{Z})$ for the quotient variety $Y = \mathbb{P}^3/G$. In this paper we use another, very simple approach.

Recall that any finite subgroup $G \subset \mathrm{Cr}_n(\Bbbk)$ is induced by a *biregular* action on a non-singular projective rational variety X (see subsection 2.1). One can easily show that the group $H^1(G, \mathrm{Pic}(X))$ is stable birational invariant and so it does not depend on the choice of X (see Corollary 2.3). In particular, if $G \subset \mathrm{Cr}_n(\Bbbk)$ is stably conjugate to a linear action, then $H^1(G, \mathrm{Pic}(X)) = 0$. This fact is not surprising: in the arithmetic case it was known for a long time (see e.g. [11, 15]). An interesting fact is that in the geometric case the group $H^1(G, \mathrm{Pic}(X))$ typically admits a good description. Note that in [10] the authors used a similar approach to construct non-stably conjugate embeddings of certain groups to $\mathrm{Cr}_n(\Bbbk)$, $n \geq 3$, [10, Example 1.36]. Our method is very elementary and can be applied to another classes of groups.

We concentrate on the case of the plane Cremona group $\mathrm{Cr}_2(\Bbbk)$. In this case the classification of finite subgroups has a long history. It was started in works of Bertini and completed recently by Dolgachev and Iskovskikh [7] (however even in this case some questions remain open). We prove the following.

Theorem 1.1.
Let a cyclic group G of prime order p act on a non-singular projective rational surface X. Assume that G fixes (point-wise) a curve of genus $g > 0$. Then

$$H^1(G, \mathrm{Pic}(X)) \simeq (\mathbb{Z}/p\mathbb{Z})^{2g}. \tag{1}$$

Taking the results of [3, 4, 8] into account (see Theorem 2.1) we get the following.

Corollary 1.2.
In the notation of Theorem 1.1 the following are equivalent:

(i) $H^1(G, \mathrm{Pic}(X)) = 0$,

(ii) *G does not fix point-wise a curve C of positive genus,*

(iii) *(X, G) is linearizable,*

(iv) *(X, G) is stably linearizable.*

In particular, we see that classical de Jonquières, Bertini and Geiser involutions are not stably linearizable. Another application of the above corollary is that the action of the simple Klein group of order 168 on del Pezzo surface of degree 2 is also not stably linearizable. More applications will be given in the forthcoming second part of the paper.

2. Preliminaries

2.1. G-varieties

Let G be a finite group. A *G-variety* is a pair (X, ρ), where X is a projective variety and ρ is an *injective* homomorphism from G to $\mathrm{Aut}(X)$. A *morphism* (resp. *rational map*) of the pairs $(X, \rho) \to (X', \rho')$ is defined to be a morphism $f\colon X \to X'$ (resp. rational map $f\colon X \dashrightarrow X'$) such that $\rho'(G') = f \circ \rho(G) \circ f^{-1}$. In particular, two subgroups of $\mathrm{Aut}(X)$ define isomorphic (resp. G-birationally equivalent) G-varieties if and only if they are conjugate inside $\mathrm{Aut}(X)$ (resp. $\mathrm{Bir}(X)$). If no confusion is likely, we will denote a G-variety by (X, G) or even by X.

Now let X be an algebraic variety and let $G \subset \mathrm{Bir}(X)$ be a finite subgroup. By shrinking X we may assume that G acts on X by biregular automorphisms. Then, replacing X with the normalization of a completion of the quotient X/G in the field $\Bbbk(X)$, we may assume that X is projective. Finally we can apply an equivariant resolution of singularities and replace X with its non-singular (projective) model. Thus $G \subset \mathrm{Bir}(X)$ is induced by a biregular action of G on a non-singular projective G-variety. In particular, this construction can be applied to finite subgroups of $\mathrm{Cr}_n(\Bbbk)$.

2.2. Minimal rational G-surfaces

Let X be a projective non-singular G-surface, where G is a finite group. It is said to be *G-minimal* if any birational G-equivariant morphism $f\colon X \to Y$ is an isomorphism. It is well known that for any G-surface there is a minimal (projective non-singular) model (see e.g. [9]). If the surface X is additionally rational, then one of the following holds [9]:

- X is a del Pezzo surface whose invariant Picard group $\operatorname{Pic}(X)^G$ is of rank 1, *or*
- X admits a structure of G-conic bundle, that is, there exists a surjective G-equivariant morphism $f\colon X \to \mathbb{P}^1$ such that $f_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^1}$, $-K_X$ is f-ample and $\operatorname{rk}\operatorname{Pic}(X)^G = 2$.

2.3. Elements of prime order

The classification of elements of prime order in the space Cremona group can be summarized as follows.

Theorem 2.1 ([3, 4, 8]).
Let $G = \langle\delta\rangle \subset \operatorname{Cr}_2(\Bbbk)$ be a cyclic subgroup of prime order p and let (X, G) be its non-singular projective model. Then the following hold.

(i) *The action (X, G) is linearizable if and only if the fixed point locus X^G does not contain any curve of positive genus.*

(ii) *If X^G contains a curve C of genus $g > 0$, then other irreducible components of X^G are either points or rational curves. In this case the minimal model $(X_{\min}, G)$ is unique up to isomorphism and there are the following possibilities:*

p	g	K_X^2	$X_{\min}$	δ
2	≥ 1	$6-2g$	*conic bundle*	*de Jonquières involution*
2	3	2	*del Pezzo surface*	*Geiser involution*
2	4	1	*del Pezzo surface*	*Bertini involution*
3	1	3	*del Pezzo surface*	*[8, A1]*
3	2	1	*del Pezzo surface*	*[8, A2]*
5	1	1	*del Pezzo surface*	*[8, A3]*

2.4. Stable conjugacy

Let (X, G) and (Y, G) be G-varieties. We say that (X, G) and (Y, G) are *stably birational* if for some n and m there exists an equivariant birational map $X \times \mathbb{P}^n \dashrightarrow Y \times \mathbb{P}^m$, where actions on $\mathbb{P}^n$ and $\mathbb{P}^m$ are trivial. This is equivalent to the existence of a $\Bbbk$-isomorphism $\Bbbk(X)(x_1, \dots, x_n) \simeq \Bbbk(Y)(y_1, \dots, y_m)$ and conjugacy of the embeddings of G to $\Bbbk$-automorphism groups of $\Bbbk(X)(x_1, \dots, x_n)$ and $\Bbbk(Y)(y_1, \dots, y_m)$.

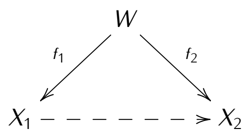
The following fact is well known in the arithmetic case (see e.g. [15]). Since we were not able to find a good reference for the present, geometric form, we provide a complete proof.

Proposition 2.2 (cf. [11, 2.2]).
Let X_1 and X_2 be projective non-singular G-varieties. Assume that (X_1, G) and (X_2, G) are stably birational. Then there are permutation G-modules Π_1 and Π_2 such that the following isomorphism of G-modules holds:

$$\operatorname{Pic}(X_1) \oplus \Pi_1 \simeq \operatorname{Pic}(X_2) \oplus \Pi_2.$$

The proof below is quite standard and depends on the resolution of singularities. There is a more sophisticated but similar proof due to Moret-Bailly which works in positive characteristic as well (see [12, 6.2], [6, 2.A.1]).

Proof. By our assumption, for some n and m, there exists a G-birational map $X_1\times\mathbb{P}^n \dashrightarrow X_2\times\mathbb{P}^m$, where the action of G on $\mathbb{P}^n$ and $\mathbb{P}^m$ is trivial. Replacing X_1 and X_2 with $X_1\times\mathbb{P}^n$ and $X_2\times\mathbb{P}^m$ respectively we may assume that there exists a G-birational map $X_1 \dashrightarrow X_2$. Consider a common G-equivariant resolution (see e.g. [1])



Then the maps f_1^* and f_2^* induce isomorphisms $\operatorname{Pic}(W) \simeq \operatorname{Pic}(X_1)\oplus\Pi_1 \simeq \operatorname{Pic}(X_2)\oplus\Pi_2$, where Π_1 (resp. Π_2) is a free $\mathbb{Z}$-module whose basis is formed by the prime f_1-exceptional (resp. f_2-exceptional) divisors. Since f_1 and f_2 are G-equivariant, the group G permutes these divisors, so Π_1 and Π_2 are permutation modules. □

Applying Shapiro's lemma, $H^1(G,\Pi_1) = H^1(G,\Pi_2) = 0$ (see e.g. [14, Chapter 1, § 2.5]), we get

Corollary 2.3.
In the notation of Proposition 2.2 we have $H^1(G,\operatorname{Pic}(X_1)) \simeq H^1(G,\operatorname{Pic}(X_2))$.

Corollary 2.4.
If in the notation of Proposition 2.2, (X,G) *is stably linearizable, then* $H^1(H,\operatorname{Pic}(X)) = 0$ *for any subgroup* $H \subset G$.

3. Proof of Theorem 1.1

Let $\delta \in G$ be a generator and let C be a (smooth) curve of fixed points with $g = g(C) > 0$. We replace (X,G) with its minimal model. First we consider the case where X is a del Pezzo surface with $\operatorname{rk}\operatorname{Pic}(X)^G = 1$. We start with more general settings.

3.1. Del Pezzo case

Let X be a del Pezzo surface and let $d = K_X^2$. Let $G \subset \operatorname{Aut}(X)$ be any finite subgroup such that $\operatorname{Pic}(X)^G \simeq \mathbb{Z}$. Denote

$$Q = \{x \in \operatorname{Pic}(X) : x\cdot K_X = 0\}.$$

Lemma 3.1.
In the above notation there exists the following natural exact sequence:

$$0 \to \mathbb{Z}/d\mathbb{Z} \to H^1(G,Q) \to H^1(G,\operatorname{Pic}(X)) \to 0.$$

Proof. By our assumption $g > 0$, we have $d \le 6$ (see e.g. Theorem 2.1). Hence, $\operatorname{Pic}(X)^G = \mathbb{Z}\cdot[K_X]$ (see [9]). Then the assertion follows from the exact sequence of G-modules

$$0 \to Q \to \operatorname{Pic}(X) \xrightarrow{\cdot K_X} \mathbb{Z} \to 0 \tag{2}$$

because $H^1(G,\mathbb{Z}) = 0$ for a finite group G. □

Lemma 3.2.
In the notation of 3.1 assume that G be a cyclic group generated by $\delta \in G$. *Then the order of* $H^1(G,\operatorname{Pic}(X))$ *equals to* $|\chi_{\delta,Q}(1)|/d$, *where* $\chi_{\delta,Q}(t)$ *is the characteristic polynomial of the action of* δ *on* Q.

The proof uses the following easy observation.

Observation 3.3.

Let G be a cyclic group of order n generated by $\delta \in G$. Let Π be a $\mathbb{Z}$-torsion free $\mathbb{Z}[G]$-module. Denote

$$N = 1 + \delta + \cdots + \delta^{n-1}, \qquad \eta = 1 - \delta \in \mathbb{Z}[G].$$

Then $H^1(G, \Pi) = (\ker N)/\eta(\Pi)$.

Proof of Lemma 3.2. Apply the above fact with $\Pi = Q$. We get $\ker N = Q$ (because $N(Q) \subset Q^\delta = 0$). Hence $[Q : \eta(Q)] = |\det \eta|$. ☐

Corollary 3.4.

In the notation of 3.1, let G be a cyclic group of prime order p. Then $d = p^j$, where $j = 0$ or 1, and

$$H^1(G, \mathrm{Pic}(X)) \simeq (\mathbb{Z}/p\mathbb{Z})^{(9-d)/(p-1)-j}.$$

Proof. Since Q has no δ-invariant vectors, the only $\mathbb{Q}$-irreducible factor of $\chi_{\delta,Q}(t)$ is the cyclotomic polynomial $t^{p-1} + \cdots + t + 1$. Hence we have

$$\chi_{\delta,Q}(t) = \left(t^{p-1} + \cdots + t + 1\right)^s, \qquad s = \frac{\mathrm{rk}\, Q}{p-1} = \frac{9-d}{p-1}.$$

On the other hand, $H^1(G, \mathrm{Pic}(X))$ is a p-torsion group because $G \simeq \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/d\mathbb{Z}$ is a subgroup of $\mathbb{Z}/p\mathbb{Z}$ by (2). Hence $d = p^j$ with $j = 0$ or 1 and $H^1(G, \mathrm{Pic}(X)) \simeq (\mathbb{Z}/p\mathbb{Z})^{s-j}$. ☐

Proof of Theorem 1.1 in the del Pezzo case. By the classification Theorem 2.1, for (p, d, g) we have one of the following possibilities: $(2, 1, 4)$, $(2, 2, 3)$, $(3, 3, 1)$, $(3, 1, 2)$, $(5, 1, 1)$. Applying Corollary 3.4 we get our equality (1). ☐

Remark.

Our proof of (1) is based on the classification Theorem 2.1. It is interesting to find a classification independent proof. The authors were not able to find it.

3.2. Conic bundle case

Now assume that X has a structure of G-equivariant conic bundle $f: X \to \mathbb{P}^1$. Then again by Theorem 2.1, δ is a *de Jonquières involution* of genus $g > 0$. Recall that this is an element $\delta \in \mathrm{Cr}_2(\Bbbk)$ of order 2 induced by an action on a (G-equivariant) relatively minimal conic bundle $f: X \to \mathbb{P}^1$ with $2g + 2$ degenerate fibers so that the locus of fixed points is a hyperelliptic curve C of genus g (elliptic curve if $g = 1$) and the restriction $f\lceil_C : C \to \mathbb{P}^1$ is a double cover (see [3, 7] for details).

3.3.

Let F be a typical fiber and let $F_i = F_i' + F_i''$, $i = 1, \ldots, 2g+2$, be all the degenerate fibers. Let $Q \subset \mathrm{Pic}(X)$ be the $\mathbb{Z}$-submodule of rank $2g+3$ generated by the components of degenerate fibers. It has a $\mathbb{Z}$-basis consisting of F and F_i', $i = 1, \ldots, 2g+2$. The action of δ is given by

$$\delta : F_i' \mapsto F - F_i', \qquad F \mapsto F.$$

Apply Observation 3.3 with $\Pi = Q$. We have

$$\ker N = \left\{ \alpha F + \sum \alpha_i F_i' \in Q \, : \, 2\alpha + \sum \alpha_i = 0 \right\}$$

and $\eta(Q)$ is generated by the classes of $2F_i' - F$. Therefore, $H^1(G, Q) = (\mathbb{Z}/2\mathbb{Z})^{2g+1}$. Since $H^1(G, \mathbb{Z}) = 0$, from the exact sequence

$$0 \to Q \to \operatorname{Pic}(X) \xrightarrow{\cdot F} \mathbb{Z} \to 0$$

we get

$$0 \to Q^G \to \operatorname{Pic}(X)^G \xrightarrow{\cdot F} \mathbb{Z} \to H^1(G, Q) \to H^1(G, \operatorname{Pic}(X)) \to 0.$$

Note that the image of $\operatorname{Pic}(X)^G$ in $\mathbb{Z}$ is generated by $K_X \cdot F = 2$. Therefore, $H^1(G, \operatorname{Pic}(X)) = (\mathbb{Z}/2\mathbb{Z})^{2g}$.

This proves Theorem 1.1.

Proof of Corollary 1.2. The implication (i) ⇒ (ii) follows by Theorem 1.1, the implication (iv) ⇒ (i) is an immediate consequence of Corollary 2.4, (iii) ⇒ (iv) is obvious, and (ii) ⇒ (iii) follows by Theorem 2.1. □

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