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Approximation of solutions to second order nonlinear Picard problems with Carathéodory right-hand side

Research Article

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Abstract: We present an approximation method for Picard second order boundary value problems with Carathéodory right-hand side. The method is based on the idea of replacing a measurable function in the right-hand side of the problem with its Kantorovich polynomial. We will show that this approximation scheme recovers essential solutions to the original BVP. We also consider the corresponding finite dimensional problem. We suggest a suitable mapping of solutions to finite dimensional problems to piecewise constant functions so that the later approximate a solution to the original BVP. That is why the presented idea may be used in numerical computations.

MSC: 34A45, 65L10, 34B24

Keywords: Picard problem • Kantorovich polynomial • Approximation of BVP solution

1. Introduction

We will consider the nonlinear boundary value problem

$$\begin{cases} u''(t) + \varphi(t, u(t)) = 0, & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

where $\varphi\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function; i.e. $\varphi(\cdot\,, s)\colon [0,1] \to \mathbb{R}$ is measurable for all $s \in \mathbb{R}$, $\varphi(t,\cdot)\colon \mathbb{R} \to \mathbb{R}$ is continuous for almost all $t \in [0,1]$, and for each $R > 0$ there exists an integrable function $m_R \in L^1(0,1)$ such that $|\varphi(t,s)| \le m_R(t)$ for almost all $t \in [0,1]$ and $|s| \le R$. We would approximate solutions to problem (1) by solutions to the

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finite dimensional problem defined below. The idea is based on Kantorovich polynomials (see [4, 12] for more details and properties, including their relation to Bernstein polynomials).

Let us assume that $\psi \in L^1(0,1)$ is an integrable function. Then n-th Kantorovich polynomial of function ψ is

$$K_n(\psi)(t) = \sum_{k=0}^{n} \psi_{kn} p_{kn}(t), \qquad \text{where} \quad p_{kn}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

and

$$\psi_{kn} = (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} \psi(s)\, ds = (n+1) \int_0^1 \chi_{[k/(n+1),(k+1)/(n+1)]}(s) \psi(s)\, ds,$$

for $k = 0, 1, \dots, n$ and χ_A denoting the characteristic function of the set A. Basic properties of the function $K_n\colon L^1(0,1) \to L^1(0,1)$ are given in [4, 12]. Let us recall the main of them.

Property 1.1 ([12, Theorem 2.1.2], [4]).

Let $\psi \in L^p(0,1)$, $p \geq 1$. Then $K_n(\psi) \in L^p(0,1)$ and the map $K_n\colon L^p(0,1) \to L^p(0,1)$ is linear, continuous, and $\|K_n\| = 1$. Moreover, $\lim_{n\to+\infty} \|K_n(\psi) - \psi\|_{L^p(0,1)} = 0$.

Consider the following problem:

$$\begin{cases} u''(t) + \big(K_n \varphi(\cdot, u(\cdot))\big)(t) = 0, & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{2_n}$$

Denote by $T\colon L^1(0,1) \to C[0,1]$ a completely continuous map such that

$$Th = u \qquad \Longleftrightarrow \qquad \begin{cases} u''(t) + h(t) = 0, & t \in (0,1), \\ u(0) = u(1) = 0. \end{cases}$$

The map T maps integrally bounded sets of $L^1(0,1)$ onto relatively compact subsets of $C^1[0,1]$ (i.e. also relatively compact subsets of $C[0,1]$). We call a subset $A \subset L^1(0,1)$ integrally bounded if there exists a function $m_A \in L^1(0,1)$ such that for all $u \in A$ the inequality $|u(t)| \leq m_A(t)$ holds for almost all $t \in [0,1]$. Let $\Phi\colon C[0,1] \to L^1(0,1)$ be the Nemytskii map for φ, i.e. for $u \in C[0,1]$ we have $\Phi(u)(t) = \varphi(t, u(t))$. Rewrite problems (1) and $(2)_n$ in the operator form as, respectively, $u = T\Phi(u)$ and

$$u = TK_n\Phi(u). \tag{2}$$

Let $f, f_n\colon C[0,1] \to C[0,1]$ be given by $f(u) = u - T\Phi(u)$ and $f_n(u) = u - TK_n\Phi(u)$ for $u \in C[0,1]$ and $n \in \mathbb{N}$. Observe that for every n the map $TK_n\Phi\colon C[0,1] \to C[0,1]$ is a completely continuous map. This is because the map $K_n\colon L^1(0,1) \to L^1(0,1)$ (for fixed $n \in \mathbb{N}$) maps integrally bounded subsets of $L^1(0,1)$ onto integrally bounded subsets. Indeed, let us assume that $A \subset L^1(0,1)$ is integrally bounded, then $|u(t)| \leq m_A(t)$ for some function $m_A \in L^1(0,1)$. Consequently,

$$|(K_n u)(t)| \leq \sum_{k=0}^{n} s|p_{k,n}(t)|(n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} m_A(s)\, ds \leq (n+1) \int_0^1 m_A(s) \left(\sum_{k=0}^{n} p_{k,n}(t) \right) ds = (n+1) \int_0^1 m_A(s)\, ds.$$

Hence for all functions u bounded by an integrable function m_A, the functions $K_n u$ are bounded (for fixed $n \in \mathbb{N}$) by the constant $(n+1) \int_0^1 m_A(s)\, ds$.

We will show that for some $u_0 \in C[0,1]$ satisfying $f(u_0) = 0$ there exists a sequence $\{u_n\}$ of solutions, $f_n(u_n) = 0$, which converges to u_0. From the topological perspective the condition, which guarantees the existence of an approximating sequence, is the nonzero value of the Leray–Schauder degree in a neighbourhood of the (isolated) zero of the map f.

Some examples referring to known existence results and showing the situations where this topological sufficient condition is satisfied will be given.

In Section 3 we will present some ideas on how we can search numerically for zeros of the maps f_n. The idea is to approximate a solution to equation (2) by some piecewise constant functions. We will show that the convergent (in $L^1(0,1)$ norm) sequence of these piecewise constant functions approximates a zero of the map f.

It has to be mentioned here that finite dimensional approximations of problems with Carathéodory right-hand side shall probably require different methods than problems with a continuous right-hand side. Regular finite difference schemes should be avoided because we cannot guarantee convergence of such methods in the case of a discontinuous right-hand side. There are many other ideas known starting from integral Euler-type methods (with practical usage limited to specific affine cases — see the discussion in [9]) up to random methods (see in particular [9, 15, 16]). We will suggest a method which does not entirely escape from the integration, but in many cases leads to classical nonlinear problems in $\mathbb{R}^k$.

2. Existence of the approximating sequence

Let us start with a series of lemmas.

Lemma 2.1.
If there exists a sequence of function $\{u_n\} \subset C[0,1]$ convergent to u_0 such that $f_n(u_n) = 0$, then u_0 is a solution to problem (1).

Proof. We can write

$$u_n = TK_n\Phi(u_n) = TK_n(\Phi(u_n) - \Phi(u_0)) + TK_n\Phi(u_0). \tag{3}$$

Further,

$$\|TK_n(\Phi(u_n) - \Phi(u_0))\| \le \|T\| \cdot \|K_n\| \cdot \|\Phi(u_n) - \Phi(u_0)\|.$$

As $\|K_n\| = 1$, $n \in \mathbb{N}$, and Φ is a continuous map, the right-side of the above inequality converges to zero. By Property 1.1, $TK_n\Phi(u_0) \to T\Phi(u_0)$ as $n \to +\infty$. Allowing $n \to +\infty$ in equation (3) we obtain $u_0 = T\Phi(u_0)$ which completes the proof. □

The next lemma will be a slight generalization of the above result.

Lemma 2.2.
Let us assume that there exists a bounded sequence $\{u_n\} \subset C[0,1]$ such that $f_n(u_n) = 0$. Then there exists a subsequence $\{u_{\gamma(n)}\}$ of $\{u_n\}$ which converges to $u_0 \in C[0,1]$ and u_0 is a solution to problem (1).

Proof. First let us show that for any bounded sequence $\{u_n\} \subset C[0,1]$, $f(u_n) = 0$, the sequence $\{TK_n\Phi(u_n)\} = \{u_n\} \subset C[0,1]$ contains a convergent subsequence. It is not obvious that the sequence $\{K_n\Phi(u_n)\} \subset L^1(0,1)$ is integrally bounded, but we will show that functions u_n are uniformly continuous.

The sequence $\{u_n\}$ is bounded in $C[0,1]$, so there exists $R > 0$ such that for all $n \in \mathbb{N}$ and $t \in [0,1]$ the inequality $|u_n(t)| \le R$ holds. Let us fix $n \in \mathbb{N}$ and denote $\psi(t) = \varphi(t, u_n(t))$. Then $|\psi(t)| \le m_R(t)$. Since $u_n = TK_n\Phi(u_n)$, it may be written as

$$u_n(t) = -\int_0^t \int_0^s (K_n\psi)(\tau)\, d\tau\, ds + t\int_0^1 \int_0^s (K_n\psi)(\tau)\, d\tau\, ds.$$

Let us take any $t_1, t_2 \in [0,1]$ such that $t_1 \leq t_2$. Then

$$|u_n(t_2) - u_n(t_1)| \leq \int_{t_1}^{t_2} \left| \int_0^s (K_n\psi)(\tau)\, d\tau \right| ds + |t_2 - t_1| \int_0^1 \int_0^1 |(K_n\psi)(\tau)|\, d\tau\, ds$$
$$\leq \int_{t_1}^{t_2} \int_0^1 |(K_n\psi)(\tau)|\, d\tau\, ds + |t_2 - t_1| \int_0^1 |(K_n\psi)(\tau)|\, d\tau.$$

As $\int_0^1 |(K_n\psi)(\tau)|\, d\tau = \|K_n\psi\|_{L^1(0,1)} \leq \|K_n\| \cdot \|\psi\|_{L^1(0,1)} \leq \|m_R\|_{L^1(0,1)}$, we have

$$|u(t_2) - u(t_1)| \leq 2|t_2 - t_1| \cdot \|m_R\|_{L^1(0,1)}.$$

By Arzelà–Ascoli's theorem functions u_n form a relatively compact subset of $C[0,1]$. So a convergent subsequence of $\{u_n\}$ can be chosen. Let us denote this subsequence by $\{u_{\gamma(n)}\}$ and its limit by $u_0 \in C[0,1]$. Let us write

$$u_{\gamma(n)} = TK_{\gamma(n)}\Phi(u_0) + T\big(K_{\gamma(n)}\Phi(u_{\gamma(n)}) - K_{\gamma(n)}\Phi(u_0)\big).$$

Similarly to the proof of Lemma 2.1, by allowing $n \to +\infty$ we obtain $u_0 = T\Phi(u_0)$. □

Lemma 2.3.

Let $u_0 \in C[0,1]$ be an isolated zero of the map f and $r > 0$ be such that $f^{-1}(0) \cap \overline{B(u_0,r)} = \{u_0\}$, where $B(u_0,r) \subset C[0,1]$ denotes the open ball centered at u_0 with radius r. Then, for almost all $n \in \mathbb{N}$, maps f_n and f are homotopic on $\overline{B(u_0,r)}$.

Proof. Define the homotopy $h_n \colon [0,1] \times \overline{B(u_0,r)} \to C[0,1]$ as

$$h_n(t,u) = (1-t)f(u) + tf_n(u) = u - T\Phi(u) - t(TK_n\Phi(u) - T\Phi(u)).$$

Let us assume that, contrary to our claim, there exists a sequence $(t_n, u_n) \in [0,1] \times \partial B(u_0, r)$ such that $h_n(t_n, u_n) = 0$ for infinitely many $n \in \mathbb{N}$. As (t_n, u_n) is bounded, we can find a convergent subsequence. Let us continue to denote this subsequence as $\{(t_n, u_n)\}$, with $t_n \to \overline{t} \in [0,1]$ and $u_n \to \overline{u} \in \partial B(u_0, r)$.

Now, we may use a representation similar to the one in (3):

$$u_n = T\Phi(u_n) - t_n\big(TK_n\Phi(u_n) - TK_n\Phi(\overline{u}) + TK_n\Phi(\overline{u}) - T\Phi(u_n)\big).$$

Similarly to Lemma 2.1, let us observe that the sequence $\{TK_n\Phi(u_n) - TK_n\Phi(\overline{u})\}$ converges to 0. On the other hand, the sequence $\{TK_n\Phi(\overline{u}) - T\Phi(u_n)\}$ can be represented as

$$TK_n\Phi(\overline{u}) - T\Phi(u_n) = TK_n\Phi(\overline{u}) - T\Phi(\overline{u}) + T\Phi(\overline{u}) - T\Phi(u_n).$$

Since Φ is a continuous operator and $K_n\Phi(\overline{u}) \to \Phi(\overline{u})$, the right-hand side of the last equation converges to 0. This means that letting $n \to +\infty$ we obtain $\overline{u} = T\Phi(\overline{u})$, which contradicts our assumption. Thus for almost all $n \in \mathbb{N}$ the homotopy h_n satisfies $h_n^{-1}(0) \cap \big([0,1] \times \partial B(u_0, r)\big) = \emptyset$, so f and f_n are homotopic. □

We can combine the two lemmas given above into the following theorem.

Theorem 2.4.

Let us assume that $u_0 \in C[0,1]$ is an isolated solution to problem (1) such that $\deg(f, B(u_0,r), 0) \neq 0$. Then there exists a sequence of solutions $\{u_n\}$ to $(2)_n$ such that $\{u_n\}$ converges to u_0 in $C[0,1]$.

Proof. Let us select any $\varepsilon_0 > 0$ such that $f^{-1}(0) \cap \overline{B(u_0, \varepsilon_0)} = \{u_0\}$. By Lemma 2.3 there exists $n_1 \in \mathbb{N}$ such that for $n \geq n_1$ the maps f_n may be joined by homotopy to the map f on $B(u_0, \varepsilon_0)$. This implies that $\deg(f_n, B(u_0, \varepsilon_0), 0) \neq 0$, hence for all $n > n_1$ there exists a zero, w_n^1 of the map f_n belonging to $B(u_0, \varepsilon_0)$.

Similarly, for any natural $k \geq 2$, there exists $n_k > n_{k-1}$ such that for $n \geq n_k$ the maps f_n may be joined by homotopy to the map f on $B(u_0, \varepsilon_0/k)$. So for each $k \in \mathbb{N}$ there exists a sequence $\{w_n^k\} \subset B(u_0, \varepsilon_0/k)$, $n \geq n_k$, such that $f_n(w_n^k) = 0$. The sequence $\{n_k\}$, $k = 1, 2, \ldots$, is increasing and obviously unbounded.

Now, we will define a sequence $\{u_n\}$ (for $n \geq n_1$) satisfying $f_n(u_n) = 0$, and convergent to u_0. To specify the value of u_n for $n \geq n_1$ we have to find a unique $k \in \mathbb{N}$ such that $n_k \leq n < n_{k+1}$. Then, we know there exists such $w_n^k \in B(u_0, \varepsilon_0/k)$ that $f_n(w_n^k) = 0$, set $u_n = w_n^k$. By construction $u_n \to u_0$ in $C[0, 1]$. □

Remark 2.5.
In Theorem 2.4 we assumed that the solution to problem (1) is isolated. However, as it can be seen from the proof, we may remove this local uniqueness assumption thus receiving a weaker conclusion of this theorem. In this case there exists a bounded sequence of solution $\{u_n\}$, $f_n(u_n) = 0$, containing a convergent subsequence (see Lemma 2.2).

The assumption that the solution to (1) has a nonzero Leray–Schauder degree may look restrictive, but there are many existence theorems where nonzero value of the Leray–Schauder degree is used in the proofs. So under conditions of those theorems the approximation sequence exists. One of such theorems is due to Granas, Guenther and Lee [6–8].

Example 2.6.
In [6–8] the authors consider the following boundary value problem:

$$\begin{cases} u''(t) = \varphi(t, u(t), u'(t)), \\ u(0) = u(1) = 0, \end{cases} \tag{4}$$

with φ being a continuous function depending on u'. It is proved that if φ satisfies the so-called Bernstein condition (or a more general Bernstein–Nagumo condition), then there exists a solution to (4). The proof is based on showing that the map f associated with the above problem can be joined by homotopy to the identity, thus the solution has a nonzero topological degree in some ball.

In our case φ does not depend on u', and the Bernstein condition sounds as there is $M > 0$ such that, for all $(t, s) \in [0, 1] \times \mathbb{R}$ satisfying $|s| > M$, the inequality $s\varphi(t, s) \geq 0$ holds. So for such φ a solution to problem (1) exists. Moreover, if the function $\varphi(t, s)$ is strictly decreasing in s for each fixed $t \in [0, 1]$, then the solution to (1) is unique.

The case of continuous right-hand side in (1) is considered in papers [2, 5]. In methods which refer to global bifurcation results, the Leray–Schauder degree is not used directly, but it serves as a tool in the Rabinowitz type theorems (see in particular [13, 14] where Carathéodory right-hand side is considered, as well as [3]). Actually in this case of global bifurcation results we can show that there must exist nontrivial solutions which have nonzero Leray–Schauder degree. The results mentioned above usually do not assume local uniqueness, but if it is not guaranteed, then we can apply a weaker version of Theorem 2.4 mentioned in Remark 2.5.

3. Finite dimensional approximations

Problems $(2)_n$ are actually finite dimensional ones because $W_n = (T \circ K_n)\big(L^1(0, 1)\big) \subset V_{n+2}$, where V_k denotes the space of polynomials which degrees do not exceed k. Moreover, W_n is an $(n+1)$-dimensional subspace of V_{n+2} consisting of polynomials $u \in V_{n+2}$ such that $u(0) = u(1) = 0$. We are interested in reducing problem $(2)_n$ to an equation $g_n(x) = 0$, where $x \in \mathbb{R}^{n+1}$ and $g_n \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a continuous map.

Before we suggest an appropriate map g_n let us write equation (2) in the expanded form

$$u(t) = \sum_{k=0}^{n} \varphi_{kn}(u)(Tp_{kn})(t), \qquad \text{where} \quad \varphi_{kn}(u) = (n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} \varphi(s, u(s))\, ds. \tag{5}$$

Integrate equation (5) on the interval $[l/(n+1), (l+1)/(n+1)]$, $l = 0, 1, \dots, n$. This leads to the following system of equations:

$$x_{ln}(u) = \sum_{k=0}^{n} \varphi_{kn}(u) a_{lkn}, \tag{6}$$

for $l = 0, 1, \dots, n$. In the above formula, $x_{ln}(u) = \int_{l/(n+1)}^{(l+1)/(n+1)} u(t)\, dt$ and $a_{lkn} = \int_{l/(n+1)}^{(l+1)/(n+1)} (Tp_{kn})(t)\, dt$. It is difficult to say that this finite system of equations is a simplification of equation (2), but looking at the very special case of the function φ, we may suggest some further ideas. Let us look at the linear case, even though the approximation of the linear case of (1) may seem quite useless. With this case we can test some ideas regarding the transformation of equation (2) given in the functional space setting into the system in the Euclidean space. At the end of the paper we will also discuss how the coefficients a_{lkn} can be calculated.

Example 3.1.

This example refers to a certain family of linear problems (1), so its practical value is rather weak, but it shows a nice transformation of the operator equation $f_n(u) = 0$ defined in the Banach space $C[0, 1]$ into the problem defined in the Euclidean space $\mathbb{R}^{n+1}$.

Let us assume that $\varphi(t, s) = \psi(t) \cdot s$, where $\psi \in L^1(0, 1)$ is constant on every interval $[l/(n+1), (l+1)/(n+1)]$ for $l = 0, 1, \dots, n$. Then

$$\varphi_{kn}(u) = (n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} \psi(s)u(s)\, ds = (n+1)\,\frac{\psi_{kn}}{n+1}\, x_{k,n}(u),$$

where ψ_{kn} denotes the value of ψ on the interval $[k/(n+1), (k+1)/(n+1)]$. Thus

$$x_{ln}(u) = \sum_{k=0}^{n} a_{lkn}\psi_{kn}x_{kn}(u)$$

or in the matrix form $x_n(u) = A_n \cdot \Phi_n(x_n(u))$, where $A_n = [a_{lkn}]_{0\le l,k\le n}$, $x_n(u) = [x_{0n}(u), x_{1n}(u), \dots, x_{nn}(u)]^T$ and $\Phi_n(x_n(u)) = [\psi_{0n}x_{0n}(u), \psi_{1n}x_{1n}(u), \dots, \psi_{nn}x_{nn}(u)]^T$.

Now, define the map $g_n \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ related to the map f_n as follows:

$$g_n(x) = x - A_n\Phi_n(x), \qquad x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1},$$

where $\Phi_n \colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is given by

$$\Phi_n(x) = (n+1)\left(\int_0^{1/(n+1)} \varphi(t, x_0)\, dt,\ \int_{1/(n+1)}^{2/(n+1)} \varphi(t, x_1)\, dt,\ \dots,\ \int_{n/(n+1)}^{1} \varphi(t, x_n)\, dt\right). \tag{7}$$

A relation between the maps g_n and f_n may be established by means of the linear map $P_n \colon L^1(0, 1) \to \mathbb{R}^{n+1}$ given by

$$P_n(u) = (x_{0n}(u), x_{1n}(u), \dots, x_{nn}(u)), \qquad \text{where} \quad x_{kn}(u) = (n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} u(s)\, ds, \quad k = 0, 1, \dots, n.$$

Actually we can treat the image of the map P_n as a set of piecewise constant functions defined as

$$\sum_{k=0}^{n} x_{kn}(u)X_{[k/(n+1),(k+1)/(n+1)]}(t).$$

Following this idea let us define the linear map $Q_n\colon C[0,1] \to L^1(0,1)$ as

$$(Q_n u)(t) = \sum_{k=0}^{n} x_{kn}(u)X_{[k/(n+1),(k+1)/(n+1)]}(t) = \sum_{k=0}^{n} (n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} u(s)\,ds \cdot X_{[k/(n+1),(k+1)/(n+1)]}(t).$$

The maps Q_n are not only linear but also uniformly bounded. Their norms do not exceed 1. Indeed, for any continuous $u \in C[0,1]$ such that $\|u\| \le 1$, for any $k = 0, 1, \dots, n$ we have

$$|x_{k,n}(u)| \le (n+1)\int_{k/(n+1)}^{(k+1)/(n+1)} |u(s)|\,ds \le 1,$$

hence $|(Q_n u)(t)| \le 1$ for almost all $t \in [0,1]$. Thus $\|Q_n u\|_{L^1(0,1)} \le 1$. And, for $u = 1$, $\|Q_n u\|_{L^1(0,1)} = 1$, hence $\|Q_n\| = 1$.

The following simple lemma will be used later.

Lemma 3.2.

If $u_n \to u_0$ in $C[0,1]$, then $Q_n(u_n) \to u_0$ in $L^1(0,1)$.

Proof. Let us start with rewriting the conclusion of the lemma:

$$Q_n u_n - u_0 = Q_n u_n - Q_n u_0 + Q_n u_0 - u_0.$$

Since $\|Q_n u_n - Q_n u_0\|_{L^1(0,1)} \le \|Q_n\| \cdot \|u_n - u_0\|_{C[0,1]} \to 0$, it is enough to prove that $Q_n u_0 \to u_0$ in $L^1(0,1)$.

$$\begin{aligned}\int_0^1 \left|(Q_n u_0)(t) - u_0(t)\right| dt &= \int_0^1 \left|\sum_{k=0}^{n} x_{kn}(u_0)X_{[k/(n+1),(k+1)/(n+1)]}(t) - u_0(t)X_{[k/(n+1),(k+1)/(n+1)]}(t)\right| \\ &\le \int_0^1 \sum_{k=0}^{n} |x_{kn}(u_0) - u_0(t)|X_{[k/(n+1),(k+1)/(n+1)]}(t)\,dt.\end{aligned}$$

As the function u_0 is uniformly continuous and $x_{kn}(u_0) = u_0(\alpha_k)$ for some $\alpha_k \in [k/(n+1),(k+1)/(n+1)]$ we know that, for an arbitrarily selected $\varepsilon > 0$ and for all $n \in \mathbb{N}$ sufficiently large, we obtain

$$|x_{kn}(u_0) - u_0(t)| \le \varepsilon,$$

for all $t \in [k/(n+1),(k+1)/(n+1)]$. This means that $\int_0^1 |(Q_n u_0)(t) - u_0(t)|\,dt \le \varepsilon$, and finally $Q_n u_0 \to u_0$ in $L^1(0,1)$. □

Let us define one more map $R_n\colon \mathbb{R}^{n+1} \to L^1(0,1)$ related to maps P_n and Q_n. Let R_n assign to the vector $x \in \mathbb{R}^{n+1}$ a piecewise constant function such that

$$(R_n x)(t) = x_k \qquad \text{for} \quad t \in \left(\frac{k}{n+1}, \frac{k+1}{n+1}\right).$$

Following this notation we can see that for any $u \in C[0,1]$ we can write

$$g_n(P_n(u)) = P_n u - P_n T K_n(\Phi(Q_n u)). \tag{8}$$

We can also see that $g_n(P_n(u)) = P_n u - P_n T K_n(\Phi(Q_n u)) = 0$ is equivalent to

$$Q_n u - Q_n T K_n(\Phi(Q_n u)) = 0, \tag{9}$$

because $P_n u = 0$ iff $Q_n u = 0$.

Below we will try to answer two important questions which appear in this context:

(i) Is it possible to prove that there exists a zero of the map g_n close (whatever it means) to the zero u_0 of the map f?

(ii) Is it possible to prove that the sequence of zeros of maps g_n approaching (whatever it means) a continuous function $u_0 \in C[0,1]$ makes this function a zero of the map f?

These two questions ought to be phrased much more precisely, but when we do it we will see that the answer to both questions is "yes".

First let us show that the superposition of the map Φ and Q_n can be well defined. The Nemytskii map Φ is defined on the space of continuous functions $C[0,1]$, while the image of Q_n lies in $L^1(0,1)$. We will change the definition of the Nemytskii map a bit. First recall a theorem of Veinberg and Krasnoselskii (see [10, 11, 17]). We will use the formulation given in [1].

Theorem A ([10]).

Let $\varphi\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ be a Carathéodory map satisfying the condition

$$|\varphi(t,s)| \le a(t) + b|s|, \qquad a \in L^1(0,1), \quad b \in \mathbb{R}. \tag{10}$$

Then the map $\Phi\colon L^1(0,1) \to L^1(0,1)$ given by $\Phi(u)(t) = \varphi(t,u(t))$ is well defined, continuous and maps bounded sets onto bounded sets.

Our map φ does not necessarily satisfy condition (10), but we can modify it in such a way that this condition is fulfilled and the sets of solutions to problems (1) and $(2)_n$ are not changed — at least in the domain of our interest, i.e. in some bounded neighbourhood of some $u_0 \in C_0[0,1]$. Let us fix $R > 0$ and define a map $\widetilde{\varphi}\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ by

$$\widetilde{\varphi}(t,s) = \begin{cases} \varphi(t,s), & |s| \le R,\\ \varphi(t,R), & s > R,\\ \varphi(t,-R), & s < -R. \end{cases}$$

This is the Carathéodory function satisfying condition (10), which makes the map $\widetilde{\Phi}\colon L^1(0,1) \to L^1(0,1)$ continuous. Moreover, for any $u \in C[0,1]$, with $\|u\| \le R$, $\widetilde{\Phi}(u) = \Phi(u)$ and $\widetilde{\Phi}(Q_n u) = \Phi(Q_n u)$. Let $\widetilde{f}_n\colon C[0,1] \to C[0,1]$ be given by

$$\widetilde{f}_n(u) = u - TK_n\big(\widetilde{\Phi}(Q_n u)\big).$$

This map is continuous and for fixed $n \in \mathbb{N}$ the superposition $TK_n\widetilde{\Phi}Q_n\colon C[0,1] \to C[0,1]$ is completely continuous. This is because the map $K_n \circ \widetilde{\Phi} \circ Q_n\colon C[0,1] \to L^1(0,1)$ maps bounded subsets of $C[0,1]$ onto integrally bounded subsets of $L^1(0,1)$ (similarly as the superposition $K_n \circ \Phi$). From the definition of the map $\widetilde{f}_n$ and relation (8) we get the following result.

Proposition 3.3.

If v is a zero of the map $\widetilde{f}_n$, then $P_n(v)$ is a zero of the map g_n.

This means that each zero of the map $\widetilde{f}_n$, corresponds to a zero of the map g_n. Hence we are quite close to answering question (i).

Lemma 3.4.

Let u_0 be an isolated zero of the map f and $r > 0$ be such that $f^{-1}(0) \cap \overline{B(u_0, r)} = \{u_0\}$. Then for $n \in \mathbb{N}$ big enough the maps f_n and $\widetilde{f}_n$ are homotopic on $B(u_0, r)$.

Proof. Let us define a homotopy $h_n : [0,1] \times C[0,1] \to C[0,1]$ by $h_n(\tau, u) = u - TK_n\widetilde{\Phi}(u) - \tau TK_n\big(\widetilde{\Phi}(Q_n u) - \widetilde{\Phi}(u)\big)$. Assume that for infinite number of $n \in \mathbb{N}$ there exist $\tau_n \in [0,1]$ and $u_n \in \partial B(u_0, r)$ such that

$$h_n(\tau_n, u_n) = u_n - TK_n\widetilde{\Phi}(u_n) - \tau_n TK_n\big(\widetilde{\Phi}(Q_n u_n) - \widetilde{\Phi}(u_n)\big) = 0. \tag{11}$$

There exist convergent subsequences of u_n and τ_n, which for simplicity we will continue to denote as $\tau_n \to \overline{\tau} \in [0,1]$ and $u_n \to \overline{u} \in \partial B(u_0, r)$. Similarly as in the proof of the Lemma 2.1 we can conclude that $TK_n\widetilde{\Phi}(u_n) \to T\widetilde{\Phi}(\overline{u})$. Let us now look at

$$\widetilde{\Phi}(Q_n u_n) - \widetilde{\Phi}(u_n) = \widetilde{\Phi}(Q_n u_n) - \widetilde{\Phi}(\overline{u}) + \widetilde{\Phi}(\overline{u}) - \widetilde{\Phi}(u_n).$$

As $u_n \to \overline{u}$ in $C[0,1]$, it converges also in $L^1(0,1)$. By continuity of $\widetilde{\Phi}$, $\widetilde{\Phi}(\overline{u}) - \widetilde{\Phi}(u_n) \to 0$. By Lemma 3.2, $Q_n u_n \to \overline{u}$ in $L^1(0,1)$, hence letting $n \to +\infty$ in (11) we obtain $\overline{u} = T\widetilde{\Phi}(\overline{u})$, $\overline{u} \in \partial B(u_0, r)$. This contradicts our assumption and completes the proof. □

From the previous lemmas it may be concluded

Theorem 3.5.

Let $u_0 \in C[0,1]$ be an isolated solution to problem (1) such that $\deg(f, B(u_0, r), 0) \neq 0$. Then there exists a sequence of zeros $x_n \in \mathbb{R}^{n+1}$, $g_n(x_n) = 0$, such that the sequence $u_n = R_n(x_n)$ of piecewise constant functions converges to u_0 in $L^1(0,1)$.

Proof. Take any $\varepsilon \in (0, r)$. By Lemmas 2.3 and 3.4 the maps f and $\widetilde{f}_n$ can be joined by a homotopy on $B(u_0, \varepsilon)$ for $n \in \mathbb{N}$ big enough. Therefore, for almost all $n \in \mathbb{N}$, there exists a function $v_n \in B(u_0, \varepsilon) \subset C[0,1]$ such that

$$v_n = TK_n\big(\widetilde{\Phi}(Q_n v_n)\big). \tag{12}$$

Consequently we have $P_n v_n = P_n TK_n\big(\widetilde{\Phi}(Q_n v_n)\big)$, which means that $x_n = P_n v_n \in \mathbb{R}^{n+1}$ is a zero of the map g_n. Next build a sequence $\{v_n\} \subset C[0,1]$ satisfying (12) and such that $v_n \to u_0$ in $C[0,1]$ as in the proof of Theorem 2.4.

Denote $u_n = Q_n v_n$. It is a piecewise constant function belonging to $L^1(0,1)$. By Lemma 3.2, if $v_n \to u_0 \in C[0,1]$ then $Q_n v_n \to u_0$ in $L^1(0,1)$. Moreover, from the definitions of Q_n and x_n, we can see that $x_{kn} = u_n\restriction_{[k/(n+1),(k+1)/(n+1)]}$. Thus $u_n = R_n(x_n)$ and $u_n \to u_0$ in $L^1(0,1)$, which completes the proof. □

Remark 3.6.

In the previous theorem we focused on piecewise constant approximation and convergence in $L^1(0,1)$. We used this representation of the function u_n because of its simplicity and its direct correspondence to x_n. As we can see in the details of the proof, there exists a sequence $\{v_n\} \subset C[0,1]$ convergent to u_0 in the norm of $C[0,1]$, which is also related to $\{x_n\}$. We can represent v_n as $v_n = TK_n\widetilde{\Phi}(u_n)$. This function is continuous and the sequence converges uniformly to u_0, but its relation to x_n is far less natural.

Now we can look at the finite dimensional problem $g_n(x) = 0$ from a different perspective. Let us assume that we have solutions $x_n \in \mathbb{R}^{n+1}$, $g_n(x_n) = 0$. We cannot directly compare different x_n, as they belong to different spaces, but we can naturally embed all of them into $L^1(0,1)$ as piecewise constant functions $u_n = R_n(x_n)$.

Lemma 3.7.

If there exists a sequence of solutions $x_n \in \mathbb{R}^{n+1}$ such that $g_n(x_n) = 0$ and $u_n = R_n(x_n) \in L^1(0,1)$ converges to u_0 in $L^1(0,1)$, then u_0 is a solution to (1).

Proof. Let

$$v_n = TK_n\widetilde{\Phi}(u_n), \tag{13}$$

$v_n \in C[0,1]$. Let us prove that v_n converges in $C[0,1]$. Similarly as in the proof of Lemma 2.1 we will show that $\{TK_n\widetilde{\Phi}(u_n)\}$ is convergent. As

$$TK_n\widetilde{\Phi}(u_n) = TK_n\big(\widetilde{\Phi}(u_n) - \widetilde{\Phi}(u_0)\big) + TK_n\widetilde{\Phi}(u_0),$$

there exists $v_0 \in C[0,1]$ such that $v_n = TK_n\widetilde{\Phi}(u_n) \to v_0$. Apply Q_n to (13),

$$Q_n(v_n) = Q_nTK_n\widetilde{\Phi}(u_n). \tag{14}$$

We know that $P_n(u_n) = x_n$ and $g_n(P_n(u_n)) = 0$ which, taking into account (9), means that $Q_nu_n - Q_nTK_n\widetilde{\Phi}(Q_nu_n) = 0$. But $Q_nu_n = u_n$, so

$$u_n = Q_nTK_n\widetilde{\Phi}(Q_nu_n). \tag{15}$$

Comparing (14) and (15) we have $u_n = Qv_n$. Rewrite (13) as $v_n = TK_n\widetilde{\Phi}(Q_nv_n)$. Letting $n \to +\infty$, we have $v_0 = T\widetilde{\Phi}(v_0)$. By Lemma 3.2, $Q_nv_n \to v_0$ in $L^1(0,1)$. Along with the assumption $u_n \to u_0$ in $L^1(0,1)$ and $Q_nv_n = u_n$ we can see that $v_0 = u_0$ and $u_0 = T\widetilde{\Phi}(u_0)$, which completes the proof. □

Lemma 3.8.

Let us assume that there exists a sequence of solutions $x_n \in \mathbb{R}^{n+1}$ such that $g_n(x_n) = 0$, $u_n = R_n(x_n) \in L^1(0,1)$ is uniformly bounded, i.e. there exists such $R > 0$ that $|u_n(t)| \le R$ for all $n \in \mathbb{N}$ and almost all $t \in [0,1]$. Then there exists a subsequence $\{u_{\gamma(n)}\} \subset \{u_n\}$, which converges to $u_0 \in L^1(0,1)$ and u_0 is a solution to problem (1).

Proof. Similarly as in the proof of Lemma 2.2 we may show that the sequence $v_n = TK_n\widetilde{\Phi}(u_n)$ contains a subsequence $v_{\gamma(n)}$ convergent to some v_0 in $C[0,1]$. Moreover, as in the previous lemma, we can observe that $u_n = Q_nv_n$, and $v_{\gamma(n)} = TK_n\widetilde{\Phi}\big(Q_nv_{\gamma(n)}\big)$. Finally, letting $n \to +\infty$ we obtain $u_0 = v_0 = T\widetilde{\Phi}(v_0)$. □

Remark 3.9.

The assumption that the sequence $u_n = R_nx_n$ is composed of uniformly bounded functions is guaranteed if we assume that all $x_n \in \mathbb{R}^{n+1}$ are uniformly bounded in the norms $|x|_{n+1} = \max\{|x_k| : k = 0,1,\dots,n\}$.

We have shown that the finite dimensional problems $g_n(x) = 0$ may be used to find numerical approximations of solutions to problem (1). Lemmas 3.7 and 3.8 prove that if we are able to solve equations $g_n(x) = 0$ and show some additional properties of the sequence of solutions $\{x_n\}$ (e.g. being uniformly bounded as mentioned in Remark 3.9 above), then the sequence x_n (or its subsequence) converges (in the sense defined above) to as solution of problem (1). In this case the solution of (1) is not necessarily essential. On the other hand, by Theorem 3.5, solutions to equations $g_n(x) = 0$ exist in neighbourhoods of essential solutions to problem (1).

Example 3.10.

Let us look at the simple example where discontinuity appears in the linear term only, which means that it is easy to provide a precise formula for the map Φ_n used in the approximation process.

$$\begin{cases} u''(t) + \lambda a(t)|u(t)| + \psi(u(t)) + b(t) = 0, \\ u(0) = u(1) = 0, \end{cases}$$

where $\lambda > 0$, $\psi\colon \mathbb{R} \to \mathbb{R}$ is a continuous function, $b \in L^1(0,1)$ and

$$a(t) = \begin{cases} 1, & t \in (0,\alpha), \\ -1, & t \in (\alpha,1), \end{cases}$$

where $\alpha \in (0,1)$ is a fixed constant. The finite dimensional approximation is given by $x = A_n\Phi_n(x)$, $x = (x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$. Let us now check what formula (7) looks like in this situation. Let us denote $b_k = \int_{k/(n+1)}^{(k+1)/(n+1)} b(t)\,dt$ for $k = 0, 1, \dots, n$. In the case $[k/(n+1), (k+1)/(n+1)] \subset [0,\alpha]$, we have

$$[\Phi_n(x_0, x_1, \dots, x_n)]_k = \lambda |x_k| + \psi(x_k) + b_k.$$

If $[k/(n+1), (k+1)/(n+1)] \subset [\alpha, 1]$, then

$$[\Phi_n(x_0, x_1, \dots, x_n)]_k = -\lambda |x_k| + \psi(x_k) + b_k.$$

And finally if $\alpha \in (k/(n+1), (k+1)/(n+1))$, then

$$[\Phi_n(x_0, x_1, \dots, x_n)]_k = \lambda |x_k| \left(\alpha - \frac{k}{n+1} - \left(\frac{k+1}{n+1} - \alpha \right) \right) + \psi(x_k) + b_k.$$

Remark 3.11.

In order to practically apply the presented finite dimensional approximation we need to know the precise formula of the matrix A_n, with the entries given by

$$a_{lkn} = \int_{l/(n+1)}^{(l+1)/(n+1)} (Tp_{kn})(t)\,dt. \tag{16}$$

Since $p_{kn}(t)$ is a known polynomial, we can say that it is easy to find the values of a_{lkn}, but the practical usage of formula (16) is limited. However, it is worth observing that we can apply the recursive formula, which simplifies the calculations. The formula originates from the following well-known recursive dependence between Bernstein basis polynomials:

$$p_{kn}(t) = (1-t)p_{k,n-1}(t) + tp_{k-1,n-1}(t), \qquad t \in [0,1], \quad n = 1, 2, \dots, \quad k = 1, 2, \dots, n-1. \tag{17}$$

Let us denote $q_{kn}(t) = (Tp_{kn})(t)$ and $Q_{kn}(t) = \int_0^t q_{kn}(s)\,ds$. We would like to find the values $a_{lkn} = Q_{k,n}((l+1)/(n+1)) - Q_{k,n}(l/(n+1))$.

From (17) through the process of successive integration by parts and we have

$$q_{kn}(t) = (1-t)q_{k,n-1}(t) + tq_{k-1,n-1}(t) + 2\big(Q_{k,n-1}(t) - Q_{k-1,n-1}(t)\big) - 2t\big(Q_{k,n-1}(1) - Q_{k-1,n-1}(1)\big). \tag{18}$$

For all $l = 1, \dots, n-2$ we obtain $q_{l,n-1} = Tp_{l,n-1}$ and having applied formula (17), we can see that for all $k = 1, \dots, n-1$ and $q_{kn}(t)$ satisfying (18), we have $q''_{k,n}(t) = -p_{kn}(t)$ and $q_{kn}(0) = q_{kn}(1) = 0$.

We should also observe that we can easily find values of Tp_{0n} and Tp_{nn} for all $n = 1, 2 \dots$ Having noted this, we can see that, with the use of formula (18) we can effectively find all polynomials $q_{kn}(t)$, and, by way of, with an easy integration step $Q_{kn}(t)$ can be found as well.

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