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Boolean algebras admitting a countable minimally acting group

Research Article

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Abstract: The aim of this paper is to show that every infinite Boolean algebra which admits a countable minimally acting group contains a dense projective subalgebra.

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1. Regular and relatively complete subalgebras

All Boolean algebras considered here are assumed to be infinite. Boolean algebraic notions, excluding symbols for Boolean operations, follow the Koppelberg's monograph [9]. In particular, if $(\mathbb{B}, \wedge, \vee, -, \mathbf{0}, \mathbf{1})$ is a Boolean algebra, then $\mathbb{B}^+ = \mathbb{B} \setminus \{\mathbf{0}\}$ denotes the set of all non-zero elements of $\mathbb{B}$. A set $\mathbb{A} \subseteq \mathbb{B}$ is a subalgebra of the Boolean algebra $\mathbb{B}$, $\mathbb{A} \leq \mathbb{B}$ for short, if $\mathbf{1}, \mathbf{0} \in \mathbb{A}$ and $\mathbb{A}$ is closed under Boolean operations or, equivalently, $u - w \in \mathbb{A}$ for all $w, u \in \mathbb{A}$. We shall write $\mathbb{A} \cong \mathbb{B}$ whenever $\mathbb{A}$ and $\mathbb{B}$ are isomorphic Boolean algebras. A non-empty set $X \subseteq \mathbb{B}^+$ is called a *partition* of $\mathbb{B}$ whenever $x \wedge y = \mathbf{0}$ for distinct $x, y \in X$ and

$$\bigvee\nolimits_{\mathbb{B}} X = \mathbf{1},$$

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i.e. the supremum of X in $\mathbb{B}$ exists and equals $\mathbf{1}$. Therefore, a partition is just a maximal set consisting of non-zero pairwise disjoint elements of a Boolean algebra. The set of all partitions of $\mathbb{B}$ will be denoted here by $\operatorname{Part}\mathbb{B}$. For a Boolean algebra $\mathbb{B}$ the symbol $\mathrm{c}(\mathbb{B})$ denotes the Souslin number of $\mathbb{B}$, i.e.

$$\mathrm{c}(\mathbb{B}) = \sup\{|\mathcal{P}| : \mathcal{P} \in \operatorname{Part}\mathbb{B}\}.$$

A subalgebra $\mathbb{A}$ of $\mathbb{B}$ is called *regular*, $\mathbb{A} \leq_{\mathrm{reg}} \mathbb{B}$ for short, whenever every partition of $\mathbb{A}$ is a partition of $\mathbb{B}$; see e.g. Koppelberg [10, p. 123], and also Heindorf and Shapiro [8, p. 14]. Let us recall that a set $X \subseteq \mathbb{B}^+$ is *dense* in a Boolean algebra $\mathbb{B}$ if for every $b \in \mathbb{B}^+$ there exists $a \in X$ such that $a \leq b$. The cardinal number

$$\pi(\mathbb{B}) = \min\left\{|X| : X \subseteq \mathbb{B}^+ \text{ and } X \text{ is dense in } \mathbb{B}\right\}$$

denotes the *density* of $\mathbb{B}$. For $\mathbb{A} \leq \mathbb{B}$ we say that $\mathbb{A}$ is a *dense subalgebra* of $\mathbb{B}$, $\mathbb{A} \leq_{\mathrm{d}} \mathbb{B}$ for short, whenever $\mathbb{A}$ is a dense subset of $\mathbb{B}$. It is easy to see that every dense subalgebra is regular, i.e. $\mathbb{A} \leq_{\mathrm{d}} \mathbb{B}$ implies $\mathbb{A} \leq_{\mathrm{reg}} \mathbb{B}$.

Let us recall that a complete Boolean algebra $\mathbb{B}^{\mathrm{c}}$ is the completion of a Boolean algebra $\mathbb{B}$ whenever $\mathbb{B}$ is a dense subalgebra in $\mathbb{B}^{\mathrm{c}}$. From the Sikorski Extension Theorem it easily follows that if $\mathbb{A}$ is isomorphic to a dense subalgebra of $\mathbb{B}$ then $\mathbb{A}^{\mathrm{c}} \cong \mathbb{B}^{\mathrm{c}}$; see e.g. Koppelberg [9]. However, there exist Boolean algebras, say $\mathbb{A}$ and $\mathbb{D}$, for which $\mathbb{A}^{\mathrm{c}} \cong \mathbb{D}^{\mathrm{c}}$ but neither $\mathbb{A}$ is isomorphic to a dense subalgebra of $\mathbb{D}$ nor $\mathbb{D}$ is isomorphic to a dense subalgebra of $\mathbb{A}$; see e.g. [5].

If $\mathbb{A} \leq \mathbb{B}$, then an element $b \in \mathbb{B}^+$ is called $\mathbb{A}$-*regular* in $\mathbb{B}$ whenever there exists an element $q(b) \in \mathbb{A}^+$ which is minimal among all the elements of $\mathbb{A}$ which are greater than b, i.e.

$$q(b) = \min\{d \in \mathbb{A} : b \leq d\};$$

see also Koppelberg [10] for an equivalent definition of $q(b)$. It is clear that if $\mathbb{A} \leq \mathbb{B}$ then every element of $\mathbb{A}$ is $\mathbb{A}$-regular in $\mathbb{B}$ since $q(b) = b$ for every $b \in \mathbb{A}$ in that case. A Boolean algebra $\mathbb{A}$ is called a *relatively complete* subalgebra of a Boolean algebra $\mathbb{B}$, $\mathbb{A} \leq_{\mathrm{rc}} \mathbb{B}$ for short, provided that every element of $\mathbb{B}$ is $\mathbb{A}$-regular. It is not difficult to show that every relatively complete subalgebra is regular; see Corollary 1.3 below. It is clear that if $\mathbb{A} \leq_{\mathrm{rc}} \mathbb{B}$ and $\mathbb{B}$ is complete, then $\mathbb{A}$ is complete as well. Indeed, if $X \subseteq \mathbb{A}$ and $u \in \mathbb{B}$ is the supremum of X in $\mathbb{B}$, then $q(u)$ is the supremum of X in $\mathbb{A}$. From the definition we obtain immediately the following lemma.

Lemma 1.1.

If $\mathbb{A} \leq \mathbb{B}$ and for some $b, c \in \mathbb{B}^+$ there exist both $q(b)$ and $q(c)$ then the following conditions hold true:

(a) *for every $a \in \mathbb{A}^+$ there exist $q(a \wedge b)$ and $q(a \wedge b) = a \wedge q(b)$,*

(b) *there exists $q(b \vee c)$ and moreover $q(b \vee c) = q(b) \vee q(c)$.*

Some of the conditions in the next proposition were proved by Koppelberg [10]; see also Balcar, Jech and Zapletal [3] or Heindorf and Shapiro [8]. For the sake of completeness we give its proof.

Proposition 1.2.

If $\mathbb{A} \leq \mathbb{B}$ then the following conditions are equivalent:

(a) $\mathbb{A} \leq_{\mathrm{reg}} \mathbb{B}$,

(b) *for every $b \in \mathbb{B}^+$ there exists $a \in \mathbb{A}^+$ such that whenever $x \in \mathbb{A}^+$ and $x \leq a$, then $x \wedge b \neq \mathbf{0}$,*

(c) *for every $b \in \mathbb{B}^+$ there exists $a \in \mathbb{A}^+$ such that $\mathbb{A}\restriction(a \wedge -b) = \{\mathbf{0}\}$,*

(d) *the set of all non-zero $\mathbb{A}$-regular elements of $\mathbb{B}$ is dense in $\mathbb{B}$,*

(e) *for every $b \in \mathbb{B}^+$ there exists $a \in \mathbb{A}^+$ such that $q(a \wedge b) = a$.*

Proof. (a) $\Rightarrow$ (b) Suppose that there exists $b \in \mathbb{B}^+$ such that below every $a \in \mathbb{A}^+$ there exists $x_a \in \mathbb{A}^+$ such that $x_a \wedge b = \mathbf{0}$. The set $X = \{x_a : a \in \mathbb{A}^+\}$ is dense in $\mathbb{A}$. By the Kuratowski–Zorn lemma there exists a maximal disjoint set $Y \subseteq X$. Since X is dense in $\mathbb{A}$, Y is a partition of $\mathbb{A}$. On the other hand, since $y \wedge b = \mathbf{0}$ for each $y \in Y$, the set Y is not a partition of $\mathbb{B}$. We have a contradiction.

(b) $\Rightarrow$ (e) Let $b \in \mathbb{B}^+$ be fixed. By condition (b) there exists $a \in \mathbb{A}^+$ such that for each $x \in \mathbb{A}^+$ we have the following implication:

$$x \leq a \qquad \Longrightarrow \qquad x \wedge b \neq \mathbf{0}. \tag{$*$}$$

In particular we have $\mathbf{0} < a \wedge b \leq a$. We shall show that $a \wedge b$ is an $\mathbb{A}$-regular element of $\mathbb{B}$. For this goal it is enough to show that

$$a = \min\{y \in \mathbb{A} : a \wedge b \leq y\}.$$

We set $Y = \{y \in \mathbb{A} : a \wedge b \leq y\}$. Since $a \in Y$, it remains to show that a is the lower bound of Y. Suppose that $a - x \neq \mathbf{0}$ for some $x \in Y$. Since $a - x \leq a$, by condition ($*$), we have $(a - x) \wedge b \neq \mathbf{0}$. On the other hand, we have $a \wedge b \wedge -x = \mathbf{0}$ because $x \in Y$. Again we get a contradiction.

(d) $\Rightarrow$ (a) Suppose $X \subseteq \mathbb{A}^+$ is a partition of $\mathbb{A}$ and there exists $b \in \mathbb{B}^+$ such that $x \wedge b = \mathbf{0}$ for every $x \in X$. By condition (d) we can assume that b is $\mathbb{A}$-regular. Clearly $\mathbf{0} < q(b)$ since $\mathbf{0} < b \leq q(b)$. Moreover, $q(b) \leq -x$ for each $x \in X$ since $b \leq -x$ for each $x \in X$. Therefore, X cannot be a partition of $\mathbb{A}$; a contradiction.

Since the equivalence (b) $\Leftrightarrow$ (c) and the implication (e) $\Rightarrow$ (d) are obvious, the proof is complete. □

Immediately from Proposition 1.2 we obtain the following corollary.

Corollary 1.3.

For each Boolean algebras $\mathbb{A}$ and $\mathbb{B}$, $\mathbb{A} \leq_{\mathrm{rc}} \mathbb{B}$ implies $\mathbb{A} \leq_{\mathrm{reg}} \mathbb{B}$.

A Boolean algebra $\mathbb{B}$ is *countably generated* over a subalgebra $\mathbb{A} \leq \mathbb{B}$ if there exists a countable set $X \subseteq \mathbb{B}$ such that $\mathbb{B} = \langle \mathbb{A} \cup X \rangle$, i.e. $\mathbb{B}$ is generated by the set $\mathbb{A} \cup X$. If $\mathbb{A}$ is countably generated over $\mathbb{C}$, then we shall write $\mathbb{C} \leq_{\mathrm{rc}\omega} \mathbb{A}$ whenever $\mathbb{C} \leq_{\mathrm{rc}} \mathbb{A}$ and we shall write $\mathbb{C} \leq_{\mathrm{reg}\omega} \mathbb{A}$ if $\mathbb{C} \leq_{\mathrm{reg}} \mathbb{A}$.

For an infinite set X the symbol $\mathrm{Fr}\, X$ denotes the free Boolean algebra generated by the set X as the set of free generators. If $|X| = |Y|$ then the Boolean algebras $\mathrm{Fr}\, X$ and $\mathrm{Fr}\, Y$ are isomorphic and are denoted by $\mathrm{Fr}\, \kappa$, where $\kappa = |X| = |Y|$. Clearly, if $X \subseteq Y$ then $\mathrm{Fr}\, X$ is a subalgebra of $\mathrm{Fr}\, Y$. In fact we have much more: if $X \subseteq Y$, then $\mathrm{Fr}\, X \leq_{\mathrm{rc}} \mathrm{Fr}\, Y$.

A Boolean algebra is called *projective* if it is a retract of a free Boolean algebra. Therefore, a Boolean algebra $\mathbb{B}$ is projective whenever there exists a cardinal number κ and homomorphisms $f \colon \mathbb{B} \to \mathrm{Fr}\, \kappa$ and $g \colon \mathrm{Fr}\, \kappa \to \mathbb{B}$ such that the composition $g \circ f$ is the identity on $\mathbb{B}$. All free Boolean algebras are obviously projective but the converse statement is not true. A first internal characterization of projective algebras was obtained in topological language by Haydon [7]. In terms of relatively complete subalgebras it can be written as follows:

Theorem 1.4 (Haydon's Theorem).

An infinite Boolean algebra $\mathbb{B}$ is projective iff there exists a sequence $\{A_\alpha : \alpha < |\mathbb{B}|\}$ of subalgebras of $\mathbb{B}$ such that the following conditions hold true:

- $\mathbb{A}_0 = \{\mathbf{0}, \mathbf{1}\}$,
- $\mathbb{A}_\alpha \leq_{\mathrm{rc}\omega} \mathbb{A}_{\alpha+1}$ *for all* $\alpha < |\mathbb{B}|$,
- $\mathbb{A}_\alpha = \bigcup\{\mathbb{A}_\beta : \beta < \alpha\}$ *whenever* $\alpha < |\mathbb{B}|$ *is a limit ordinal,*
- $\mathbb{B} = \bigcup\{\mathbb{A}_\beta : \beta < |\mathbb{B}|\}$.

An algebraic proof of Haydon's Theorem can be found in Koppelberg [11] and also in Heindorf and Shapiro [8].

A Boolean algebra $\mathbb{C}$ is called a *Cohen algebra* if the completion of $\mathbb{C}$ is isomorphic to the completion of the product of countably many free Boolean algebras. The notion of a Cohen algebra is due to Koppelberg [11], motivated by the Cohen forcing. A topological theorem proved by Shapiro [12] says that every subalgebra of a free Boolean algebra is a Cohen algebra. In particular, a subalgebra of a projective Boolean algebra is a Cohen algebra; see e.g. [8, p. 133].

On the other hand, dense subalgebra of a projective Boolean algebra need not be projective; see e.g. Koppelberg [10]. However, another topological result obtained by Shapiro [13] implies that every Cohen algebra has to contain a dense projective subalgebra. In much simpler way the same theorem follows from a nice characterization of Cohen algebras given by Koppelberg [11]. For the sake of this characterization Koppelberg introduced the notion of the Cohen skeleton. A collection $\mathcal{S}$ of subalgebras of a Boolean algebra $\mathbb{B}$ is called a *Cohen skeleton* if it satisfies the following conditions:

- $\mathbb{A} \leq_{\mathrm{reg}} \mathbb{B}$ for every $\mathbb{A} \in \mathcal{S}$,
- an element of $\mathcal{S}$ contains a dense countable subalgebra,
- the union of every nonempty chain in $\mathcal{S}$ is a dense subset of a member of $\mathcal{S}$,
- for every $\mathbb{A} \in \mathcal{S}$ and every countable set $X \subseteq \mathbb{B}$ there exists $\mathbb{C} \in \mathcal{S}$ such that $\mathbb{A} \cup X \subseteq \mathbb{C}$ and a dense subalgebra of $\mathbb{C}$ is countably generated over $\mathbb{A}$.

Then the Koppelberg characterization reads as follows.

Theorem 1.5 (Koppelberg's Theorem).
If a Boolean algebra $\mathbb{B}$ satisfies the countable chain condition, then the following conditions are equivalent:

- *$\mathbb{B}$ is a Cohen algebra,*
- *$\mathbb{B}$ has a Cohen skeleton,*
- *$\mathbb{B}$ contains a dense projective subalgebra.*

2. Automorphisms group acting on a Boolean algebra

We say that a group $\mathcal{H}$ of automorphisms of a Boolean algebra $\mathbb{B}$ *acts minimally* on $\mathbb{B}$ if for each $b \in \mathbb{B}^+$ there exist $h_1, \dots, h_n \in \mathcal{H}$ such that

$$h_1(b) \vee \dots \vee h_n(b) = \mathbf{1}. \tag{1}$$

see e.g. [1]. Clearly, if $\mathbb{B}$ is a homogeneous Boolean algebra then the group of all automorphisms acts minimally on $\mathbb{B}$. In particular, for every $\kappa \geq \omega$ the group of all automorphisms of $\mathrm{Fr}\,\kappa$, the free Boolean algebra of size κ, acts minimally on $\mathrm{Fr}\,\kappa$. Since the Boolean algebra $\mathrm{Fr}\,\kappa$ is homogeneous, the size of this group is 2^κ. However, by homogeneity, there is a group $\mathcal{H}$ of automorphisms of algebra $\mathrm{Fr}\,\kappa$ of size κ which acts minimally on $\mathrm{Fr}\,\kappa$. Moreover, Turek [15] has shown that there exists an infinite cyclic group of automorphisms acting minimally on $\mathrm{Fr}\,2^\omega$.

Lemma 2.1.
If a group $\mathcal{H}$ of automorphisms acts minimally on a Boolean algebra $\mathbb{B}$, then $\mathrm{c}(\mathbb{B}) \leq |\mathcal{H}|$.

Proof. Suppose $\mathcal{P} \in \mathrm{Part}\,\mathbb{B}$ is of cardinality greater than $|\mathcal{H}|$ and choose an ultrafilter p on $\mathbb{B}$. For every $h \in \mathcal{H}$ the set $\{h(u) : u \in \mathcal{P}\}$ is a partition of $\mathbb{B}$. Hence, there exists at most one element $u_h \in \mathcal{P}$ such that $h(u_h) \in p$. Choose an arbitrary $w \in \mathcal{P} \setminus \{u_h : h \in \mathcal{H}\}$. Then $h(w) \notin p$ for every $h \in \mathcal{H}$. On the other hand, since $\mathcal{H}$ acts minimally on $\mathbb{B}$, there exist $h_1, \dots, h_n \in \mathcal{H}$ such that

$$\mathbf{1} = h_1(w) \vee \dots \vee h_n(w).$$

Since p is an ultrafilter, there exists $i \leq n$ such that $h_i(w) \in p$; we get a contradiction. □

If $\mathcal{H}$ is a group of automorphisms of a Boolean algebra $\mathbb{B}$, then a subalgebra $\mathbb{A} \leq \mathbb{B}$ is called to be an *$\mathcal{H}$-proper* subalgebra of $\mathbb{B}$, shortly $\mathbb{A} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$, whenever for every $a \in \mathbb{A}$ and every $h \in \mathcal{H}$ there exists $X \subseteq \mathbb{A}$ such that

$$h(a) = \bigvee\nolimits_{\mathbb{B}} X.$$

Clearly, if $h{\restriction}\mathbb{A}$ is an automorphism of $\mathbb{A}$ for every $h \in \mathcal{H}$, then $\mathbb{A}$ is an $\mathcal{H}$-proper subalgebra of $\mathbb{B}$. We get the following proposition.

Proposition 2.2.

If $\mathcal{H}$ is a group of automorphisms of $\mathbb{B}$ such that $\mathcal{H}$ acts minimally on $\mathbb{B}$ and $\mathbb{A} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$, then $\mathbb{A} \leq_{\text{reg}} \mathbb{B}$.

Proof. Suppose $T \in \operatorname{Part}\mathbb{A}$ and there exists $b \in \mathbb{B}$ such that

$$b \wedge t = \mathbf{0} \tag{2}$$

for every $t \in T$. There exist $h_1, \dots, h_n \in \mathcal{H}$ with (1). Since $\mathbb{A} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$, for every $k \leq n$ and every $t \in T$ there exists a set $X_{t,k} \subset \mathbb{A}$ such that

$$h_k(t) = \bigvee\nolimits_{\mathbb{B}} X_{t,k}. \tag{3}$$

Now, for every $k \leq n$ we set $X_k = \bigcup\{X_{t,k} : t \in T\}$. We claim that

$$\bigvee\nolimits_{\mathbb{A}} X_k = \mathbf{1} \tag{4}$$

for every $k \leq n$. For this goal we fix $a \in \mathbb{A}^+$ and $k \leq n$. Since $h_k^{-1} \in \mathcal{H}$ and $\mathbb{A}$ is an $\mathcal{H}$-proper subalgebra of $\mathbb{B}$, there exists $z \in \mathbb{A}^+$ such that $z \leq h_k^{-1}(a)$. Since $T \in \operatorname{Part}\mathbb{A}$, there exists $t \in T$ such that $z \wedge t \neq \mathbf{0}$. Therefore $h_k(z) \wedge h_k(t) \neq \mathbf{0}$ and hence $a \wedge h_k(t) \neq \mathbf{0}$. Then, by condition (3) we get $a \wedge x \neq \mathbf{0}$ for some $x \in X_{t,k}$. This completes the proof of condition (4). By this condition, for every $k \leq n$ we can choose $x_k \in X_k$ in such a way that

$$x_1 \wedge \cdots \wedge x_n \neq \mathbf{0}. \tag{5}$$

On the other hand, by conditions (2) and (3), we get $x \wedge h_k(b) = \mathbf{0}$ for every $k \leq n$ and every $x \in X_k$. Hence, by condition (1), we obtain

$$x_1 \wedge \cdots \wedge x_n = x_1 \wedge \cdots \wedge x_n \wedge (h_1(b) \vee \cdots \vee h_n(b)) \leq (x_1 \wedge h_1(b)) \vee \cdots \vee (x_n \wedge h_n(b)) = \mathbf{0},$$

which leads to a contradiction with (5). □

We shall also use the following property of Boolean algebras with a group of automorphisms which acts minimally.

Lemma 2.3.

If a group of automorphisms $\mathcal{H}$ acts minimally on a Boolean algebra $\mathbb{B}$, then $\pi(\mathbb{B}) = \pi(\mathbb{B}{\restriction}u)$ for every $u \in \mathbb{B}^+$.

Proof. If there exists a dense set $X \subseteq \mathbb{B}{\restriction}u$ of size $\kappa \geq \omega$, then for every $h \in \mathcal{H}$ there exists in $\mathbb{B}{\restriction}h(u)$ a dense set of the same size κ. By the assumptions, there exist $h_1, \dots, h_n \in \mathcal{H}$ such that $h_1(u) \vee \cdots \vee h_n(u) = \mathbf{1}$. Hence $\mathbb{B}$ admits a dense set of size κ. □

Using Propositions 2.2 and 1.2 we can modify a bit the definition of $\mathcal{H}$-proper subalgebras.

Lemma 2.4.

Assume that a group $\mathcal{H}$ of automorphisms of a Boolean algebra $\mathbb{B}$ acts minimally on $\mathbb{B}$. Then $\mathbb{A} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$ iff for every $h \in \mathcal{H}$ and every $a \in \mathbb{A}$ there exists a set $X \subseteq \mathbb{A}^+$ of pairwise disjoint elements such that

$$h(a) = \bigvee\nolimits_{\mathbb{B}} X.$$

Proof. Assume that $X \subseteq \mathbb{A}^+$ and $h(a) = \bigvee_{\mathbb{B}} X$. Let $Y \subseteq \mathbb{A}^+$ be a maximal in $\mathbb{A}^+$ disjoint set such that every element of Y is below some element of X. Then $\bigvee_{\mathbb{B}} X = \bigvee_{\mathbb{B}} Y$. Otherwise one can choose $u \in \mathbb{B}^+$ such that $u \leq \bigvee_{\mathbb{B}} X$ and $u \wedge y = \mathbf{0}$ for every $y \in Y$. By Propositions 1.2 and 2.2, we can assume that u is $\mathbb{A}$-regular in $\mathbb{B}$ and $u \leq x$ for some $x \in X$. Since $u \leq -y$ for every $y \in Y$, we have $q(u) \leq -y$ for every $y \in Y$. We get a contradiction with the maximality of Y since $q(u) \in \mathbb{A}^+$ and $q(u) \leq x$. The opposite implication is trivial. □

3. The main result

The next lemma gives a rather technical but very useful property of regular subalgebras.

Lemma 3.1.

Assume that $\mathbb{A} \leq_{\text{reg}} \mathbb{B}$ and $\pi(\mathbb{B}\restriction b) > \pi(\mathbb{A})$ for each $b \in \mathbb{B}^+$. Then for every $b \in \mathbb{B}^+$ there exists an $\mathbb{A}$-regular element $c \in \mathbb{B}^+$ such that $c \leq b$ and $x \wedge b \neq \mathbf{0}$ implies $x \wedge (b - c) \neq \mathbf{0}$ for all $x \in \mathbb{A}^+$.

Proof. Since $\pi(\mathbb{B}\restriction b) > \pi(\mathbb{A})$, the set $\{x \wedge b : x \in \mathbb{A}^+\}$ cannot be dense in $\mathbb{B}\restriction b$. Hence, there exists $c \in (\mathbb{B}\restriction b)^+$ such that $x \wedge b \wedge -c \neq \mathbf{0}$ whenever $x \in \mathbb{A}^+$ and $x \wedge b \neq \mathbf{0}$. By Proposition 1.2 we can assume that c is $\mathbb{A}$-regular since $\mathbb{A}$ is a regular subalgebra of $\mathbb{B}$. □

The last lemma can be extended as follows.

Lemma 3.2.

Assume $\mathbb{A} \leq_{\text{reg}} \mathbb{B}$ and $\pi(\mathbb{B}\restriction b) > \pi(\mathbb{A})$ for every $b \in \mathbb{B}^+$. Then for every $b \in \mathbb{B}^+$ there exists an infinite $\mathcal{P} \in \operatorname{Part}(\mathbb{B}\restriction b)$ which consists of $\mathbb{A}$-regular elements and the following condition holds true: if $\mathcal{R} \subseteq \mathcal{P}$ is finite, then

$$x \wedge b \neq \mathbf{0} \quad \Longrightarrow \quad x \wedge \left(b - \bigvee \mathcal{R}\right) \neq \mathbf{0} \tag{6}$$

for each $x \in \mathbb{A}^+$.

Proof. Let us consider the family Σ of all sets $\mathcal{S} \subseteq (\mathbb{B}\restriction b)^+$ of disjoint $\mathbb{A}$-regular elements of $\mathbb{B}$ which satisfy the following property:

$(**)$ for every $x \in \mathbb{A}^+$ and every finite subfamily $\mathcal{R} \subseteq \mathcal{S}$ we have (6).

By Lemma 3.1, there exists an $\mathbb{A}$-regular element $c \in \mathbb{B}$ such that the set $\mathcal{S} = \{c\}$ fulfills the condition $(**)$. Hence the family Σ is non-empty. By the Kuratowski–Zorn lemma there exists a maximal family $\mathcal{P} \in \Sigma$. It remains to show that $\bigvee_{\mathbb{B}} \mathcal{P} = b$. Suppose that there exists $d \in (\mathbb{B}\restriction b)^+$ such that $d \wedge p = \mathbf{0}$ for each $p \in \mathcal{P}$. Again, by Lemma 3.1, we obtain an $\mathbb{A}$-regular element $c \leq d$ such that $x \wedge d \neq \mathbf{0}$ implies $x \wedge (d - c) \neq \mathbf{0}$ for every $x \in \mathbb{A}^+$.

To get a contradiction it is enough to show that $\mathcal{P} \cup \{c\} \in \Sigma$. For this goal assume that $\mathcal{R} \subseteq \mathcal{P}$ is finite and $x \wedge b \neq \mathbf{0}$. We shall show that

$$x \wedge \left(b - \bigvee (\mathcal{R} \cup \{c\})\right) \neq \mathbf{0}.$$

We have two cases. If $x \wedge d = \mathbf{0}$, then $x \leq -c$ since $c \leq d$. Since $\mathcal{P} \in \Sigma$, we get

$$\mathbf{0} < x \wedge b \wedge -\bigvee \mathcal{R} \leq x \wedge b \wedge -\bigvee \mathcal{R} \wedge -c = x \wedge b \wedge -\left(\bigvee \mathcal{R} \vee c\right) = x \wedge b \wedge -\bigvee (\mathcal{R} \cup \{c\}).$$

If $x \wedge d \neq \mathbf{0}$, then $x \wedge d \wedge -c > \mathbf{0}$. Since $d \leq b$ and $d \wedge \bigvee \mathcal{R} = \mathbf{0}$, we have $d \leq b \wedge -\bigvee \mathcal{R}$. Therefore we get

$$\mathbf{0} < x \wedge d \wedge -c \leq x \wedge b \wedge -\bigvee \mathcal{R} \wedge -c = x \wedge b \wedge -\bigvee (\mathcal{R} \cup \{c\}),$$

which completes the proof. □

If $\mathcal{R}_1, \mathcal{R}_2 \in \operatorname{Part} \mathbb{B}$ then we say that $\mathcal{R}_2$ is a *refinement* of $\mathcal{R}_1$, $\mathcal{R}_1 \prec \mathcal{R}_2$ for short, if for every $u \in \mathcal{R}_2$ there exists $v \in \mathcal{R}_1$ such that $u \leq v$.

Proposition 3.3.

Assume $\mathrm{c}(\mathbb{B}) = \omega$ *and* $\mathbb{A} \leq_{\mathrm{reg}} \mathbb{B}$ *and* $\pi(\mathbb{B} \restriction b) > \pi(\mathbb{A})$ *for every* $b \in \mathbb{B}^+$. *Then for every finite collection* $\mathcal{R}_1, \dots, \mathcal{R}_n \in \operatorname{Part} \mathbb{B}$, *there exists* $\mathcal{R} \in \operatorname{Part} \mathbb{B}$ *consisting of* $\mathbb{A}$*-regular elements such that*

(a) $\mathcal{R}_i \prec \mathcal{R}$ *for every* $i \leq n$,

(b) *for every* $w \in \mathcal{R}_1 \cup \dots \cup \mathcal{R}_n$ *the set* $\{u \in \mathcal{R} : u \leq w\}$ *is a countable infinite partition of* $\mathbb{B} \restriction w$,

(c) *for every* $w \in \mathcal{R}_1 \cup \dots \cup \mathcal{R}_n$ *and for every finite set* $\mathcal{P} \subseteq \{u \in \mathcal{R} : u \leq w\}$ *and every* $x \in \mathbb{A}^+$ *we have*

$$x \wedge w \neq \mathbf{0} \quad \Longrightarrow \quad x \wedge \left(w - \bigvee \mathcal{P}\right) \neq \mathbf{0}.$$

Proof. Assume $\mathcal{R}_1, \dots, \mathcal{R}_n \in \operatorname{Part} \mathbb{B}$. Let $\mathcal{Q} = \{w_1 \wedge w_2 \wedge \dots \wedge w_n : w_i \in \mathcal{R}_i,\ i \leq n\} \setminus \{\mathbf{0}\}$. It is easy to check that $\mathcal{Q} \in \operatorname{Part} \mathbb{B}$ and $\mathcal{R}_i \prec \mathcal{Q}$ for every $i \leq n$. Clearly, for every $w \in \mathcal{R}_j$ the set

$$\left\{w_1 \wedge \dots \wedge w_{j-1} \wedge w \wedge w_{j+1} \wedge \dots \wedge w_n : w_i \in \mathcal{R}_i,\ i \leq n,\ i \neq j\right\} \setminus \{\mathbf{0}\} \subseteq \mathcal{Q}$$

is a partition of $\mathbb{B} \restriction w$. By Lemma 3.2, for every $b \in \mathcal{Q}$ we obtain a partition $\mathcal{R}_b$ of $\mathbb{B} \restriction b$ consisting of $\mathbb{A}$-regular elements such that whenever $\mathcal{P} \subseteq \mathcal{R}_b$ is finite and $x \in \mathbb{A}^+$ then $x \wedge b \neq \mathbf{0}$ implies $x \wedge (b - \bigvee \mathcal{P}) \neq \mathbf{0}$. By Lemma 3.2, $|\mathcal{R}_b| \leq \omega$ since $\mathrm{c}(\mathbb{B}) \leq \omega$. To complete the proof it is enough to set $\mathcal{R} = \bigcup \{\mathcal{R}_b : b \in \mathcal{Q}\}$. □

We are ready to prove our main result. Assume $\mathcal{H}$ is a countable group of automorphisms acting minimally on an infinite Boolean algebra $\mathbb{B}$. We shall show that the collection

$$\left\{\mathbb{A} \subseteq \mathbb{B} : \mathbb{A} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}\right\}$$

of subalgebras of $\mathbb{B}$ has got some properties which are similar to those of the Cohen skeleton. It appears that these properties determine that a Boolean algebra $\mathbb{B}$ is a Cohen algebra.

Theorem 3.4.

Assume $\mathcal{H}$ *is a countable group of automorphisms acting minimally on an infinite Boolean algebra* $\mathbb{B}$. *Then for every Boolean algebra* $\mathbb{A}$ *such that* $\pi(\mathbb{A}) < \pi(\mathbb{B})$ *and* $\mathbb{A} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$ *and for every* $b \in \mathbb{B}^+$ *there exists a Boolean algebra* $\mathbb{C}$ *such that*

$$\mathbb{A} \leq_{\mathrm{rc}\omega} \mathbb{C} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$$

and $c \leq b$ *for some* $c \in \mathbb{C}^+$.

Proof. We can assume that $\mathcal{H} = \{h_n : n = 1, 2, \dots\}$ and h_1 is the identity on $\mathbb{B}$. By Proposition 2.2 we have $\mathbb{A} \leq_{\mathrm{reg}} \mathbb{B}$. Since $\mathcal{H}$ acts minimally on $\mathbb{B}$ and $|\mathcal{H}| \leq \omega$, by Lemma 2.1 we have $\mathrm{c}(\mathbb{B}) \leq \omega$ and by Lemma 2.3 we also have $\pi(\mathbb{B}) = \pi(\mathbb{B} \restriction u)$ for every $u \in \mathbb{B}^+$.

Let us consider the set $\mathrm{Seq} = \bigcup \{{}^n\omega : n < \omega\}$ of all functions from $n = \{0, 1, \dots, n-1\}$ into ω, $n < \omega$. Let

$$\mathrm{Seq} = \{s_n : n \in \omega\},$$

where s_0 is the empty function. For $g, f \in \mathrm{Seq}$, we say that f extends g whenever $g \subseteq f$. Hence, f extends g iff $\operatorname{dom} g \subseteq \operatorname{dom} f$ and $f \restriction \operatorname{dom} g = g$. If $g \in {}^n\omega$ and $k \in \omega$ then the symbol $g^\frown k$ denotes the sequence of length $n+1$ that extends g and whose last term is k, i.e. $g^\frown k$ is the function $f \colon n+1 \to \omega$ such that $f|n = g$ and $f(n) = k$. We shall construct a sequence $\{\mathcal{P}_n \colon n < \omega\}$ of partitions of $\mathbb{B}$. Every $\mathcal{P}_n$ consists of $\mathbb{A}$-regular elements of $\mathbb{B}$ and are indexed by finite sequences of the length n, i.e. $\mathcal{P}_n = \{u_g : g \in {}^n\omega\}$, and the following conditions hold true:

(i) $\mathcal{P}_0 = \{u_\emptyset\}$, where $u_\emptyset = \mathbf{1}$ and $\{b, -b\} \prec \mathcal{P}_1$,

(ii) $\bigvee\{u_{g^\frown i} : i < \omega\} = u_g$ for every $g \in {}^n\omega$,

(iii) $u_{g^\frown i} \wedge u_{g^\frown j} = \mathbf{0}$ whenever $g \in {}^n\omega$ and $i \neq j$,

(iv) for every $i \in \{1, \dots, n\}$ and every $u \in \mathcal{P}_{n-1}$ there exists an infinite family $\mathcal{P} \subseteq \mathcal{P}_n$ such that $h_i(u) = \bigvee \mathcal{P}$,

(v) for every $u \in \mathcal{P}_{n-1}$, every finite subfamily $\mathcal{P} \subseteq \mathcal{P}_n \cap \mathbb{B}\restriction u$ and every $a \in \mathbb{A}^+$, $a \wedge u \neq \mathbf{0}$ implies $a \wedge (u - \bigvee \mathcal{P}) \neq \mathbf{0}$.

To obtain $\mathcal{P}_1$ we consider the family $\mathcal{R}_1 = \{b, -b\}$. From Proposition 3.3 we get a countable infinite partition $\mathcal{R}$ of the algebra $\mathbb{B}$ consisting of $\mathbb{A}$-regular elements such that the following conditions hold true:

- $\mathcal{R}_1 \prec \mathcal{R}$,
- if $w \in \mathcal{R}_1$ the set $\{u \in \mathcal{R} : u \leq w\}$ is a countable infinite partition of $\mathbb{B}\restriction w$,
- if $w \in \mathcal{R}_1$ and $\mathcal{P} \subseteq \{u \in \mathcal{R} : u \leq w\}$ is a finite set and $x \in \mathbb{A}^+$ then $x \wedge w \neq \mathbf{0}$ implies $x \wedge (w - \bigvee \mathcal{P}) \neq \mathbf{0}$.

Let $\mathcal{P}_1 = \mathcal{R}$. We enumerate all elements of the family $\mathcal{P}_1$ by elements of the set ${}^1\omega$. Assume that we defined the partitions $\mathcal{P}_1, \dots, \mathcal{P}_n$. Applying again Proposition 3.3 for partitions

$$\mathcal{R}_i = \{h_i(u) : u \in \mathcal{P}_n\},$$

where $i \in \{1, \dots, n\}$, we obtain a partition $\mathcal{R} \in \operatorname{Part} \mathbb{B}^+$ which consists of $\mathbb{A}$-regular elements and satisfies the following conditions:

(a) $\mathcal{R}_i \prec \mathcal{R}$ for every $i \leq n$,

(b) for every $w \in \mathcal{R}_1 \cup \dots \cup \mathcal{R}_n$ the set $\{u \in \mathcal{R} : u \leq w\}$ is a countable infinite partition of $\mathbb{B}\restriction w$,

(c) for every $w \in \mathcal{R}_1 \cup \dots \cup \mathcal{R}_n$ and every finite set $\mathcal{P} \subseteq \{u \in \mathcal{R} : u \leq w\}$ and every $x \in \mathbb{A}^+$, $x \wedge w \neq \mathbf{0}$ implies $x \wedge (w - \bigvee \mathcal{P}) \neq \mathbf{0}$.

We set $\mathcal{P}_{n+1} = \mathcal{R}$. By condition (b), the partition $\mathcal{P}_{n+1}$ can be indexed by elements of ${}^{n+1}\omega$ in such way that for every $g \in {}^n\omega$,

$$\{v \in \mathcal{R} : v \leq u_g\} = \{u_{g^\frown n} : n < \omega\}.$$

Let us observe that $\mathcal{P}_{n+1}$ satisfies conditions (ii)–(v). In fact, since h_1 is the identity, $\mathcal{R}_1 = \mathcal{P}_n$ and hence $\mathcal{P}_n \prec \mathcal{P}_{n+1}$. Condition (b) implies (ii) and (iv) and condition (c) implies (v). The induction is complete.

For any $f, g \in \mathrm{Seq}$ we denote $f \perp g$ whenever neither $g \subseteq f$ nor $f \subseteq g$. By conditions (ii) and (iii), for every $g, h \in \mathrm{Seq}$ we get the following:

(vi) $g \subseteq h$ and $g \neq h$ imply $u_h < u_g$,

(vii) $g \perp f$ implies $u_f \wedge u_g = \mathbf{0}$.

Let us recall that $\{s_n : n \in \omega\}$ is the fixed enumeration of the set Seq of all finite sequences of natural numbers. Now we consider a sequence of algebras $\{\mathbb{A}_n : n \in \omega\}$ where $\mathbb{A}_n$ is the subalgebra of $\mathbb{B}$ generated by $\mathbb{A} \cup \{u_f : f \subseteq s_i,\ i \leq n\}$.

Finally we set $\mathbb{C} = \bigcup\{\mathbb{A}_n : n \in \omega\}$. By (vi) and (vii) we have the following claim.

Claim 1. *Every element of $\mathbb{C}^+$ is a finite sum of elements of the form $a \wedge u_f \wedge -u_{g_1} \wedge \dots \wedge -u_{g_p}$, where $a \in \mathbb{A}$ and $g_i \perp g_j$ for distinct $i, j \leq p$ and $f \subsetneq g_i$ for all $i \leq p$ and $g_1, \dots, g_p \in \{g : g \subseteq s_i,\ i \leq n\}$ for some $n \in \omega$.*

By condition (i) there exists an element $c \in \mathbb{C}^+$ such that $c \leq b$. It is easy to see that $\mathbb{C}$ is countably generated over $\mathbb{A}$. We shall prove that $\mathbb{A} \leq_{\mathrm{rc}} \mathbb{C}$. For this goal let us fix an element $x \in \mathbb{C}^+$. There exists some $n \in \omega$ such that $x \in \mathbb{A}_n$. We have to prove that there exists

$$q(x) = \min\{d \in \mathbb{A} : x \leq d\}.$$

By Lemma 1.1 (b), and Claim 1 we can assume that

$$x = a \wedge u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p},$$

where $a \in \mathbb{A}$, $g_i \perp g_j$ for all distinct $i, j \leq p$ and $f \subseteq g_i$ for all $i \leq p$.

Now, by Lemma 1.1 (a), it suffices to prove that there exists

$$q\left(u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p}\right).$$

Since for each $m \in \omega$ the partition $\mathcal{P}_m$ consists of $\mathbb{A}$-regular elements, there exists $q(u_f)$. We shall show that $q\left(u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p}\right) = q(u_f)$. Clearly we have $q(u_f) \in \mathbb{A}$ and $u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p} \leq q(u_f)$. Suppose that there exists some $y \in \mathbb{A}^+$ such that $q(u_f) - y \neq \mathbf{0}$. Then we have $\mathbf{0} \neq -y \wedge u_f$ and by condition (v), we get

$$\mathbf{0} < -y \wedge \left(u_f - \bigvee\{u_{g_i|\operatorname{dom} f+1} : i \leq p\}\right),$$

since for every $i \leq p$ we have $f \subsetneq g_i$ and thus $\{u_{g_i|\operatorname{dom} f+1} : i \leq p\}$ is a finite subfamily of $\mathcal{P}_{\operatorname{dom} f+1}$. Since $u_{g_i} \leq u_{g_i|\operatorname{dom} f+1}$ we obtain

$$\mathbf{0} < -y \wedge \left(u_f - \bigvee\{u_{g_i} : i \leq p\}\right) = u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p} \wedge -y.$$

Hence the element y cannot be an upper bound of $u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p}$. To complete the proof of the theorem it remains to show that $\mathbb{C} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$. Let $c \in \mathbb{C}^+$ and $h_i \in \mathcal{H}$ be fixed. We shall show that there exists a set $T \subseteq \mathbb{C}$ such that $h_i(c) = \bigvee_{\mathbb{B}} T$. For this goal we fix some $e \in \mathbb{B}^+$ such that $e \leq h_i(c)$. There exists $n \in \omega$ such that $c \in \mathbb{A}_n$. By Claim 1 we can assume that

$$c = a \wedge u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p},$$

where $a \in \mathbb{A}^+$, and $f, g_1, \ldots, g_p \in \{f : f \subseteq s_i,\ i \leq n\}$ and $g_i \perp g_j$ for all distinct $i, j \leq p$ and $f \subsetneq g_i$ for all $i \leq p$.

We shall need the following claim.

Claim 2. *If $\mathcal{R}_1, \mathcal{R}_2 \in \operatorname{Part}\mathbb{B}$ and $\mathcal{R}_1 \prec \mathcal{R}_2$, then for every $v \in \mathcal{R}_1$ and every $b \in \mathbb{B}^+$ such that $-v \wedge b \neq \mathbf{0}$ there exists $u \in \mathcal{R}_2$ such that $\mathbf{0} \neq u \wedge b \leq -v \wedge b$.*

In fact, since $\mathcal{R}_1$ is a partition and $\neg(b \leq v)$, there exists $v' \in \mathcal{R}_1$ such that $v \wedge v' = \mathbf{0}$ and $b \wedge v' \neq \mathbf{0}$. Since $v' = \bigvee\{x \in \mathcal{R}_2 : x \leq v'\}$, there exists $u \in \mathcal{R}_2$ such that $u \leq v'$ and $u \wedge b \neq \mathbf{0}$. This completes the proof of the claim.

Now we return to the proof that $\mathbb{C} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$. Since $e > \mathbf{0}$ and $e \leq h_i(c)$, we have

$$u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p} \wedge a \wedge h_i^{-1}(e) \neq \mathbf{0}.$$

Hence, by Claim 2 there exists $m \in \omega$ and an element $u_k \in \mathcal{P}_m$ such that $\operatorname{dom} k \geq i$ and $u_k \leq u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p}$ and

(viii) $\mathbf{0} \neq u_k \wedge a \wedge h_i^{-1}(e) \leq u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p} \wedge a \wedge h_i^{-1}(e)$.

Since $\mathbb{A} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$, by condition (iv) there exist families $T_{u_k}, T_a \subset \mathbb{A}^+$ of disjoint elements such that $\bigvee_{\mathbb{B}} T_a = h_i(a)$ and $\bigvee_{\mathbb{B}} T_{u_k} = h_i(u_k)$. By distributivity lows we have $\bigvee_{\mathbb{B}} T_a \wedge \bigvee_{\mathbb{B}} T_{u_k} = \bigvee_{\mathbb{B}} T$, where $T = \{x \wedge y : x \in T_a,\, y \in T_{u_k}\}$. Since $u_f \wedge -u_{g_1} \wedge \cdots \wedge -u_{g_p} \wedge a \wedge h_i^{-1}(e) = c \wedge a \wedge h_i^{-1}(e)$, by condition (viii) we have

$$\mathbf{0} \neq h_i(u_k) \wedge h_i(a) \wedge e \leq h_i(c) \wedge h_i(a) \wedge e \leq h_i(c) \wedge e.$$

There exists $t \in T$ such that $\mathbf{0} \neq t \wedge e \leq h_i(u_k \wedge a) \wedge e$. Since $e \in \mathbb{B}^+$ was chosen arbitrarily so that $e \leq h_i(c)$, we get

$$h_i(c) = \bigvee\nolimits_{\mathbb{B}} T.$$

Hence we get $\mathbb{C} \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$, which completes the proof. □

As a immediate consequence of the last theorem we obtain the following theorem.

Theorem 3.5.

If a countable group of automorphisms acts minimally on a Boolean algebra $\mathbb{B}$, then $\mathbb{B}$ contains a dense projective algebra of size $\pi(\mathbb{B})$.

Proof. Let $\tau = \pi(\mathbb{B})$ and let $\{b_\alpha : \alpha < \tau\}$ be a dense subset of $\mathbb{B}^+$. Since countable Boolean algebras are projective, we can assume that $\tau > \omega$. By transfinite induction we define a sequence of Boolean algebras $\{\mathbb{A}_\alpha : \alpha < \tau\}$ such that $\mathbb{A}_\alpha \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$ for every $\alpha < \tau$ and the following conditions hold true:

(a) $\mathbb{A}_0 = \{\mathbf{0}, \mathbf{1}\}$,

(b) $\mathbb{A}_\alpha \leq_{\mathrm{rc}\omega} \mathbb{A}_{\alpha+1}$ for all $\alpha < \tau$,

(c) $\mathbb{A}_\alpha = \bigcup\{\mathbb{A}_\beta : \beta < \alpha\}$ whenever α is a limit ordinal,

(d) for every $\alpha < \tau$ there exists $a \in \mathbb{A}^+_{\alpha+1}$ such that $a \leq b_\alpha$.

If the Boolean algebras $\{\mathbb{A}_\alpha : \alpha < \gamma\}$ satisfying conditions (a)–(d) have been constructed for some $\gamma < \tau$ and γ is a limit ordinal we set $\mathbb{A}_\gamma = \bigcup\{\mathbb{A}_\alpha : \alpha < \gamma\}$. Definition of the $\mathcal{H}$-proper subalgebras easily implies that $\mathbb{A}_\alpha \leq_{\mathcal{H}\text{-prop}} \mathbb{B}$.

Assume that γ is a successor ordinal, e.g. $\gamma = \mu + 1$ and the conditions (a)–(c) are fulfilled for all $\beta \leq \mu$. It is clear that $\pi(\mathbb{A}_\mu) \leq |\mu| + \omega < \pi(\mathbb{B})$. Then, by Theorem 3.4 there exists a Boolean algebra $\mathbb{C}$ such that

$$\mathbb{A}_\mu \leq_{\mathrm{rc}\omega} \mathbb{C} \leq_{\mathcal{H}-\mathrm{prop}} \mathbb{B}$$

and $a \leq b_\mu$ for some $a \in \mathbb{C}^+$. Then we set $\mathbb{A}_\gamma = \mathbb{C}$. Now, by conditions (a)–(c) and Haydon's Theorem (Theorem 1.4), we conclude that

$$\mathbb{D} = \bigcup\{\mathbb{A}_\alpha : \alpha < \tau\}$$

is a projective Boolean algebra. From condition (d) it follows that $\mathbb{D}$ is a dense subalgebra of $\mathbb{B}$ since the set $\{b_\alpha : \alpha < \tau\}$ is dense in $\mathbb{B}$. □

Remark 3.6.

The above theorem can also be proved by the use of Koppelberg's characterization of Cohen algebras; see Theorem 1.5. For this purpose one has to show that the family of all those subalgebras of $\mathbb{B}$ which are invariant with respect to a countable group of automorphisms constitute a Cohen skeleton.

In case of complete Boolean algebras we obtain the following corollary.

Corollary 3.7.

If $\mathbb{B}$ is a complete Boolean algebra and there exists a countable group of automorphisms acting minimally on $\mathbb{B}$, then $\mathbb{B} \cong (\mathrm{Fr}\,\kappa)^{\mathrm{c}}$, where $\kappa = \pi(\mathbb{B})$.

Proof. It is an immediate consequence of Theorem 3.5. Indeed, let a projective Boolean algebra $\mathbb{A}$ be a dense subalgebra of $\mathbb{B}$. Then $\mathbb{B} \cong \mathbb{A}^c$ and, by Lemma 2.1, $\pi(\mathbb{A}) = \pi(\mathbb{A}\restriction u)$ for every $u \in \mathbb{A}^+$. From a theorem of Shapiro [12] it follows that if $\mathbb{A}$ is a projective Boolean algebra and $\pi(\mathbb{A}\restriction u)$ is the same for every $u \in \mathbb{A}^+$, then $\mathbb{A}^c$ is isomorphic to the completion of a free Boolean algebra; see also [8, p. 116]. This completes the proof since $\kappa = \pi(\mathbb{B})$. □

Remark 3.8.
The above theorem was proved for the first time in [1] and next it was strongly improved by Balcar and Franěk [2]. They proved that if $\mathbb{B}(S)$ is the clopen algebra of the phase space of the universal minimal dynamical system over a semigroup S (see [2] for definitions) and $\mathbb{B}(S)$ is atomless and G is either concellative or has a minimal left ideal or is commutative, then $\mathbb{B}(S)$ is a Cohen algebra. In particular, if S is a countable group, then $\mathbb{B}(S)$ is a complete Boolean algebra which admits a countable group of automorphisms acting minimally on it (see also Bandlow [4], Turek [14], Geschke [6]).

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