Cent. Eur. J. Math. • 12(1) • 2014 • 97-113
DOI: 10.2478/s11533-013-0331-8

VERSITA

Central European Journal of **Mathematics**

Curvature properties of φ-null Osserman Lorentzian $\mathcal{S}$-manifolds

Research Article

Letizia Brunetti[1]*, Angelo V. Caldarella[1]†

1 *Department of Mathematics, University of Bari, Via Edoardo Orabona 4, Bari 70125, Italy*

Received 6 October 2012; accepted 27 March 2013

Abstract: We expound some results about the relationships between the Jacobi operators with respect to null vectors on a Lorentzian $\mathcal{S}$-manifold and the Jacobi operators with respect to particular spacelike unit vectors. We study the number of the eigenvalues of such operators on Lorentzian $\mathcal{S}$-manifolds satisfying the φ-null Osserman condition, under suitable assumptions on the dimension of the manifold. Then, we provide in full generality a new curvature characterization for Lorentzian $\mathcal{S}$-manifolds and we use it to obtain an algebraic decomposition for the Riemannian curvature tensor of φ-null Osserman Lorentzian $\mathcal{S}$-manifolds.

MSC: 53C50, 53C15, 53B30, 53C25

Keywords: Osserman condition • Jacobi operator • Lorentzian $\mathcal{S}$-manifold • Lorentz manifold

1. Introduction

The Jacobi operator is one of the main objects of study in Riemannian and semi-Riemannian Geometry due to the consequences that its behaviour may have on several geometrical properties of a manifold.

Let (M, g) be an n-dimensional semi-Riemannian manifold with metric g of signature (h, k), where $h + k = n$ and where we first put the plus signs and then the minus signs. If $k = 0$ then M is a Riemannian manifold, while if $k = 1$, M is Lorentzian. For any $p \in M$, the unit spacelike and timelike tangent spheres at p are the sets $S_p^+(M) = \{z \in T_pM : g_p(z,z) = +1\}$ and $S_p^-(M) = \{z \in T_pM : g_p(z,z) = -1\}$, respectively. These are the fibers of the unit spacelike and timelike sphere bundles $S^+(M) = \bigcup_{p\in M} S_p^+(M)$ and $S^-(M) = \bigcup_{p\in M} S_p^-(M)$, respectively. We put $S_p(M) = S_p^+(M) \cup S_p^-(M)$ and $S(M) = \bigcup_{p\in M} S_p(M)$. Note that for a Riemannian manifold one has $S_p(M) = S_p^+(M)$ and $S(M) = S^+(M)$. If $p \in M$, for any $z \in S_p(M)$ the *Jacobi operator with respect to* z is the endomorphism $R_z \colon z^\perp \to z^\perp$ defined by $R_z(\cdot) = R_p(\cdot\,, z)z$ (see, for example, [26]), where R is the $(1,3)$-type curvature tensor field of (M, g).

* *E-mail: letizia.brunetti@uniba.it*
† *E-mail: angelo.caldarella@uniba.it*

The Jacobi operator is a self-adjoint map and, in the Riemannian case, it is diagonalizable. The study of its eigenvalues has a special interest: for example, they indicate the extreme values of the sectional curvatures of the manifold. The Jacobi operator is involved in geodesic deformations and it plays a central role in the Jacobi equations. In the semi-Riemannian context, one prefers to deal with the coefficients of the characteristic polynomial of the Jacobi operators rather than with the eigenvalues, because of diagonalization problems.

Clearly, in the Riemannian case, the eigenvalues of the Jacobi operators R_z may depend both on the vector $z \in S_p(M)$ and on the point $p \in M$. It is easy to see that a Riemannian manifold (M, g) has constant sectional curvature c if and only if the Jacobi operators R_z have exactly one constant eigenvalue $\lambda = c$, for any $z \in S(M)$. Thus, it appears natural to study the curvature properties of the manifold when the Jacobi operators admit more than one eigenvalue, independent of both the vector $z \in S_p(M)$ and the point $p \in M$, that is when (M, g) is a (*globally*) *Osserman manifold*. A Riemannian manifold is said to be *pointwise Osserman* when, for every point $p \in M$, the Jacobi operators have eigenvalues independent of $z \in S_p(M)$. For a general account on the geometry of Osserman manifolds, we refer the reader to [26].

It is known that any locally flat or rank-one symmetric space (two-point homogeneous space) is a (globally) Osserman manifold. The converse statement constitutes the well-known Osserman Conjecture, proposed in [40] (see also [39]). Several results have been obtained in search of an answer to this conjecture.

In the Riemannian setting, many cases, depending on the dimension n of the manifold, were solved by Chi [16–18], who proved the Osserman Conjecture for any dimension $n \neq 4k$, $k \geqslant 2$. More recently, in a series of works by Nikolayevsky [33–35], one can find answers to the conjecture for almost every remaining case and, as far as the authors know, only few cases in dimension $n = 16$ still remain open. It is worth noting that in dimension $n = 4$ there are examples of manifolds which are pointwise but not globally Osserman, so that they are not locally isometric to a rank-one symmetric space [29].

In the indefinite setting, a semi-Riemannian manifold (M, g) is said to be (globally) *spacelike Osserman* if the characteristic polynomial of R_z is independent of both $z \in S_p^+(M)$ and $p \in M$; analogously, (M, g) is called (globally) *timelike Osserman* if the characteristic polynomial of R_z is independent of both $z \in S_p^-(M)$ and $p \in M$. In [25] it is proved that (M, g) being spacelike Osserman is equivalent to (M, g) being timelike Osserman.

In the case of metrics with indefinite (non-Lorentzian) signature, there are several counterexamples to the conjecture (see for example [8, 9, 27]).

The Osserman problem was also considered in the Lorentzian setting by García-Río, Kupeli and Vázquez-Abal [23, 24], who together with Blažić, Bokan and Gilkey [7] gave a complete affirmative answer to the Osserman Conjecture, proving that a Lorentzian manifold is Osserman if and only if it has constant sectional curvature. In their original proof, they considered the spacelike and timelike cases separately, but now a different proof for both cases, simultaneously treated, is provided in [26].

Finally, very recently, some new conditions of Osserman-type have been introduced and studied for manifolds with degenerate metrics in [1, 3]. There, the authors, using the results of [2], deal with suitable Jacobi operators defined with respect to the degenerate metric of lightlike hypersurfaces and submanifolds of a semi-Riemannian manifold.

Now, we concentrate our interest in the Lorentzian case. Here the Osserman problem is completely solved, and so new Osserman-type conditions can be considered. This richness is offered by the different possibilities for the causal character of a tangent vector, together with special features of Lorentzian geometry. Namely, in [24] the attention has been focused on Osserman conditions related to timelike vectors, the so-called *null Osserman conditions*, by using Jacobi operators defined on suitable sets made of null (lightlike) vectors (see also [26]).

Lorentzian manifolds M endowed with a compatible almost contact metric structure (φ, ξ, η, g) are interesting with relation to the null Osserman conditions, since the characteristic vector field ξ has to be necessarily timelike. It is known that any Lorentzian Sasaki manifold $(M, \varphi, \xi, \eta, g)$ is globally null Osserman with respect to ξ if the manifold has constant φ-sectional curvature. Nevertheless, as seen in [10], a similar result fails when one considers Lorentzian $\mathcal{S}$-manifolds with constant φ-sectional curvature, although Lorentzian $\mathcal{S}$-manifolds generalize Lorentzian Sasaki manifolds. In Section 3 we prove that, even more generally, no Lorentzian $\mathcal{S}$-manifold can be null Osserman, removing the hypothesis of constant φ-sectional curvature. On the other side, a Lorentzian $\mathcal{S}$-manifold can never be Osserman, since such manifolds do not have constant sectional curvature. This has led the first author to introduce and study in [10] a new kind of null Osserman condition, called the *φ-null Osserman condition*, which seems to be more convenient for Lorentzian $\mathcal{S}$-manifolds and, more generally, for manifolds carrying Lorentzian globally framed f-structures. It is easy to see that it reduces to the null Osserman condition when one considers Lorentzian almost contact metric structures. A first study of the relationships among the three kinds of Osserman conditions, the classical, the null-Osserman and the φ-null Osserman, has been recently conducted in [12], using well-known constructions of principal bundles involving Lorentzian $\mathcal{S}$-manifolds, (indefinite) Kähler manifolds and Lorentzian Sasaki manifolds [14].

In this paper we are going to proceed further in the study of the φ-null Osserman condition. As main results, in Section 3, we obtain a link between the behaviour of the Jacobi operators with respect to null vectors on a Lorentzian $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ and the Jacobi operators with respect to unit vectors in $\operatorname{Im}\varphi$. Then we use it to prove a result on the number of the eigenvalues of these operators for a φ-null Osserman Lorentzian $\mathcal{S}$-manifold, under suitable assumptions on the dimension of the manifold, in analogy to some results obtained by Chi [16] for Osserman Riemannian manifolds. In Section 4 we give an extension of the curvature results of [10] to the case of an arbitrary φ-null Osserman Lorentzian $\mathcal{S}$-manifold. We first provide a new characterization for a class of curvature-like maps on a Lorentzian g.f.f-manifold, which includes the Riemannian curvature tensor of such manifolds, and then, using this result, we prove a curvature characterization for φ-null Osserman Lorentzian $\mathcal{S}$-manifolds with an arbitrary number of characteristic vector fields. Finally, in Section 5, we apply this curvature characterization in the context of the fibrations considered in [12], linking the eigenvalue structure of the φ-null Osserman Lorentzian $\mathcal{S}$-manifold as total space with the corresponding eigenvalue structure of the Kähler and the Lorentzian Sasaki metrics on the base manifolds.

In what follows, all manifolds, tensor fields and maps are assumed to be smooth. Moreover, all manifolds are supposed to be connected and, according to [32], for the Riemannian curvature tensor R of a semi-Riemannian manifold (M, g) we put

$$R(X,Y,Z,W) = g(R(Z,W)Y,X) = g\left(\left([\nabla_Z,\nabla_W] - \nabla_{[Z,W]}\right)Y,X\right),$$

for any vector fields X, Y, Z, W on M. It is well known that the following fundamental symmetries hold:

$$R(X,Y,Z,W) = R(Z,W,X,Y), \qquad R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z),$$
$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0,$$

for any $X, Y, Z, W \in \Gamma(TM)$. More generally, if $p \in M$, any multilinear map $F\colon (T_pM)^4 \to \mathbb{R}$ is said to be a *curvature-like map* if it satisfies the above symmetries [38]. For any linearly independent vectors $x, y \in T_pM$ spanning a non-degenerate plane $\pi = \operatorname{span}(x,y)$, that is $\Delta(\pi) = g_p(x,x)g_p(y,y) - g_p^2(x,y) \neq 0$, the *sectional curvature of F with respect to π* is, by definition, the real number

$$K(\pi) = K(x,y) = \frac{F(x,y,x,y)}{\Delta(\pi)}.$$

It is well known that the sectional curvature $K(\pi)$ is independent of the non-degenerate plane π if and only if $F(x,y,z,w) = k(g(x,z)g(y,w) - g(y,z)g(x,w))$, for all $x,y,z,w \in T_pM$, with $K(\pi) = k \in \mathbb{R}$ [38, p. 80]. More generally, a special feature of semi-Riemannian manifolds is that the sectional curvature on non-degenerate planes can be linked to the behaviour of the curvature with respect to degenerate planes, as provided in [19, Theorem 5] and [38, p. 229]. Since the proof of the cited results involves only the algebraic symmetries of the Riemannian curvature tensor, we report the result here, for later use, stated for any curvature-like map on a Lorentzian manifold.

Lemma 1.1.

Let (M,g) be a Lorentzian manifold and $F\colon (T_pM)^4 \to \mathbb{R}$ a curvature-like map, $p \in M$. The following conditions are equivalent:

(a) *$F(x,y,z,w) = k(g(x,z)g(y,w) - g(y,z)g(x,w))$, for all $x,y,z,w \in T_pM$, with $k \in \mathbb{R}$;*

(b) *$F(x,y,x,y) = 0$, for any degenerate plane $\pi = \operatorname{span}\{x,y\}$ in T_pM.*

2. Preliminaries

Let us recall some basic definitions and facts about contact and $\mathcal{S}$-structures which we will need in the rest of the paper. Following [5], an *almost contact structure* on a $(2n+1)$-dimensional manifold M is, by definition, a triple (φ, ξ, η), where φ is a $(1,1)$-type tensor field on M, ξ a vector field and η a 1-form such that $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$, where $I\colon TM \to TM$ is the identity mapping. From the previous conditions one deduces that $\varphi\xi = 0$ and $\eta \circ \varphi = 0$. Moreover, the endomorphism φ has constant rank $2n$ and, for any $p \in M$, one has the splitting $T_pM = (\operatorname{Im}\varphi_p) \oplus (\operatorname{span}\xi_p)$. The condition $\eta = 0$ defines the $2n$-dimensional non-integrable distribution $\ker\eta = \operatorname{Im}\varphi$, called the *contact distribution*, while ξ is called the *characteristic vector field* of the almost contact structure. An almost contact manifold (M, φ, ξ, η) is said to be *normal* if the $(1,2)$-type tensor field $N = [\varphi, \varphi] + 2d\eta \otimes \xi$ vanishes identically, where $[\varphi, \varphi]$ is the Nijenhuis tensor field of φ, defined by $[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$, for any $X, Y \in \Gamma(TM)$.

An indefinite metric tensor g on an almost contact manifold (M, φ, ξ, η) is said to be *compatible* with the almost contact structure (φ, ξ, η) if

$$g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \tag{1}$$

for all $X, Y \in \Gamma(TM)$, where $\varepsilon = g(\xi, \xi) = \pm 1$. Then, the manifold M is said to be an *indefinite almost contact metric manifold* with structure (φ, ξ, η, g). From (1) it follows easily that $g(X, \xi) = \varepsilon\eta(X)$ and $g(X, \varphi Y) = -g(\varphi X, Y)$, for any $X, Y \in \Gamma(TM)$, as well as that $\operatorname{Im}\varphi$ is orthogonal to the distribution spanned by ξ. The 2-form Φ on M defined by $\Phi(X, Y) = g(X, \varphi Y)$ is called the *fundamental*, or the *Sasakian* 2*-form* of the indefinite almost contact metric manifold. If $\Phi = d\eta$, the manifold $(M, \varphi, \xi, \eta, g)$ is said to be an *indefinite contact metric manifold*. Finally, a normal indefinite contact metric manifold is, by definition, an *indefinite Sasakian manifold*. It is known that an indefinite almost contact metric manifold is indefinite Sasakian if and only if the covariant derivative of φ satifies $(\nabla_X\varphi)Y = g(X, Y)\xi - \varepsilon\eta(Y)X$, with $\varepsilon = g(\xi, \xi) = \pm 1$. It follows easily that $\nabla_X\xi = -\varepsilon\varphi X$ and that ξ is a Killing vector field. Standard reference for contact structures in the Riemannian case is [5], while for the indefinite case the reader is referred to [20, 42].

In [36, 37], Nakagawa introduced a generalization of the above structures by means of the notion of framed f-manifold, later developed and studied by Goldberg and Yano [30, 31] and others with the denomination of globally framed f-manifolds. A manifold M is said to be a *globally framed f-manifold* (briefly g.f.*f-manifold*) if it carries a globally framed f-structure, that is a non-vanishing $(1,1)$-type tensor field φ on M of constant rank satisfying $\varphi^3 + \varphi = 0$, and such that the subbundle associated with the distribution $\ker\varphi$ is parallelizable. If $\dim\ker\varphi = s \geqslant 1$, this is equivalent to the existence of s linearly independent vector fields ξ_α and 1-forms η^α, $\alpha \in \{1, \dots, s\}$, such that

$$\varphi^2 = -I + \eta^\alpha \otimes \xi_\alpha \qquad \text{and} \qquad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \tag{2}$$

where I is the identity mapping. Clearly, if $s = 1$ we get an almost contact structure. From (2) it follows that $\varphi\xi_\alpha = 0$ and $\eta^\alpha \circ \varphi = 0$, for any $\alpha \in \{1, \dots, s\}$. The conditions $\eta^1 = \ldots = \eta^s = 0$ define the $2n$-dimensional distribution $\operatorname{Im}\varphi$ on which φ acts as an almost complex tensor field. One has the splitting $T_pM = (\operatorname{Im}\varphi_p) \oplus \operatorname{span}((\xi_1)_p, \dots, (\xi_s)_p)$, for any $p \in M$, and $\dim M = 2n + s$. Each ξ_α is said to be a *characteristic vector field* of the structure. A g.f.f-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha)$, $\alpha \in \{1, \dots, s\}$, is called *normal* if the $(1,2)$-type tensor field $N = [\varphi, \varphi] + 2d\eta^\alpha \otimes \xi_\alpha$ vanishes identically.

From now on, a $(2n+s)$-dimensional g.f.f-manifold will be simply denoted by $(M, \varphi, \xi_\alpha, \eta^\alpha)$, leaving the condition $\alpha \in \{1, \dots, s\}$ understood.

Following [13], an indefinite metric g on a g.f.f-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha)$ is said to be *compatible* with the g.f.f-structure $(\varphi, \xi_\alpha, \eta^\alpha)$ if

$$g(\varphi X, \varphi Y) = g(X, Y) - \sum_{\alpha=1}^{s} \varepsilon_\alpha\eta^\alpha(X)\eta^\alpha(Y), \tag{3}$$

for all $X, Y \in \Gamma(TM)$, where $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$. Then, the manifold M is said to be an *indefinite metric* g.f.*f-manifold* with structure $(\varphi, \xi_\alpha, \eta^\alpha, g)$. From (3) it follows that $g(X, \xi_\alpha) = \varepsilon_\alpha\eta^\alpha(X)$ and $g(X, \varphi Y) = -g(\varphi X, Y)$, for any $X, Y \in \Gamma(TM)$, as well as that $\operatorname{Im}\varphi$ is orthogonal to the distribution spanned by all the characteristic vector fields. The 2-form Φ on M defined by $\Phi(X, Y) = g(X, \varphi Y)$ is called the *fundamental* 2*-form* of the indefinite metric g.f.f-manifold. If $\Phi = d\eta^\alpha$, for any $\alpha \in \{1, \dots, s\}$, the manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ is said to be an *indefinite almost $\mathcal{S}$-manifold*. Finally, an *indefinite $\mathcal{S}$-manifold* is, by definition, a normal indefinite almost $\mathcal{S}$-manifold. In [13] it is proved that in an indefinite $\mathcal{S}$-manifold the covariant derivative of φ satisfies $(\nabla_X\varphi)Y = g(X,Y)\widetilde{\xi} + \widetilde{\eta}(Y)\varphi^2 X$, where $\widetilde{\xi} = \sum_{\alpha=1}^{s}\xi_\alpha$ and $\widetilde{\eta} = \sum_{\alpha=1}^{s}\varepsilon_\alpha\eta^\alpha$. It follows that $\nabla_X\xi_\alpha = -\varepsilon_\alpha\varphi X$ and $\nabla_{\xi_\alpha}\xi_\beta = 0$, for any $\alpha,\beta\in\{1,\dots,s\}$, and that each ξ_α is a Killing vector field. In particular, for $s=1$ one finds again the notion of indefinite Sasakian manifold [42].

For further properties on $\mathcal{S}$-manifolds, in the Riemannian context, we refer the reader to [4, 6, 15] and to [21], where the notion of almost $\mathcal{S}$-manifold is introduced, while the generalization to the semi-Riemannian setting is given in [13].

We have the following useful formulas for the Riemannian curvature tensor of an indefinite $\mathcal{S}$-manifold [11].

Lemma 2.1.

Let $(M,\varphi,\xi^\alpha,\eta_\alpha,g)$ be a $(2n+s)$-dimensional indefinite $\mathcal{S}$-manifold, $s\geqslant 1$. The following identities hold, for any $X,Y,Z\in\Gamma(TM)$, any $U,V\in\operatorname{span}(\xi_1,\dots,\xi_s)$ and any $\alpha,\beta,\gamma\in\{1,\dots,s\}$:

(a) $R(X,Y,\xi_\alpha,Z) = \varepsilon_\alpha\big\{\widetilde{\eta}(X)g(\varphi Y,\varphi Z) - \widetilde{\eta}(Y)g(\varphi X,\varphi Z)\big\}$,

(b) $R(\xi_\beta,Y,\xi_\alpha,Z) = \varepsilon_\beta\varepsilon_\alpha g(\varphi Y,\varphi Z)$,

(c) $R(\xi_\beta,\xi_\gamma,\xi_\alpha,Z) = 0$,

(d) $R(\varphi X,\varphi Y,\xi_\alpha,Z) = 0$,

(e) $R(U,Y,V,Z) = \widetilde{\eta}(U)\widetilde{\eta}(V)g(\varphi Y,\varphi Z)$,

where, $\varepsilon_\alpha = g(\xi_\alpha,\xi_\alpha) = \pm 1$ for any $\alpha\in\{1,\dots,s\}$, and $\widetilde{\eta} = \sum\limits_{\alpha=1}^{s}\varepsilon_\alpha\eta^\alpha$.

In particular, we have

$$R(X,Y,Z,\xi_\alpha) = 0 \qquad \text{and} \qquad R(X,\xi_\alpha,Y,\xi_\beta) = \varepsilon_\alpha\varepsilon_\beta g(X,Y), \tag{4}$$

for any $X,Y,Z\in\operatorname{Im}\varphi$ and any $\alpha,\beta\in\{1,\dots,s\}$.

3. Lorentzian $\mathcal{S}$-manifolds and the φ-null Osserman condition

Let (M,g) be a Lorentzian manifold and $p\in M$. Following [24, 26], if $u\in T_pM$ is a lightlike (or null) vector, that is $u\neq 0$ and $g_p(u,u)=0$, since $\operatorname{span}u\subset u^\perp$, we consider the quotient space $\overline{u}^\perp = u^\perp/\operatorname{span}u$, together with the canonical projection $\pi\colon u^\perp\to\overline{u}^\perp$. A positive definite inner product $\overline{g}$ can be defined on $\overline{u}^\perp$ by putting $\overline{g}(\overline{x},\overline{y}) = g_p(x,y)$, where $\pi(x)=\overline{x}$ and $\pi(y)=\overline{y}$, so that $(\overline{u}^\perp,\overline{g})$ becomes a Euclidean vector space. The *Jacobi operator with respect to u* is the endomorphism $\overline{R}_u\colon\overline{u}^\perp\to\overline{u}^\perp$ defined by $\overline{R}_u(\overline{x}) = \pi(R_p(x,u)u)$, for all $\overline{x}=\pi(x)\in\overline{u}^\perp$. It is easy to see that $\overline{R}_u$ is self-adjoint, hence diagonalizable.

Any subspace $W\subset u^\perp$ such that $u^\perp = (\operatorname{span}u)\oplus W$ is called a *geometrical realization* of $\overline{u}^\perp$. It is a non-degenerate subspace and $\pi\restriction_W\colon(W,g_p)\to(\overline{u}^\perp,\overline{g})$ is an isometry, so that we can identify $(\overline{u}^\perp,\overline{g})\cong(W,g_p)$.

If $z\in T_pM$ is a unit timelike vector, the *null congruence set* of z at p is

$$N(z) = \{u\in T_pM : g_p(u,u)=0,\ g_p(u,z)=-1\}.$$

Since z is timelike and the space (T_pM,g_p) is Lorentzian, we have $g_p(u,z)\neq 0$ for any lightlike vector $u\in T_pM$ [38], hence $N(z)$ is a non-empty set. Note that the set $N(z)$ is in one-to-one correspondence with the set $S(z)=\{v\in z^\perp : g_p(v,v)=1\}$, called the *celestial sphere of z*, via the map $\psi\colon N(z)\to S(z)$ such that $\psi(u)=u-z$ (see [26]).

Definition 3.1 ([24, 26]).

Let (M,g) be a Lorentzian manifold, $p\in M$ and $z\in T_pM$ a timelike unit vector. Then, (M,g) is said to be *null Osserman with respect to z* if the eigenvalues of $\overline{R}_u$ and their multiplicities are independent of $u\in N(z)$.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian g.f.f-manifold, with $\dim M = 2n + s$ and $s \geqslant 1$. It is always possible to consider a local orthonormal φ-adapted frame $(X_i, \varphi X_i, \xi_\alpha)$, with $1 \leqslant i \leqslant n$ and $1 \leqslant \alpha \leqslant s$, and since φ acts as an Hermitian structure on $\operatorname{Im}\varphi$, we easily see that exactly one of the characteristic vector fields has to be timelike. Thus, it is natural to study the null Osserman condition of the manifold with respect to this timelike vector. There is no loss of generality in assuming that ξ_1 is the unit timelike vector field, and from now on, unless otherwise stated, we will always make such assumption. It is known that Lorentz Sasakian manifolds with constant φ-sectional curvature are null Osserman with respect to the characteristic vector field. We are going to see that this result is no more true when we pass to Lorentzian $\mathcal{S}$-manifolds with more than one characteristic vector field.

Proposition 3.2.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $\mathcal{S}$-manifold, with $\dim M = 2n+s$ and $s \geqslant 2$. Then for any $p \in M$, the null Osserman condition with respect to $(\xi_1)_p$ does not hold.

Proof. Fix $p \in M$. For brevity we drop the subscript p from the notations. Let $u = \xi_1 + \xi_\beta = \psi^{-1}(\xi_\beta) \in N(\xi_1)$, for some $\beta \in \{2, \dots, s\}$, and $\overline{R}_u$ be the Jacobi operator. It is easy to see that

$$W' = (\operatorname{Im}\varphi) \oplus \operatorname{span}\left(\xi_2, \dots, \widehat{\xi}_\beta, \dots, \xi_s\right)$$

is a geometrical realization of $\overline{u}^\perp$, where $\widehat{\xi}_\beta$ means that the vector ξ_β is omitted. Identifying $\overline{u}^\perp \cong W'$, and putting $\overline{w} = \pi(w)$, for any $w \in W'$, by the definition of $\overline{R}_u$ and using (4), if $y, z \in \operatorname{Im}\varphi$, we get

$$\begin{aligned}\overline{g}\left(\overline{R}_u(\overline{y}), \overline{z}\right) &= \overline{g}\left(\pi(R(y,u)u), \pi(z)\right) = g(R(y,u)u, z) \\ &= R(\xi_1, y, \xi_1, z) + R(\xi_\beta, y, \xi_\beta, z) + R(\xi_1, y, \xi_\beta, z) + R(\xi_\beta, y, \xi_1, z) = 0;\end{aligned}$$

if $y \in \operatorname{Im}\varphi$ and $\gamma \in \{2, \dots, s\} - \{\beta\}$:

$$\begin{aligned}\overline{g}\left(\overline{R}_u(\overline{y}), \overline{\xi}_\gamma\right) &= \overline{g}\left(\pi(R(y,u)u), \pi(\xi_\gamma)\right) = g(R(y,u)u, \xi_\gamma) \\ &= R(\xi_1, y, \xi_1, \xi_\gamma) + R(\xi_\beta, y, \xi_\beta, \xi_\gamma) + R(\xi_1, y, \xi_\beta, \xi_\gamma) + R(\xi_\beta, y, \xi_1, \xi_\gamma) = 0;\end{aligned}$$

if $\alpha, \gamma \in \{2, \dots, s\} - \{\beta\}$:

$$\begin{aligned}\overline{g}\left(\overline{R}_u(\overline{\xi}_\alpha), \overline{\xi}_\gamma\right) &= \overline{g}\left(\pi(R(\xi_\alpha,u)u), \pi(\xi_\gamma)\right) = g(R(\xi_\alpha,u)u, \xi_\gamma) \\ &= R(\xi_1, \xi_\alpha, \xi_1, \xi_\gamma) + R(\xi_\beta, \xi_\alpha, \xi_\beta, \xi_\gamma) + R(\xi_1, \xi_\alpha, \xi_\beta, \xi_\gamma) + R(\xi_\beta, \xi_\alpha, \xi_1, \xi_\gamma) = 0.\end{aligned}$$

It follows that $\overline{R}_u = 0$. Equivalently, the only eigenvalue of $\overline{R}_u$ is $\lambda = 0$, and assuming that M is null Osserman with respect to ξ_1, we get $\overline{R}_u = 0$ for all $u \in N(\xi_1)$. Let us choose now $v = \xi_1 + x$, with $x \in \operatorname{Im}\varphi$, $g(x,x) = 1$. Then $x \in S(\xi_1)$, and $v = \psi^{-1}(x) \in N(\xi_1)$, thus we can consider the Jacobi operator $\overline{R}_v$. If $V = x^\perp \cap \operatorname{Im}\varphi$, then $W'' = V \oplus \operatorname{span}(\xi_2, \dots, \xi_s)$ is a geometrical realization of $\overline{v}^\perp$. Identifying $\overline{v}^\perp \cong W''$ and using again (4), for any $\alpha, \beta \in \{2, \dots, s\}$, we obtain

$$\overline{g}\left(\overline{R}_v(\overline{\xi}_\alpha), \overline{\xi}_\beta\right) = R(v, \xi_\alpha, v, \xi_\beta) = R(x, \xi_\alpha, x, \xi_\beta) = 1; \tag{5}$$

if $y \in V$ and $\beta \in \{2, \dots, s\}$, using also the Bianchi Identity, we have

$$\begin{aligned}\overline{g}\left(\overline{R}_v(\overline{y}), \overline{\xi}_\beta\right) = R(v, y, v, \xi_\beta) &= R(\xi_1, y, x, \xi_\beta) + R(x, y, \xi_1, \xi_\beta) + R(x, y, x, \xi_\beta) \\ &= R(x, \xi_1, y, \xi_\beta) - R(x, \xi_\beta, y, \xi_1) = 0.\end{aligned} \tag{6}$$

But (5) and (6) imply that $\overline{R}_v(\overline{\xi}_\beta) = \sum_{\alpha=2}^{s} \overline{\xi}_\alpha \neq 0$, which contradicts the assumption that M is null Osserman, and the claim follows. □

From the above proof it is clear that the Jacobi operators $\overline{R}_u$, with $u = \xi_1 + \xi_\beta$, have a trivial behaviour with respect to the eigenvalues, since they vanish identically. Therefore, in [10] the following new condition is introduced.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian g.f.f-manifold, with $\dim M = 2n + s$ and $s \geqslant 1$. If $p \in M$, the *φ-celestial sphere* of $(\xi_1)_p$ is, by definition, the set

$$S_\varphi\big((\xi_1)_p\big) = S\big((\xi_1)_p\big) \cap \operatorname{Im}\varphi_p,$$

while

$$N_\varphi\big((\xi_1)_p\big) = \psi^{-1}\big(S_\varphi\big((\xi_1)_p\big)\big),$$

is called the *φ-null congruence set* of $(\xi_1)_p$.

Definition 3.3.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian g.f.f-manifold, and $p \in M$. We say that M is *φ-null Osserman with respect to* $(\xi_1)_p$ if the eigenvalues of $\overline{R}_u$ and their multiplicities are independent of $u \in N_\varphi\big((\xi_1)_p\big)$.

When M is a Lorentzian almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ one easily sees that the φ-null Osserman condition reduces to the null Osserman one, since $N_\varphi(\xi_p) = N(\xi_p)$. Moreover, in [10] it is proved that any Lorentzian $\mathcal{S}$-manifold with constant φ-sectional curvature is φ-null Osserman with respect to $(\xi_1)_p$, at any point, generalizing the similar known result for Lorentz Sasakian space forms.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian g.f.f-manifold, with $\dim M = 2n + s$ and $s \geqslant 1$. Fix $p \in M$ and consider $u \in N_\varphi\big((\xi_1)_p\big)$. Writing $u = (\xi_1)_p + x$, with $x \in S_\varphi\big((\xi_1)_p\big)$, we can consider the Jacobi operator $R_x \colon x^\perp \to x^\perp$ corresponding to $\overline{R}_u \colon \overline{u}^\perp \to \overline{u}^\perp$, and vice-versa. We are going to find out the link between these two operators.

Proposition 3.4.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $\mathcal{S}$-manifold, with $\dim M = 2n+s$ and $s \geqslant 1$, and $p \in M$. Then, M is φ-null Osserman with respect to $(\xi_1)_p$ if and only if the eigenvalues of R_x with their multiplicities are independent of $x \in S_\varphi\big((\xi_1)_p\big)$.

Proof. Throughout the proof we fix $p \in M$ and, for simplicity, we omit it from the notations. Let us first suppose $s \geqslant 2$. Choose $u \in N_\varphi(\xi_1)$ and put $u = \xi_1 + x$, with $x \in S_\varphi(\xi_1)$. If $V = x^\perp \cap \operatorname{Im}\varphi$, then $W_1 = V \oplus \operatorname{span}(\xi_2, \dots, \xi_s)$ is a geometrical realization of $\overline{u}^\perp$ and $W_2 = V \oplus \operatorname{span}(\xi_1, \xi_2, \dots, \xi_s) = x^\perp$. Identifying $\overline{u}^\perp \cong W_1$, and putting $\overline{w} = \pi(w)$, $w \in W_1$, from (5) and (6), for any $y \in V$ and any $\alpha, \beta \in \{2, \dots, s\}$, we get

$$\overline{g}\big(\overline{R}_u(\overline{\xi}_\alpha), \overline{\xi}_\beta\big) = g(R_x(\xi_\alpha), \xi_\beta) = 1, \qquad \overline{g}\big(\overline{R}_u(\overline{y}), \overline{\xi}_\beta\big) = g(R_x(y), \xi_\beta) = 0. \tag{7}$$

Using (4), for any $y, z \in V$, one gets

$$\overline{g}\big(\overline{R}_u(\overline{y}), \overline{z}\big) = R(u, y, u, z) = R(\xi_1, y, \xi_1, z) + R(x, y, x, z) + R(\xi_1, y, x, z) + R(x, y, \xi_1, z) = g(y, z) + g(R_x(y), z). \tag{8}$$

Finally, for any $y \in V$ and any $\alpha \in \{1, \dots, s\}$,

$$g(R_x(y), \xi_1) = 0, \qquad g(R_x(\xi_1), \xi_\alpha) = \begin{cases} 1, & \alpha = 1 \\ -1, & \alpha \neq 1. \end{cases} \tag{9}$$

It follows that V is an invariant subspace with respect to the action of both $\overline{R}_u$ and R_x. Choosing any orthonormal base $\mathcal{B}$ for V, then $\mathcal{B}_1 = \mathcal{B} \cup \{\xi_2, \dots, \xi_s\}$ and $\mathcal{B}_2 = \mathcal{B} \cup \{\xi_1, \xi_2, \dots, \xi_s\}$ are orthonormal bases for W_1 and $x^\perp$, respectively. If we denote by A the $(2n-1)$-square matrix of $\overline{R}_u\restriction_V$ with respect to the base $\mathcal{B}$, then (7), (8) and (9) imply that

$$C = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

is the matrix of $\overline{R}_u$ with respect to the base $\overline{\mathcal{B}}_1 = \pi(\mathcal{B}_1)$ of $\overline{u}^{\perp}$, and

$$D = \begin{pmatrix} & 0 & \\ A - I_{2n-1} & \vdots & 0 \\ & 0 & \\ 0 \cdots 0 & -1 & 1 \cdots 1 \\ & -1 & \\ 0 & \vdots & B \\ & -1 & \end{pmatrix}$$

is the matrix of R_x with respect to $\mathcal{B}_2$, where, in both matrices, B is the $(s-1)$-square matrix with all elements equal to 1. Computing the characteristic polynomials of C and D, we get

$$p_C(\lambda) = p_A(\lambda)(-1)^{s-1}\lambda^{s-2}(\lambda-(s-1)), \qquad p_D(\lambda) = p_A(\lambda+1)(-1)^{s}\lambda^{s-1}(\lambda-(s-2)),$$

from which the statement follows.

The same proof also works in the case $s = 1$, with only straightforward modifications. Namely, for $s = 1$, the matrix B disappears, hence $C = A$ and

$$D = \begin{pmatrix} A - I_{2n-1} & 0 \\ 0 & -1 \end{pmatrix},$$

thus obtaining $p_C(\lambda) = p_A(\lambda)$ and $p_D(\lambda) = -(\lambda+1)p_A(\lambda+1)$, from which the statement again follows. □

Remark 3.5.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $\mathcal{S}$-manifold. Since it is clear that $S_\varphi\big((\xi_1)_p\big) = S_\varphi\big(-(\xi_1)_p\big)$, for any $p \in M$, from the above proposition it follows that M is φ-null Osserman with respect to $(\xi_1)_p$ if and only if it is φ-null Osserman with respect to $-(\xi_1)_p$. Therefore, from now on, any Lorentzian $\mathcal{S}$-manifold satisfying the φ-null Osserman condition with respect to $(\xi_1)_p$ will be simply said to be *φ-null Osserman at the point p*.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $\mathcal{S}$-manifold, with $\dim M = 2n + s$ and $s \geqslant 1$. Fix $p \in M$, consider $u \in N_\varphi\big((\xi_1)_p\big)$ and put $\psi(u) = u - (\xi_1)_p = x \in S_\varphi\big((\xi_1)_p\big)$.

As before, if $s \geqslant 2$, putting $V = x^{\perp} \cap \operatorname{Im}\varphi_p$ and $U = \operatorname{span}\big((\xi_2)_p, \dots, (\xi_s)_p\big)$, we can consider the geometrical realization $W = V \oplus U$ of $\overline{u}^{\perp}$, and identify $\overline{u}^{\perp} \cong W$. From the proof of Proposition 3.4, it is clear that we can decompose $\overline{R}_u = \overline{R}_u\!\restriction_V \circ p_V + \overline{R}_u\!\restriction_U \circ p_U$, where p_V and p_U are the projections of W onto V and U, respectively. Clearly, $\overline{R}_u\!\restriction_U \circ p_U$ always admits the eigenvalues $\lambda_0 = 0$ and $\lambda_1 = s - 1$, with multiplicity $m_0 = s - 2$ and $m_1 = 1$, independent of $u \in N_\varphi\big((\xi_1)_p\big)$. If $s = 1$, the subspace U simply disappears.

In any case, we can fix our attention only on the behaviour of the endomorphism $\overline{R}_u\!\restriction_V \circ p_V$, which we denote, from now on, by $\overline{R}_u^{\varphi}$.

Remark 3.6.

From the proof of the previous proposition, it is clear that $c \in \mathbb{R}$ is an eigenvalue of the endomorphism $\overline{R}_u^{\varphi}$ if and only if $c - 1$ is an eigenvalue of $R_x\!\restriction_V \circ p_V$, where $u \in N_\varphi\big((\xi_1)_p\big)$ and $\psi(u) = u - (\xi_1)_p = x \in S_\varphi\big((\xi_1)_p\big)$, for any $p \in M$.

Proposition 3.7.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $\mathcal{S}$-manifold, with $\dim M = (4m+2) + s$ and $s \geqslant 1$, where $\dim \operatorname{Im}\varphi = 4m + 2$. If M is φ-null Osserman at $p \in M$, then for all $u \in N_\varphi\big((\xi_1)_p\big)$ only one of the following two cases can occur:

(i) *$\overline{R}_u^{\varphi}$ admits exactly one eigenvalue c with multiplicity $4m + 1$;*
(ii) *$\overline{R}_u^{\varphi}$ admits exactly two eigenvalues c_1 and c_2 with multiplicities 1 and $4m$.*

Proof. Fix $p \in M$. Identifying $\operatorname{Im}\varphi_p \cong \mathbb{R}^{4m+2}$, we consider $S_\varphi\big((\xi_1)_p\big) \cong \mathbb{S}^{4m+1}$ and endow $N_\varphi\big((\xi)_p\big)$ with the smooth structure such that $\psi\colon N_\varphi\big((\xi)_p\big) \to \mathbb{S}^{4m+1}$ is a diffeomorphism. Hence, for any $u \in N_\varphi\big((\xi_1)_p\big)$, putting $x = \psi(u)$, we can identify $V = x^\perp \cap \operatorname{Im}\varphi_p \cong T_x\mathbb{S}^{4m+1}$ and, under this identification, $\overline{R}_u^\varphi$ induces a unique endomorphism $\mathcal{R}_x\colon T_x\mathbb{S}^{4m+1} \to T_x\mathbb{S}^{4m+1}$ defined by $\mathcal{R}_x(y) = \pi^{-1}\big(\overline{R}_u^\varphi(\overline{y})\big)$, for all $y \in V$. It is clear that $\mathcal{R}_x$ and $\overline{R}_u^\varphi$ admit the same eigenvalues, and as we let u vary in $N_\varphi\big((\xi_1)_p\big)$ we obtain a bundle homomorphism $\mathcal{R} = (\mathcal{R}_x)_{x\in\mathbb{S}^{4m+1}}\colon T\mathbb{S}^{4m+1} \to T\mathbb{S}^{4m+1}$. Let $c \in \mathbb{R}$ be an eigenvalue of $\overline{R}_u^\varphi$, independent of $u \in N_\varphi\big((\xi)_p\big)$. Putting $\mathcal{D}_x = \ker(\mathcal{R}_x - cI)$, we get a distribution $\mathcal{D} = (\mathcal{D}_x)_{x\in\mathbb{S}^{4m+1}}$ on $\mathbb{S}^{4m+1}$. By a well-known result about the maximal dimensions of distributions on spheres (see [41, p. 155]), the only possibilities for $\dim\mathcal{D}$ are $1, 4m$ or $4m+1$. If $\dim\mathcal{D} = 4m+1$, then c is the only eigenvalue of each $\overline{R}_u^\varphi$, with multiplicity $4m+1$, and (i) follows. If $\dim\mathcal{D} = 1$, for another eigenvalue c' with distribution $\mathcal{D}' = (\mathcal{D}'_x)_{x\in\mathbb{S}^{4m+1}}$, $\mathcal{D}'_x = \ker(\mathcal{R}_x - c'I)$, we must have $\dim\mathcal{D}' = 4m$, since otherwise we get a 2-dimensional distribution $\mathcal{D}\oplus\mathcal{D}'$ on $\mathbb{S}^{4m+1}$, and (ii) follows. □

Note that in case (i), M has constant φ-sectional curvature at the point p, and if we suppose that M is φ-null Osserman at each $p \in M$, and that the eigenvalues of $\overline{R}_u$ do not depend on the point p, then M is a Lorentzian $\mathcal{S}$-space form.

4. The curvature of φ-null Osserman Lorentzian $\mathcal{S}$-manifolds

In this section we study in full generality the curvature tensor of those φ-null Osserman Lorentzian $\mathcal{S}$-manifolds whose Jacobi operators $\overline{R}_u^\varphi$ admit exactly two distinct eigenvalues, thus including the case (ii) of Proposition 3.7. In [10] it has been proved a curvature result for the special subclass made of the φ-null Osserman Lorentzian $\mathcal{S}$-manifolds with two characteristic vector fields. Here, we provide an extension of that result to the general case of φ-null Osserman Lorentzian $\mathcal{S}$-manifolds with an arbitrary number of characteristic vector fields.

The key point of the proof is the special algebraic characterization for curvature-like maps provided by Lemma 4.1, which generalizes the analogue result of [10]. We begin by finding some useful expressions for lightlike vectors which take advantage of the presence of the Lorentzian $\mathcal{S}$-structure.

Throughout what follows, we fix a point p in the manifold and, for simplicity, we omit it from the notations (except for T_pM).

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $\mathcal{S}$-manifold, with $\dim M = 2n+s$ and $s \geqslant 2$, and u a lightlike tangent vector at p. Since $T_pM = (\operatorname{Im}\varphi) \oplus \operatorname{span}(\xi_1, \dots, \xi_s)$ and ξ_1 is unit timelike, we can always express the vector u, up to a non-vanishing factor, either in the form

$$u = tx + \xi_1 + \sum_{\alpha=2}^{s} k_\alpha \xi_\alpha, \tag{10}$$

where $x \in \operatorname{Im}\varphi$ with $g(x,x) = 1$, and $t, k_2, \dots, k_s \in \mathbb{R}$ such that $t \neq 0$ and $t^2 + \sum_{\alpha=2}^{s} k_\alpha^2 = 1$, or in the form

$$u = \xi_1 + \sum_{\alpha=2}^{s} h_\alpha \xi_\alpha, \tag{11}$$

where $h_2, \dots, h_s \in \mathbb{R}$ with $\sum_{\alpha=2}^{s} h_\alpha^2 = 1$.

A lightlike vector $u \in T_pM$ will be said to be a vector *of the first kind* and *of the second kind*, if it has the form (10) and (11), respectively. Any degenerate plane π in T_pM can be written in the form $\pi = \operatorname{span}(u, y)$, where $u \in T_pM$ is a lightlike vector either of the first or of the second kind, and $y \in u^\perp$. Let us see how to express $u^\perp$, according to the kind of u.

Suppose u is of the first kind and put $w_0 = \xi_1 + \sum_{\alpha=2}^{s} k_\alpha \xi_\alpha$. Since $g(w_0, w_0) < 0$, we can consider an orthonormal base $\mathcal{B} = (w_1, w_2, \dots, w_{s-1})$ of the Euclidean space $E = w_0^\perp \cap \operatorname{span}(\xi_1, \dots, \xi_s)$ and we have

$$u^\perp = \operatorname{span}(u, \varphi x, x_2, \varphi x_2, \dots, x_n, \varphi x_n, w_1, \dots, w_{s-1}),$$

where $(x, \varphi x, x_2, \varphi x_2, \dots, x_n, \varphi x_n)$ is any orthonormal base of $\operatorname{Im}\varphi$. Hence, any $y \in u^\perp$ can be written as

$$y = au + by' + \lambda^i w_i, \tag{12}$$

where $a, b, \lambda^1, \dots, \lambda^{s-1} \in \mathbb{R}$, and $y' \in \operatorname{span}(\varphi x, x_2, \varphi x_2, \dots, x_n, \varphi x_n)$, $g(y', y') = 1$.

Suppose u is of the second kind and put $\xi' = \sum_{\alpha=2}^{s} h_\alpha \xi_\alpha$. We can consider any orthonormal base $\mathcal{B}' = (w'_1, \dots, w'_{s-2})$ of $\xi'^\perp \cap \operatorname{span}(\xi_2, \dots, \xi_s)$ and we get

$$u^\perp = \operatorname{span}(u) \oplus (\operatorname{Im}\varphi) \oplus \operatorname{span}(w'_1, \dots, w'_{s-2}).$$

Hence, any $y \in u^\perp$ can be written as

$$y = au + by' + \mu^i w'_i, \tag{13}$$

with $a, b, \mu^1, \dots, \mu^{s-2} \in \mathbb{R}$ and $y' \in \operatorname{Im}\varphi$, $g(y', y') = 1$.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be an indefinite g.f.f-manifold with $\dim M = 2n+s$ and $s \geqslant 1$. From now on we put $\widetilde{\xi} = \sum_{\alpha=1}^{s} \xi_\alpha$ and $\widetilde{\eta} = \sum_{\alpha=1}^{s} \varepsilon_\alpha \eta^\alpha$, with $\varepsilon_\alpha = g(\xi_\alpha, \xi_\alpha) = \pm 1$, according to the causal character of ξ_α. Consider the $(1,3)$-type algebraic curvature tensors T and S on M defined by

$$\begin{aligned} T(X,Y)Z &= g(\varphi Y, \varphi Z)\widetilde{\eta}(X)\widetilde{\xi} - g(\varphi X, \varphi Z)\widetilde{\eta}(Y)\widetilde{\xi} - \widetilde{\eta}(Y)\widetilde{\eta}(Z)\varphi^2 X + \widetilde{\eta}(X)\widetilde{\eta}(Z)\varphi^2 Y, \\ S(X,Y)Z &= g(\varphi X, \varphi Z)\varphi^2 Y - g(\varphi Y, \varphi Z)\varphi^2 X, \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$.

Now, we are in a position to prove the following result which characterizes a special class of curvature-like maps on a Lorentzian g.f.f-manifold by means of the behaviour with respect to suitable degenerate planes.

Lemma 4.1.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian g.f.f-manifold, $\dim M = 2n + s$ and $s \geqslant 1$. Fix $p \in M$, and let $F\colon (T_pM)^4 \to \mathbb{R}$ be a curvature-like map satisfying

$$F(x, \xi_\alpha, y, \xi_\beta) = \varepsilon_\alpha \varepsilon_\beta g(\varphi x, \varphi y) \qquad \text{and} \qquad F(\varphi x, \varphi y, \varphi z, \xi_\alpha) = 0, \tag{14}$$

for any $x, y, z \in T_pM$ and $\alpha, \beta \in \{1, \dots, s\}$. The following statements are equivalent:

(a) *$F(u, y, u, y) = 0$ on any degenerate 2-plane $\pi = \operatorname{span}\{u, y\}$ with $u \in N_\varphi((\xi_1)_p)$ and $y \in u^\perp \cap \operatorname{Im}\varphi_p$,*

(b) *$F(x, y, z, w) = g_p(S_p(x, y)z, w) - g_p(T_p(x, y)z, w)$, for any $x, y, v, w \in T_pM$.*

Proof. We treat separately the cases $s = 1$ and $s \geqslant 2$. If $s = 1$, M reduces to a Lorentzian almost contact metric manifold $(M, \varphi, \xi, \eta, g)$, with $g(\xi, \xi) = -1$. Hence, $\widetilde{\eta} = -\eta$, $\widetilde{\xi} = \xi$ and, by (1), $g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y)$, for any $X, Y \in \Gamma(TM)$. Using this, it is easy to see that, for any $x, y, z \in T_pM$,

$$\begin{aligned} S(x,y)z &= g(y,z)x - g(y,z)\eta(x)\xi + \eta(y)\eta(z)x - g(x,z)y + g(x,z)\eta(y)\xi - \eta(x)\eta(z)y, \\ T(x,y)z &= -g(y,z)\eta(x)\xi + g(x,z)\eta(y)\xi + \eta(y)\eta(z)x - \eta(x)\eta(z)y, \end{aligned}$$

from which it follows that the statement (b) is equivalent to

$$F(x, y, z, w) = g(x, w)g(y, z) - g(y, w)g(x, z). \tag{15}$$

On the other hand, it is clear that, up to a non-vanishing multiplicative factor, any lightlike vector $u \in T_pM$ can be written as $u = \xi \pm x$, with $x \in \operatorname{Im}\varphi$, $g(x, x) = 1$. This is equivalent to saying that $N(\xi) = N_\varphi(\xi)$ is the set of all the lightlike vector of T_pM. Therefore, any degenerate plane π in T_pM is spanned by $u \in N(\xi)$ and $y \in u^\perp \cap \operatorname{Im}\varphi$, hence the statement (a) is equivalent to requiring that F vanishes on any degenerate plane in T_pM. By Lemma 1.1, this is equivalent to requiring that there exists $k \in \mathbb{R}$ such that

$$F(x, y, z, w) = k\{g(y, w)g(x, z) - g(x, w)g(y, z)\}, \tag{16}$$

for any $x, y, z \in T_pM$. The above formula and (14) yield $-k = F(x, \xi, x, \xi) = 1$, for any unit $x \in \operatorname{Im}\varphi$. Thus, (16) is equivalent to (15), that is (a) $\Leftrightarrow$ (b).

Now suppose $s \geqslant 2$. It is obvious that (b) implies (a), since

$$g(T(u, y)u, y) = \widetilde{\eta}(u)\widetilde{\eta}(u)g\left(\varphi^2 y, y\right) = -g(y, y), \qquad g(S(u, y)u, y) = g(\varphi u, \varphi u)g\left(\varphi^2 y, y\right) = -g(y, y),$$

for any $u \in N_\varphi(\xi_1)$ and $y \in \operatorname{Im}\varphi$. Conversely, assume that (a) holds and let $H\colon (T_pM)^4 \to \mathbb{R}$ be the curvature-like map such that, for any $x, y, z, w \in T_pM$,

$$H(x, y, z, w) = F(x, y, z, w) - g(S(x, y, v), w) + g(T(x, y, v), w). \tag{17}$$

We are going to see that $H = 0$, by using Lemma 1.1. Therefore, we have to calculate H on any degenerate plane $\pi = \operatorname{span}(u, y)$, where the lightlike vector u can be either of the first kind or of the second kind, and $y \in u^\perp$. First, note that (14) clearly implies that

$$F(x, v', y, v'') = \widetilde{\eta}(v')\widetilde{\eta}(v'')g(\varphi x, \varphi y) \qquad \text{and} \qquad F(x, \xi_\alpha, \xi_\beta, \xi_\gamma) = 0, \tag{18}$$

for any $x, y \in T_pM$, any $v', v'' \in \operatorname{span}(\xi_1, \ldots, \xi_s)$ and any $\alpha, \beta, \gamma \in \{1, \ldots, s\}$. Now, suppose that u is of the first kind and that $y \in u^\perp$ is given by (12). Since, by (14) and the First Bianchi Identity, $F(u, y', u, w_i) = 0$, using (18) we have

$$\begin{aligned} F(u, y, u, y) &= b^2 F(u, y', u, y') + \lambda^i\lambda^j F(u, w_i, u, w_j) + 2\lambda^i b F(u, y', u, w_i) \\ &= b^2 F(u, y', u, y') + \lambda^i\lambda^j \widetilde{\eta}(w_i)\widetilde{\eta}(w_j)g(\varphi u, \varphi u) = b^2 F(u, y', u, y') + g(\varphi u, \varphi u)\big(\widetilde{\eta}(y) - a\widetilde{\eta}(u)\big)^2. \end{aligned} \tag{19}$$

Now, using (10) and (18), we get

$$\begin{aligned} F(u, y', u, y') &= t^2 F(x, y', x, y') + F(w_0, y', w_0, y') = g(\varphi u, \varphi u)F(x, y', x, y') + F(w_0, y', w_0, y') \\ &= g(\varphi u, \varphi u)F(x, y', x, y') + \widetilde{\eta}^2(w_0)g(y', y') = g(\varphi u, \varphi u)F(x, y', x, y') + \widetilde{\eta}^2(u)g(y', y'). \end{aligned}$$

Substituting the above formula in (19), we get

$$F(u, y, u, y) = b^2 g(\varphi u, \varphi u)F(x, y', x, y') + b^2\widetilde{\eta}^2(u)g(y', y') + g(\varphi u, \varphi u)\big(\widetilde{\eta}(y) - a\widetilde{\eta}(u)\big)^2.$$

Let us now put $u' = x + \xi_1$. We see that $u' \in N_\varphi(\xi_1)$ and that $y' \in u'^\perp \cap \operatorname{Im}\varphi$. Since $F(u', y', u', y') = F(x, y', x, y') + g(y', y')$, from the above formula we obtain

$$F(u, y, u, y) = b^2 g(\varphi u, \varphi u)F(u', y', u', y') - b^2 g(y', y')g(\varphi u, \varphi u) + b^2\widetilde{\eta}(u)^2 g(y', y') + g(\varphi u, \varphi u)\big(\widetilde{\eta}(y) - a\widetilde{\eta}(u)\big)^2. \tag{20}$$

Expanding $g(S(u, y)u, y)$, we get

$$g(S(u, y)u, y) = g(\varphi u, \varphi u)\big\{a^2 g(\varphi^2 u, u) + b^2 g(\varphi^2 y', y')\big\} - a^2 g(\varphi u, \varphi u)g\left(\varphi^2 u, u\right) = -b^2 g(\varphi u, \varphi u)g(y', y'). \tag{21}$$

Analogously, expanding $g(T(u,y)u,y)$, we have

$$\begin{aligned} g(T(u,y)u,y) &= -\widetilde{\eta}^2(u)\{a^2 g(\varphi u,\varphi u) + b^2 g(\varphi y',\varphi y')\} + 2a g(\varphi u,\varphi u)\widetilde{\eta}(u)\widetilde{\eta}(y) - g(\varphi u,\varphi u)\widetilde{\eta}^2(y) \\ &= -b^2 g(y',y')\widetilde{\eta}^2(u) - g(\varphi u,\varphi u)\{a^2\widetilde{\eta}^2(u) - 2a\widetilde{\eta}(u)\widetilde{\eta}(y) + \widetilde{\eta}^2(y)\} \\ &= -b^2 g(y',y')\widetilde{\eta}^2(u) - g(\varphi u,\varphi u)\big(a\widetilde{\eta}(u) - \widetilde{\eta}(y)\big)^2. \end{aligned} \tag{22}$$

Now (20), (21) and (22) imply $H(u,y,u,y) = b^2 g(\varphi u,\varphi u)F(u',y',u',y')$, where $u' \in N_\varphi(\xi_1)$ and $y' \in u'^\perp \cap \operatorname{Im}\varphi$. Thus (a) yields $H(u,y,u,y) = 0$ when u is a lightlike vector of the first kind.

Now, suppose that u is a lightlike vector of the second kind and that $y \in u^\perp$ is given by (13). Using (11), with analogous computations as above, we find

$$F(u,y,u,y) = b^2 F(u,y',u,y') + \mu^i\mu^j F(u,w_i',u,w_j') + 2\mu^i b F(u,y',u,w_i') = b^2 g(y',y')\widetilde{\eta}^2(u).$$

Moreover, $\varphi u = 0$, hence $g(S(u,y)u,y) = 0$ and, with straightforward calculations,

$$g(T(u,y)u,y) = \widetilde{\eta}(u)\widetilde{\eta}(u)g\big(\varphi^2 y,y\big) = -b^2 g(y',y')\widetilde{\eta}^2(u).$$

The above identities imply $H(u,y,u,y) = 0$ when u is a lightlike vector of the second kind.

Thus, H vanishes on any degenerate 2-plane. By Lemma 1.1, there exists $k \in \mathbb{R}$ such that $H(x,y,z,w) = k\{g(y,z)g(x,w) - g(x,z)g(y,w)\}$, for any $x,y,z,w \in T_pM$. It follows that $H(x,\xi_\alpha,x,\xi_\alpha) = -k\varepsilon_\alpha$, for any unit $x \in \operatorname{Im}\varphi$ and any $\alpha \in \{1,\dots,s\}$. On the other hand, (17) yields $H(x,\xi_\alpha,x,\xi_\alpha) = 0$, hence $H(x,y,z,w) = 0$, for any $x,y,z,w \in T_pM$, and (b) follows. □

Remark 4.2.

It is worth noting that the Riemannian curvature tensor of a Lorentzian $\mathcal{S}$-manifold verifies the conditions in (14), which follow from (b) and (d) of Lemma 2.1. Therefore, the previous result gives a new curvature characterization for such manifolds as a corollary.

Remark 4.3.

We point out that the analogous result stated in [10] for a Lorentzian $\mathcal{S}$-manifold having exactly two characteristic vector fields is a particular case of Lemma 4.1. Indeed, using the notations of [10], the tensor S_* is nothing but S, and a long, but straightforward calculation shows that the tensor T reduces to S^*, when the manifold has only two characteristic vector fields.

Given an even-dimensional Osserman Riemannian manifold (M,g) whose Jacobi operators R_x, for any $x \in S(M)$, admit exactly two eigenvalues $c_1, c_2 \in \mathbb{R}$, with algebraic multiplicities 1 and $\dim M - 2$, respectively, it is possible to define an almost Hermitian structure J by associating to each $x \in S(M)$ the unit eigenvector Jx of R_x with respect to the eigenvalue c_1 with multiplicity 1 (see [26, 29]). In [28, Lemma 3.5.1, p. 204] the above construction of J is provided in a purely algebraic setting. In [10] this construction is adapted to the case of a $(2n+s)$-dimensional φ-null Osserman Lorentzian $\mathcal{S}$-manifold $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$ at a point $p \in M$, with Jacobi operators $\overline{R}_u^\varphi$, $u \in N_\varphi\big((\xi_1)_p\big)$, admitting eigenvalues as said. In this way, an almost Hermitian structure is obtained on $\operatorname{Im}\varphi_p$. Now, let us use that construction here. For any $x \in T_pM$, let us denote by x' its projection onto $\operatorname{Im}\varphi_p$, that is, in view of the first equation in (2), $x' = -\varphi^2 x$. Consider the $(1,3)$-type tensor R^0 and R^J on T_pM, defined by

$$R^0(x,y)z = g(y',z')x' - g(x',z')y', \qquad R^J(x,y)z = g(Jy',z')Jx' - g(Jx',z')Jy' + 2g(x',Jy')Jz'$$

for any $x,y,z \in T_pM$. Using exactly the same proof as in [10], except for the use of Lemma 4.1, of course, and using the properties of Lemma 2.1, we obtain the following curvature characterization.

Theorem 4.4.

Let $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$ be a Lorentzian $\mathcal{S}$-manifold, $\dim M = 2n + s$, with $n > 1$ and $s \geqslant 1$. The following statements are equivalent.

(a) *M is φ-null Osserman at a point p, and for any $u \in N_\varphi\big((\xi_1)_p\big)$ the Jacobi operators $\overline{R}^\varphi_u$ admit exactly two distinct eigenvalues $c_1, c_2 \in \mathbb{R}$, with multiplicities 1 and $2n - 2$, respectively.*

(b) *There exists an almost Hermitian structure J on $\operatorname{Im}\varphi_p$ and $c_1, c_2 \in \mathbb{R}$, such that for any $x, y, z \in T_pM$,*

$$R_p(x, y)z = T_p(x, y)z - S_p(x, y)z + c_2 R^0(x, y)z + \frac{c_1 - c_2}{3} R^J(x, y)z.$$

In particular, for $s = 1$, when the φ-null Osserman condition at a point is nothing but the null Osserman condition on a Lorentzian almost contact metric manifold, we have seen that $T_p(x, y)z - S_p(x, y)z = g_p(x, z)y - g_p(y, z)x$, for any $x, y, z \in T_pM$. Using this, we obtain the following final result as a corollary.

Corollary 4.5.

Let $(M, \varphi, \xi, \eta, g)$ be a Lorentz Sasakian manifold, with $\dim M = 2n + 1$ and $n > 1$. The following statements are equivalent.

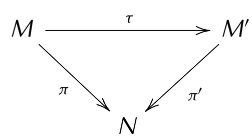
(a) *M is null Osserman at a point p, and for any $u \in N_\varphi(\xi_p)$ the Jacobi operators $\overline{R}_u$ admit exactly two distinct eigenvalues $c_1, c_2 \in \mathbb{R}$, with multiplicities 1 and $2n - 2$, respectively.*

(b) *There exists an almost Hermitian structure J on $\operatorname{Im}\varphi_p$ and $c_1, c_2 \in \mathbb{R}$, such that for any $x, y, z \in T_pM$,*

$$R_p(x, y)z = g_p(x, z)y - g_p(y, z)x + c_2 R^0(x, y)z + \frac{c_1 - c_2}{3} R^J(x, y)z.$$

5. An application: canonical torus bundles

In this section we give a simple application of the above two characterizations, by considering the canonical torus bundles constructed in [14]. It is known that a compact, connected and regular Lorentzian (or, more generally, indefinite) $\mathcal{S}$-manifold $(M, \varphi, \xi_\alpha, \eta^\alpha, g)$, with $\dim M = 2n + s$, $s \geqslant 2$, projects itself onto a $2n$-dimensional compact (indefinite) Kähler manifold (N, J', G') and onto a $(2n+1)$-dimensional compact and regular Lorentz (indefinite) Sasakian manifold $(M', \varphi', \xi', \eta', g')$, giving rise to the following commutative diagram:



(23)

Here, all the maps are semi-Riemannian submersions with totally geodesic fibres, and more precisely:

- τ is the projection of a principal $\mathbb{T}^{s-1}$-bundle over M',
- π' is the projection of a principal $\mathbb{S}^1$-bundle over N,
- π is the projection of a principal $\mathbb{T}^s$-bundle over N.

where $\mathbb{T}^k$ is the k-dimensional torus, for any $k \in \mathbb{N}$, $k \geqslant 1$.

Let us first consider the semi-Riemannian submersion $\pi\colon M \to N$. At any point $p \in M$, the horizontal space is $\mathcal{H}_p = \operatorname{Im}\varphi_p$ and the vertical space is $\mathcal{V}_p = \operatorname{span}\big((\xi_1)_p, \dots, (\xi_s)_p\big)$. If we denote by v and h the canonical projections of TM onto $\mathcal{V}$ and $\mathcal{H}$, respectively, the second O'Neill tensor of π is the $(1, 2)$-type tensor field A defined by $A(X, Y) = A_X Y = h(\nabla_{hX} vY) + v(\nabla_{hX} hY)$, for any $X, Y \in TM$. Following [22], and omitting from now on the point p for simplicity, let us denote by R^* the $(1, 3)$-type curvature tensor on the horizontal space such that $R^*(x, y)z$ is the horizontal lift of $R'(x',y')z'$, where R' is the $(1,3)$-type curvature tensor of (N,J',G'), and $x'=\pi_*(x)$, $y'=\pi_*(y)$, $z'=\pi_*(z)$, for any $x,y,z\in\mathcal{H}$. Using the second identity in (1.28) of [22, p. 13], and the general properties of the tensor A (see [22, p. 9, Lemma 1.1]), we get

$$R^*(x,y)z=R(x,y)z+2A_zA_xy-A_xA_yz+A_yA_xz, \tag{24}$$

for any $x,y,z\in\mathcal{H}$. Since $A_xA_yz=(s-2)g(y,\varphi z)\varphi x$, for any $x,y,z\in\mathcal{H}$, (see [12, p. 7, Lemma 4.2]) then (24) becomes

$$R^*(x,y)z=R(x,y)z+(s-2)R^{\varphi}(x,y)z, \tag{25}$$

where we put $R^{\varphi}(x,y)z=2g(x,\varphi y)\varphi z-g(y,\varphi z)\varphi x+g(x,\varphi z)\varphi y$. Under the hypotheses (a) of Theorem 4.4, using the expression of the curvature tensor given in (b) of the same theorem, (25) becomes

$$R^*(x,y)z=(c_2-1)R^0(x,y)z+\frac{c_1-c_2}{3}R^J(x,y)z+(s-2)R^{\varphi}(x,y)z, \tag{26}$$

since it is easy to check that $T(x,y)z=0$ and $S(x,y)z=R^0(x,y)z$, for any $x,y,z\in\mathcal{H}$. It is worth noting that the above identity holds even in the case of the principal $\mathbb{S}^1$-bundle $\pi'\colon M'\to N$, putting $s=1$, where M' is a null Osserman Lorentz Sasakian manifold, as one can easily see using (b) of Corollary 4.5.

Identity (26) characterizes the curvature tensor of the Kähler manifold N as the base manifold of the fibration, by means of its horizontal lift. In particular, if we suppose, as in [12, Proposition 4.4], that for any $x\in S_{\varphi}\big((\xi_1)_p\big)$, φx is the eigenvector of R_x related to the eigenvalue c_1 with multiplicity one, then we get $\varphi=J$, so that J is projectable onto J' at the fixed point $p\in M$, and (26) gives

$$R'(x',y')z'=(c_2-1)R'^0(x',y')z'+\frac{c_1+3(s-2)-c_2}{3}R^{J'}(x',y')z',$$

for any $x',y',z'\in T_{p'}N$, $p'=\pi(p)$, where R'^0 and $R^{J'}$ are the projections of R^0 and R^J, respectively, on N. On account of Remark 3.6, the above identity agrees with the fact that N is an Osserman manifold, with the Jacobi operator $R'_{x'}$ admitting the eigenvalues $c'_1=c_1+3(s-2)-1$ and $c'_2=c_2-1$, with multiplicity 1 and $2n-2$, as observed at the end of the proof of [12, Proposition 4.4].

Referring to the commutative diagram (23), let us now consider the semi-Riemannian submersion $\tau\colon M\to M'$, between the $(2n+s)$-dimensional Lorentzian $\mathcal{S}$-manifold $(M,\varphi,\xi_\alpha,\eta^\alpha,g)$ and the $(2n+1)$-dimensional Lorentz Sasakian manifold $(M',\varphi',\xi',\eta',g')$. At any point $p\in M$, the horizontal space is $\mathcal{H}_p=(\operatorname{Im}\varphi_p)\oplus\operatorname{span}\big((\xi_1)_p\big)$ and the vertical space is $\mathcal{V}_p=\operatorname{span}\big((\xi_2)_p,\dots,(\xi_s)_p\big)$. Suppressing again the explicit dependence on p in the notation, since $A_xA_yz=(s-1)g(y,\varphi z)\varphi x$, for any $x,y,z\in\mathcal{H}$, (see [12, p. 8, Lemma 4.6]), then (1.28) of [22, p. 13] gives

$$g(R^*(x,y)z,w)=g(R(x,y)z,w)+(s-1)\big\{2g(x,\varphi y)g(\varphi z,w)-g(y,\varphi z)g(\varphi x,w)+g(x,\varphi z)g(\varphi y,w)\big\}, \tag{27}$$

for any $x,y,z,w\in\mathcal{H}$. Under the hypotheses (a) of Theorem 4.4, using the expression of R given in (b) of the same theorem, if $x,y,z\in\mathcal{H}$ and $w\in\operatorname{Im}\varphi$ we get

$$\begin{aligned}g(R(x,y)z,w)&=-\tilde\eta(y)\tilde\eta(z)g\big(\varphi^2x,w\big)+\tilde\eta(x)\tilde\eta(z)g\big(\varphi^2y,w\big)-g(\varphi x,\varphi z)g\big(\varphi^2y,w\big)\\&\quad+g(\varphi y,\varphi z)g\big(\varphi^2x,w\big)+c_2g\big(R^0(x,y)z,w\big)+\frac{c_1-c_2}{3}g\big(R^J(x,y)z,w\big),\end{aligned}$$

and using this, from (27) it follows

$$R^*(x,y)z\!\restriction_{\operatorname{Im}\varphi}=-\eta^1(y)\eta^1(z)\varphi^2x+\eta^1(x)\eta^1(z)\varphi^2y+(c_2-1)R^0(x,y)z+\frac{c_1-c_2}{3}R^J(x,y)z+(s-1)R^{\varphi}(x,y)z, \tag{28}$$

for any $x,y,z\in\mathcal{H}$. Analogously, using again the expression of R given by (b) of Theorem 4.4, if $w=\xi_1$, we have

$$g(R(x,y)z,\xi_1)=g(\varphi y,\varphi z)\eta^1(x)-g(\varphi x,\varphi z)\eta^1(y), \tag{29}$$

for any $x, y, z \in \mathcal{H}$. Summarizing, using (29) in (27), from (28), we have

$$R^*(x,y)z = (c_2-1)R^0(x,y)z + \frac{c_1-c_2}{3}R^J(x,y)z + (s-1)R^{\varphi}(x,y)z \\ - \eta^1(y)\eta^1(z)\varphi^2 x + \eta^1(x)\eta^1(z)\varphi^2 y + \{g(\varphi x, \varphi z)\eta^1(y) - g(\varphi y, \varphi z)\eta^1(x)\}\,\xi_1, \tag{30}$$

for any $x, y, z \in \mathcal{H}$. Using (2) and (3), an easy computation shows that

$$R^0(x,y)z + g(x,z)y - g(y,z)x = -\eta^1(y)\eta^1(z)\varphi^2 x + \eta^1(x)\eta^1(z)\varphi^2 y + \{g(\varphi x, \varphi z)\eta^1(y) - g(\varphi y, \varphi z)\eta^1(x)\}\,\xi_1,$$

thus, from (30) it follows that

$$R^*(x,y)z = g(x,z)y - g(y,z)x + c_2 R^0(x,y)z + \frac{c_1-c_2}{3}R^J(x,y)z + (s-1)R^{\varphi}(x,y)z, \tag{31}$$

for any $x, y, z \in \mathcal{H}$.

Now, (31) characterizes the curvature tensor of the Lorentz Sasakian manifold $(M', \varphi', \xi', \eta', g')$ as the base manifold of the fibration $\tau\colon M \to M'$, by means of its horizontal lift. In particular, if we suppose again, as in [12, Proposition 4.7], that for any $x \in S_{\varphi}\big((\xi_1)_p\big)$, φx is the eigenvector of R_x related to the eigenvalue c_1 with multiplicity one, then we get $\varphi = J$, so that J is projectable onto φ' at the fixed point $p \in M$, and (31) gives

$$R'(x',y')z' = g'(x',z')y' - g'(y',z')x' + c_2 R'^0(x',y')z' + \frac{c_1+3(s-1)-c_2}{3}R^{\varphi'}(x',y')z',$$

for any $x', y', z' \in T_{p'}M'$, $p' = \pi(p)$, where R'^0 and $R^{\varphi'}$ are the projections of R^0 and R^J, respectively, on M'. According to Remark 3.6, the above identity agrees both with the expression of curvature given by (b) of Corollary 4.5, for a null Osserman Lorentz Sasakian manifold, and with the fact that the Jacobi operator $R'_{x'}$ admits the eigenvalues $c'_1 = c_1 + 3(s-1) - 1$ and $c'_2 = c_2 - 1$, with multiplicity 1 and $2n-2$, as observed in the proof of [12, Proposition 4.7].

In a forthcoming paper we are going to study the global version of the φ-null Osserman condition and, in providing some results on the relationships between the pointwise and the global versions of this condition, we will see some consequences on the φ-sectional curvature of the manifold.

Acknowledgements

The authors wish to express their heartly thanks to the referee for his/her valuable comments, which the last section is based on.

References

[1] Atindogbe C., Duggal K.L., Pseudo–Jacobi operators and Osserman lightlike hypersurfaces, Kodai Math. J., 2009, 32(1), 91–108

[2] Atindogbe C., Ezin J.-P., Tossa J., Pseudoinversion of degenerate metrics, Int. J. Math. Math. Sci., 2003, 55, 3479–3501

[3] Atindogbe C., Lugiambudila O., Tossa J., Lightlike Osserman submanifolds of semi-Riemannian manifolds, Afr. Mat., 2011, 22(2), 129–151

[4] Blair D.E., Geometry of manifolds with structural group $\mathcal{U}(n) \times \mathcal{O}(s)$, J. Differential Geometry, 1970, 4, 155–167

[5] Blair D.E., Riemannian Geometry of Contact and Symplectic Manifolds, 2nd ed., Progr. Math., 203, Birkhäuser, Boston, 2010

[6] Blair D.E., Ludden G.D., Yano K., Differential geometric structures on principal toroidal bundles, Trans. Amer. Math. Soc., 1973, 181, 175–184

[7] Blažić N., Bokan N., Gilkey P., A note on Osserman Lorentzian manifolds, Bull. London Math. Soc., 1997, 29(2), 227–230

[8] Blažić N., Bokan N., Rakić Z., A note on the Osserman conjecture and isotropic covariant derivative of curvature, Proc. Amer. Math. Soc., 2000, 128(1), 245–253

[9] Bonome A., Castro R., García-Río E., Hervella L., Vázquez-Lorenzo R., Nonsymmetric Osserman indefinite Kähler manifolds, Proc. Amer. Math. Soc., 1998, 126(9), 2763–2769

[10] Brunetti L., An Osserman-type condition on g.f.f-manifolds with Lorentz metric, preprint available at http://arxiv.org/abs/1106.6317v2

[11] Brunetti L., The η-Einstein condition on indefinite $\mathcal{S}$-manifolds, preprint available at http://arxiv.org/abs/1202.3289

[12] Brunetti L., Caldarella A.V., Principal torus bundles of Lorentzian $\mathcal{S}$-manifolds and the φ-null Osserman condition, Kodai Math. J., 2013, 36(2), 197–208

[13] Brunetti L., Pastore A.M., Curvature of a class of indefinite globally framed f-manifolds, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 2008, 51(99)(3), 183–204

[14] Brunetti L., Pastore A.M., Examples of indefinite globally framed f-structures on compact Lie groups, Publ. Math. Debrecen, 2012, 80(1-2), 215–234

[15] Cabrerizo J.L., Fernández L.M., Fernández M., The curvature tensor fields on f-manifolds with complemented frames, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Secţ. I a Mat., 1990, 36(2), 151–161

[16] Chi Q.-S., A curvature characterization of certain locally rank-one symmetric space, J. Differential Geom., 1988, 28(2), 187–202

[17] Chi Q.-S., Curvature characterization and classification of rank-one symmetric spaces, Pacific J. Math., 1991, 150(1), 31–42

[18] Chi Q.-S., Quaternionic Kaehler manifolds and a curvature characterization of two-point homogeneous spaces, Illinois J. Math., 1991, 35(3), 408–418

[19] Dajczer M., Nomizu K., On sectional curvature of indefinite metrics. II, Math. Ann., 1980, 247(3), 279–282

[20] Duggal K.L., Space time manifolds and contact structures, Int. J. Math. Math. Sci., 1990, 13(3), 545–554

[21] Duggal K.L., Ianus S., Pastore A.M., Maps interchanging f-structures and their harmonicity, Acta Appl. Math., 2001, 67(1), 91–115

[22] Falcitelli M., Ianus S., Pastore A.M., Riemannian Submersions and Related Topics, World Scientific, River Edge, 2004

[23] García-Río E., Kupeli D.N., Four-Dimensional Osserman Lorentzian manifolds, In: New Developments in Differential Geometry, Debrecen, July 26–30, 1994, Math. Appl., 350, Kluwer, Dordrecht, 1996, 201–211

[24] García-Río E., Kupeli D.N., Vázquez-Abal M.E., On a problem of Osserman in Lorentzian geometry, Differential Geom. Appl., 1997, 7(1), 85–100

[25] García-Río E., Kupeli D.N., Vázquez-Abal M.E., Vázquez-Lorenzo R., Affine Osserman connections and their Riemann extensions, Differential Geom. Appl., 1999, 11(2), 145–153

[26] García-Río E., Kupeli D.N., Vázquez-Lorenzo R., Osserman Manifolds in Semi-Riemannian Geometry, Lecture Notes in Math., 1777, Springer, Berlin, 2002

[27] García-Río E., Vázquez-Abal M.E., Vázquez-Lorenzo R., Nonsymmetric Osserman pseudo-Riemannian manifolds, Proc. Amer. Math. Soc., 1998, 126(9), 2771–2778

[28] Gilkey P., Geometric Properties of Natural Operators Defined by the Riemann Curvature Tensor, World Scientific, River Edge, 2001

[29] Gilkey P., Swann A., Vanhecke L., Isoparametric geodesic spheres and a conjecture of Osserman concerning the Jacobi operator, Quart. J. Math. Oxford, 1995, 46(3), 299–320

[30] Goldberg S.I., Yano K., On normal globally framed f-manifolds, Tôhoku Math. J., 1970, 22(3), 362–370

[31] Goldberg S.I., Yano K., Globally framed f-manifolds, Illinois J. Math., 1971, 15(3), 456–474

[32] Kobayashi S., Nomizu K., Foundations of Differential Geometry, I, II, Interscience Tracts in Pure and Applied Mathematics, 15(I,II), Interscience, New York–London, 1963, 1969

[33] Nikolayevsky Yu., Osserman manifolds of dimension 8, Manuscripta Math., 2004, 115(1), 31–53

[34] Nikolayevsky Yu., Osserman conjecture in dimension $n \neq 8, 16$, Math. Ann., 2005, 331(3), 505–522

[35] Nikolayevsky Yu., On Osserman manifolds of dimension 16, In: Contemporary Geometry and Related Topics, Belgrade, June 26–July 2, 2005, Univ. Belgrade Fac. Math., Belgrade, 2006, 379–398
[36] Nakagawa H., f-structures induced on submanifolds in spaces, almost Hermitian or Kaehlerian, Kōdai Math. Sem. Rep., 1966, 18(2), 161–183
[37] Nakagawa H., On framed f-manifolds, Kōdai Math. Sem. Rep., 1966, 18(4), 293–306
[38] O'Neill B., Semi-Riemannian Geometry, Pure Appl. Math., 103, Academic Press, New York, 1983
[39] Osserman R., Curvature in the eighties, Amer. Math. Monthly, 1990, 97(8), 731–756
[40] Osserman R., Sarnak P., A new curvature invariant and entropy of geodesic flows, Invent. Math., 1984, 77(3), 455–462
[41] Steenrod N., The Topology of Fibre Bundles, Princeton Math. Ser., 14, Princeton University Press, Princeton, 1951
[42] Takahashi T., Sasakian manifold with pseudo-Riemannian metric, Tôhoku Math. J., 1969, 21(2), 271–290