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The L^q spectra and Rényi dimension of generalized inhomogeneous self-similar measures

Research Article

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Abstract: Very recently bounds for the L^q spectra of inhomogeneous self-similar measures satisfying the Inhomogeneous Open Set Condition (IOSC), being the appropriate version of the standard Open Set Condition (OSC), were obtained. However, if the IOSC is not satisfied, then almost nothing is known for such measures. In the paper we study the L^q spectra and Rényi dimension of generalized inhomogeneous self-similar measures, for which we allow an infinite number of contracting similarities and probabilities depending on positions. As an application of the results, we provide a systematic approach to obtaining non-trivial bounds for the L^q spectra and Rényi dimension of inhomogeneous self-similar measures not satisfying the IOSC and of homogeneous ones not satisfying the OSC. We also provide some non-trivial bounds without any separation conditions.

MSC: 28A80

Keywords: L^q spectra • Rényi dimension • Inhomogeneous self-similar measure • Homogeneous self-similar measure

1. Introduction

There is a huge body of literature (see [9] and references therein) investigating different aspects of the homogeneous self-similar measures satisfying

$$\mu_0 = \sum_{i=1}^{N} p_i \mu_0 \circ S_i^{-1}, \tag{1}$$

where p_i are probabilities and $S_i\colon \mathbb{R}^d \to \mathbb{R}^d$ are contracting similarities. It is also well known (see [8] or [13], for instance) that there exists a unique, non-empty and compact subset $K_\emptyset$ of $\mathbb{R}^d$ which satisfies $K_\emptyset = \bigcup_{i=1}^N S_i(K_\emptyset)$. Such sets are called homogeneous self-similar sets and there is a connection between them and the measures satisfying (1). Namely, the support of μ_0 is equal to the set $K_\emptyset$.

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It is easily observed that a measure μ_0 satisfying (1) can be viewed as the solution of the following equation:

$$\mu - \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} = 0.$$

This viewpoint suggests to investigate the corresponding inhomogeneous equation. Specifically, by making a simple transformation, it would be of interest to investigate measures which are solutions of the following inhomogeneous equation:

$$\mu = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} + p\nu, \tag{2}$$

where ν is a fixed probability measure on $\mathbb{R}^d$ and its support is a compact set $C \subset \mathbb{R}^d$. Measures that satisfy (2) are called inhomogeneous self-similar measures. Inhomogeneous self-similar measures were introduced by Barnsley et al. in the 1980s, along with inhomogeneous self-similar sets of the form

$$K = \bigcup_{i=1}^{N} S_i(K) \cup C. \tag{3}$$

These measures and sets were introduced as tools for image compression and are mentioned in various monographs (for instance, see [4, 5] or [22]). For some examples of inhomogeneous self-similar measures, we refer the reader to [6]. It is also worth mentioning that inhomogeneous self-similar sets (3) are closely related to the measures μ that satisfy (2). Specifically, it is proved in [19, Proposition 1.2] that the support of μ is equal to the set K.

In [19] the first study of the L^q spectra and Rényi dimensions of (2), under the assumption that the sets $(S_1K, \ldots, S_NK, C)$ are pairwise disjoint, was intiated. When examining the L^q spectra of inhomogeneous self-similar measures, the assumption of the disjointness of these sets is clearly unsatisfactory. This fact was stated by the authors of [19], and they asked (see [19, Question 2.7]) whether the results obtained in [19, Section 2.1] are true when only the Inhomogeneous Open Set Condition (IOSC), which is the appropriate version of the standard Open Set Condition (OSC), is assumed. In the recent paper [18], we answered this question affirmatively in relation to the main theorem of [19, Section 2.1] and we also improved estimates from [19, Theorem 2.1].

This paper was motivated by the fact that the form of inhomogeneous self-similar measures given by (2) is a particular case of the following measures:

$$\mu(A) = \sum_{i \in I} \int_{S_i^{-1}(A)} p_i(x)\, d\mu(x) + p\nu(A), \tag{4}$$

where $p_i(x)$ are place dependent probabilities and the set of indexes I is at most countable. Such composition generalizes so-called iterated function systems with probabilities depending on positions for systems consisting of contracting similarities. It would be of interest to generalize our previous results to this more general form which has not been studied yet in the literature (see [6, 19, 21, 22]).

In the first part of the paper we will provide estimates for the L^q spectra of inhomogeneous self-similar measures given by (4). As a consequence, we will obtain also estimates for the Rényi dimension of (4) and, in particular, we will give a partial answer to another question from [19], namely Question 2.13. In the second part, we will present some applications of our results. We will try to apply the results obtained in the first part in order to go a step further and to obtain some non-trivial estimates relaxing the assumed separability condition. Thus, we will focus on the problem of providing non-trivial estimates for the L^q spectra and Rényi dimension of inhomogeneous and homogeneous self-similar measures not satisfying separability conditions like the IOSC and OSC. If the OSC is not satisfied then we can find only sporadic studies of various special classes of measures (see [10, 11, 16, 17, 24, 25]) for which something is known about the L^q spectra or other multifractal properties. For the inhomogeneous case, to the best of our knowledge, no investigation was performed so far. In the general case, failure to meet such separability assumptions significantly impedes the calculation of dimensions and the study of other properties. Applying our main result, we provide the first systematic approach to obtaining non-trivial bounds for the L^q spectra and, consequently, to obtaining some non-trivial bounds for the Rényi dimension of inhomogeneous self-similar measures that do not satisfy the IOSC. This approach will be further extended to homogeneous measures that do not satisfy the OSC. We also obtain some non-trivial bounds for such measures without any separation conditions.

As an application of (4), we will turn our attention to non-linear self-similar measures. In the spirit of (4), we will present a more general form of these measures than that introduced by Glickenstein and Strichartz [12] and considered further in papers by Olsen and Snigireva [20]. Next, we will provide non-trivial bounds for the L^q spectra and Rényi dimension for them. Most of our results complement the study of multifractal properties of inhomogeneous self-similar measures from [21].

2. Preliminaries

Let $(S_i)_{i\in I}\colon \mathbb{R}^d \to \mathbb{R}^d$ be contracting similarities and r_i denote the contraction ratio of S_i. We assume that the set of indexes I is at most countable. Let $C \subset \mathbb{R}^d$ be a fixed, non-empty, compact set. Our considerations are carried out under the assumption of Inhomogeneous Open Set Condition, which, throughout the paper, will be abbreviated as IOSC. The IOSC states: there exists a non-empty and bounded open set U such that the following conditions are satisfied.

(I1) $C \subseteq \overline{U}$.

(I2) $S_i(U) \subseteq U$, $i \in I$.

(I3) $S_i(U) \cap S_j(U) = \emptyset$, $i \neq j$, $i, j \in I$.

(I4) $S_i(U) \cap C = \emptyset$, $i \in I$.

The Open Set Condition (OSC) assumes that only the conditions (I2) and (I3) are satisfied. We will discuss relaxation of conditions (I3) and (I4) in Section 5.

It is well known (see [4] or [18]) that there exists a unique inhomogeneous self-similar set K such that

$$K = \bigcup_{i\in I} S_i(K) \cup C. \tag{5}$$

K is non-empty, compact and $K \subseteq \overline{U}$. If I is infinite then by $K_{|n}$ we will denote the following subset of K:

$$K_{|n} = \bigcup_{i=1}^{n} S_i(K) \cup C, \qquad n \in \mathbb{N}.$$

Let $\big((p_i(x))_{i\in I}, p\big)\colon \mathbb{R}^d \to [0,1]$ be a place dependent probability vector with positive constant probability p and let $I^+ = \{i \in I : \inf_{x\in\mathbb{R}^d} p_i(x) > 0\}$. Let also $\overline{p}_i = \sup_{x\in\mathbb{R}^d} p_i(x)$ and $\underline{p}_i = \inf_{x\in\mathbb{R}^d} p_i(x)$. Denote by $\mathcal{M}_1(\mathbb{R}^d)$ the space that consists of all probability measures, i.e., let $\mu(\mathbb{R}^d) = 1$ for $\mu \in \mathcal{M}_1(\mathbb{R}^d)$. Let $B(x,r)$ denote a closed ball with the centre at x and the radius r, $\operatorname{int} A$ denote the interior of a set A, $\overline{A}$ or $\operatorname{cl} A$ denote the closure of A, and $\mathcal{B}_X$ denote the σ algebra of the Borel subsets of X; $\# A$ stands for the cardinality of a set A and by $\overline{\mathbb{R}}$ we mean $\mathbb{R} \cup \infty$.

3. The L^q spectra

In this section we establish the main result of this paper, apart of applications. The result gives estimates for the L^q spectra of the generalized form of inhomogeneous self-similar measures (4). To reduce the size of the paper, lemmas and propositions in this section are mainly stated without proofs. They are generalizations of statements in [18, 19] and their proofs can be obtained in a similar way. We start with the following theorem which generalizes [18, Theorem 4.1], for details see [18, Section 4].

Theorem 3.1.

Let ν be a Borel probability measure with compact support $C \subset \mathbb{R}^d$, let $\big((p_i(x))_{i\in I}, p\big)\colon \mathbb{R}^d \to [0,1]$ be a probability vector with positive constant probability p and let $(S_i)_{i\in I}\colon \mathbb{R}^d \to \mathbb{R}^d$ be contracting similarities. Assume that $\sum_{i\in I}\overline{p}_i < 1$. Then, the Markov operator $M\colon \mathcal{M}_1(\mathbb{R}^d) \to \mathcal{M}_1(\mathbb{R}^d)$ defined by the formula

$$M\mu(A) = \sum_{i\in I}\int_{S_i^{-1}(A)} p_i(x)\,d\mu(x) + p\,\nu(A)$$

is strongly asymptotically stable. In particular, there exists a unique probability measure μ that satisfies

$$\mu(A) = \sum_{i\in I}\int_{S_i^{-1}(A)} p_i(x)\,d\mu(x) + p\,\nu(A). \tag{6}$$

The proof follows from the observation that M is contractive in the total variation norm with a ratio $\sum\limits_{i\in I}\overline{p}_i$.

Remark.

Theorem 3.1 gives only one case in which we can show the existence of invariant measures (6), namely, when $\sum_{i\in I}\overline{p}_i < 1$. If this assumption is not satisfied, the existence and uniqueness of such measures follows, e.g., from an argument similar to the one in [19, Proposition 1.1].

The next theorem generalizes [19, Proposition 1.2] and [18, Theorem 4.2] for the case of the generalized form of inhomogeneous self-similar measures (6).

Theorem 3.2.

Let μ be a unique inhomogeneous self-similar measure given by (6) and let K be a unique, non-empty, compact set satisfying (5). Then, $\operatorname{supp}\mu = K$.

Proof. It is enough to show that $\operatorname{supp}\mu$ satisfies (5). The inclusion $\operatorname{supp}\mu \subseteq \bigcup_{i\in I} S_i(\operatorname{supp}\mu)\cup C$ can be shown in a similar way as in the proof of [19, Proposition 1.2]; therefore is omitted. However, for the opposite inclusion the method from [19] is not applicable as $\sum_{i\in I}\overline{p}_i + p$ can be greater than 1. To prove the opposite inclusion, observe that $C \subseteq \operatorname{supp}\mu$ and hence it is enough to show that $\operatorname{supp}\mu \subseteq S_i^{-1}(\operatorname{supp}\mu)$ for all $i \in I$. Suppose the contrary, that $\operatorname{supp}\mu \subsetneq S_j^{-1}(\operatorname{supp}\mu)$ for some $j \in I$. Then

$$\begin{aligned}
1 = \mu(\operatorname{supp}\mu) &= \sum_{i\in I}\int_{S_i^{-1}(\operatorname{supp}\mu)} p_i(x)\,d\mu(x) + p\,\nu(\operatorname{supp}\mu) = \sum_{\substack{i\in I\\ i\neq j}}\int_{S_i^{-1}(\operatorname{supp}\mu)} p_i(x)\,d\mu(x) + \int_{S_j^{-1}(\operatorname{supp}\mu)} p_j(x)\,d\mu(x) + p\\
&= \sum_{\substack{i\in I\\ i\neq j}}\int_{S_i^{-1}(\operatorname{supp}\mu)\cap\operatorname{supp}\mu} p_i(x)\,d\mu(x) + \int_{S_j^{-1}(\operatorname{supp}\mu)\cap\operatorname{supp}\mu} p_j(x)\,d\mu(x) + p\\
&\le \sum_{\substack{i\in I\\ i\neq j}}\int_{\operatorname{supp}\mu} p_i(x)\,d\mu(x) + \int_{S_j^{-1}(\operatorname{supp}\mu)\cap\operatorname{supp}\mu} p_j(x)\,d\mu(x) + p = 1 - p - \int_{\operatorname{supp}\mu} p_j(x)\,d\mu(x) + \int_{S_j^{-1}(\operatorname{supp}\mu)\cap\operatorname{supp}\mu} p_j(x)\,d\mu(x) + p.
\end{aligned}$$

Hence,

$$\int_{\operatorname{supp}\mu} p_j(x)\,d\mu(x) \le \int_{S_j^{-1}(\operatorname{supp}\mu)\cap\operatorname{supp}\mu} p_j(x)\,d\mu(x).$$

By the above inequality it follows that $\operatorname{supp}\mu = S_j^{-1}(\operatorname{supp}\mu)\cap\operatorname{supp}\mu$. It, in turn, implies $\operatorname{supp}\mu \subseteq S_j^{-1}(\operatorname{supp}\mu)$ and we come to a contradiction. □

From now on, we assume that $I^+ = I$. However, we will discuss the necessity of this assumption later in this section. Let us define some functions related to the L^q spectra. Namely,

- For $q \geq 1$, define the functions $\overline{\beta}_I(q)\colon \mathbb{R} \to \mathbb{R}$ and $\underline{\beta}_I(q)\colon \mathbb{R} \to \mathbb{R}$ by the formulas

$$\sum_{i\in I} \overline{p}_i^{\,q} r_i^{\overline{\beta}_I(q)} = 1, \qquad \sum_{i\in I} \underline{p}_i^{\,q} r_i^{\underline{\beta}_I(q)} = 1.$$

- For $q < 1$, define the functions $\overline{\beta}_I(q-1)\colon \mathbb{R} \to \overline{\mathbb{R}}$ and $\underline{\beta}_I(q-1)\colon \mathbb{R} \to \overline{\mathbb{R}}$ by the formulas

$$\sum_{i\in I} \overline{p}_i \underline{p}_i^{\,q-1} r_i^{\overline{\beta}_I(q-1)} = 1, \qquad \sum_{i\in I} \underline{p}_i \overline{p}_i^{\,q-1} r_i^{\underline{\beta}_I(q-1)} = 1.$$

A particular form of such functions was considered in [18] and [19]. If $\# I = n$ for some $n \in \mathbb{N}$ then we will write $\overline{\beta}_n(q), \underline{\beta}_n(q), \overline{\beta}_n(q-1)$ and $\underline{\beta}_n(q-1)$. If $\# I = \mathbb{N}$ then we will omit n and simply write $\overline{\beta}(q)$, $\underline{\beta}(q)$, $\overline{\beta}(q-1)$ and $\underline{\beta}(q-1)$.

Let us recall the following notation introduced in [19]: for $l, m \in \mathcal{M}_1(\mathbb{R}^d)$, $q \in \mathbb{R}$ and $A \subseteq \operatorname{supp} m$, write

$$I_m(q,r) = \int_{\operatorname{supp} m} m(B(x,r))^{q-1}\, dm(x), \qquad I_{m|A}(q,r) = \int_{\operatorname{supp} m\cap A} m(B(x,r))^{q-1}\, dm(x),$$
$$I_{l,m,A}(q,r) = \int_A l(B(x,r))^{q-1}\, dm(x).$$

From this point and forwards, we fix an inhomogeneous self-similar measure μ satisfying (6).

Lemma 3.3.

Assume that the IOSC is satisfied. Then, for all $q \in \mathbb{R}$,

$$I_\mu(q,r) \le \sum_{i\in I} \overline{p}_i I_{\mu,\mu\circ S_i^{-1},S_iK}(q,r) + p\, I_{\mu,\nu,C}(q,r), \qquad \sum_{i\in I} \underline{p}_i I_{\mu,\mu\circ S_i^{-1},S_iK}(q,r) + p\, I_{\mu,\nu,C}(q,r) \le I_\mu(q,r).$$

Proof. Fix any $q \in \mathbb{R}$ and $r > 0$. From (6) we have

$$I_\mu(q,r) \le \sum_{i\in I} \overline{p}_i I_{\mu,\mu\circ S_i^{-1},K}(q,r) + p\, I_{\mu,\nu,K}(q,r), \qquad \sum_{i\in I} \underline{p}_i I_{\mu,\mu\circ S_i^{-1},K}(q,r) + p\, I_{\mu,\nu,K}(q,r) \le I_\mu(q,r).$$

The assertion follows now immediately from the fact that

$$I_{\mu,\mu\circ S_i^{-1},K}(q,r) = I_{\mu,\mu\circ S_i^{-1},K\cap \operatorname{supp}\mu\circ S_i^{-1}}(q,r) = I_{\mu,\mu\circ S_i^{-1},S_iK}(q,r), \qquad I_{\mu,\nu,K}(q,r) = I_{\mu,\nu,K\cap\operatorname{supp}\nu}(q,r) = I_{\mu,\nu,C}(q,r). \qquad \square$$

Let us introduce the following notation. For $l, m \in \mathcal{M}_1(\mathbb{R}^d)$, $A \subseteq \operatorname{supp} m$ and $x \in A$, write

$$\overline{J}_{i,m,A}(x,r) = \sum_{j\ne i} \overline{p}_j\, m\bigl(S_j^{-1}(B(x,r)\cap S_j(\operatorname{supp} m))\bigr) + p\,\nu(B(x,r)\cap C),$$
$$\overline{J}_{C,m,A}(x,r) = \sum_{i\in I} \overline{p}_i\, m\bigl(S_i^{-1}(B(x,r)\cap S_i(\operatorname{supp} m))\bigr),$$
$$\overline{F}_{i,l,m,A}(q,r) = \int_A \left(l\left(B\left(x, \frac{r}{r_i}\right)\right) + \frac{\overline{J}_{i,l,A}(S_i x, r)}{\overline{p}_i}\right)^{q-1} dm(x).$$

In a similar manner we define $\underline{J}_{i,m,A}(x,r), \underline{J}_{C,m,A}(x,r), \underline{F}_{i,l,m,A}(q,r)$ by replacing $\bar{\cdot}$ with $\underline{\cdot}$ in the right sides. We will simply write $\overline{J}_i(x,r)$ if $m=\mu$ and $A=K$ in $\overline{J}_{i,m,A}(x,r)$. Analogously, $\overline{J}_C(x,r)$ stands for $\overline{J}_{C,m,A}(x,r)$ when $m=\mu$ and $A=C$. Finally, if $l=m=\mu$ and $A=K$ in $\overline{F}_{i,l,m,A}(q,r)$ then we will simply write $\overline{F}_i(q,r)$. We use the same rule for the bottom line values.

Now let us make an important observation. It is easily seen from (6) that under the assumption of the IOSC we have:

- For all $q \geq 1$ and $r > 0$,

$$\mu(B(x,r))^{q-1} \leq \begin{cases} \left(\overline{p}_i \mu\left(B\left(S_i^{-1}x, \frac{r}{r_i}\right)\right) + \overline{J}_i(x,r)\right)^{q-1} & \text{for} \quad x \in S_i K, \\ \left(p\,\nu(B(x,r)\cap C) + \overline{J}_C(x,r)\right)^{q-1} & \text{for} \quad x \in C, \end{cases}$$

$$\mu(B(x,r))^{q-1} \geq \begin{cases} \left(\underline{p}_i \mu\left(\left(S_i^{-1}x, \frac{r}{r_i}\right)\right) + \underline{J}_i(x,r)\right)^{q-1} & \text{for} \quad x \in S_i K, \\ \left(p\,\nu(B(x,r)\cap C) + \underline{J}_C(x,r)\right)^{q-1} & \text{for} \quad x \in C. \end{cases}$$

- For all $q < 1$ and $r > 0$, we have to change the inequalities to the opposite ones.

For greater clarity: in the above formulas, we have

$$\overline{J}_i(x,r) = \sum_{j\neq i} \overline{p}_j \mu\big(S_j^{-1}(B(x,r)\cap S_j K)\big) + p\,\nu(B(x,r)\cap C), \qquad \overline{J}_C(x,r) = \sum_{i\in I} \overline{p}_i \mu\big(S_i^{-1}(B(x,r)\cap S_i K)\big).$$

Analogously, in $\underline{J}_i(x,r)$ and $\underline{J}_C(x,r)$, we have, respectively, $\underline{p}_j$ and $\underline{p}_i$. Let us also introduce the following notation:

$$\overline{F}_{C,\nu,m,A}(q,r) = \int_A \left(\nu(B(x,r)) + \frac{\overline{J}_{C,\mu,A}(x,r)}{p}\right)^{q-1} dm(x).$$

In the particular case when $m = \nu$ and $A = C$, we simply write

$$\overline{F}_C(q,r) = \int_C \left(\nu(B(x,r)) + \frac{\overline{J}_C(x,r)}{p}\right)^{q-1} d\nu(x).$$

In the same way we denote $\underline{F}_{C,\nu,m,A}(q,r)$ and $\underline{F}_C(q,r)$ in which we have, respectively, $\underline{J}_{C,\mu,A}(x,r)$ and $\underline{J}_C(x,r)$.

The next proposition resembles Lemma 3.3 but goes a step further.

Proposition 3.4.

Assume that the IOSC is satisfied. For all $q \geq 1$,

$$\sum_{i\in I} \underline{p}_i^q \underline{F}_i(q,r) + p^q \underline{F}_C(q,r) \leq I_\mu(q,r) \leq \sum_{i\in I} \overline{p}_i^q \overline{F}_i(q,r) + p^q \overline{F}_C(q,r).$$

For all $q < 1$,

$$\sum_{i\in I} \underline{p}_i \overline{p}_i^{q-1} \overline{F}_i(q,r) + p^q \overline{F}_C(q,r) \leq I_\mu(q,r) \leq \sum_{i\in I} \overline{p}_i \underline{p}_i^{q-1} \underline{F}_i(q,r) + p^q \underline{F}_C(q,r).$$

Proof. We will provide the proof only for the right inequality "$\leq$" and for $q \geq 1$. For the left inequality "$\leq$" and for both inequalities for $q < 1$ the proof is similar. Fix $q \geq 1$ and let $r > 0$. We have,

$$\begin{aligned} I_{\mu,\mu\circ S_i^{-1},S_iK}(q,r) &\leq \int_{S_iK} \left(\overline{p}_i \mu\left(B\left(S_i^{-1}x, \frac{r}{r_i}\right)\right) + \overline{J}_i(x,r)\right)^{q-1} d(\mu\circ S_i^{-1})(x) \\ &= \int_K \left(\overline{p}_i \mu\left(B\left(x, \frac{r}{r_i}\right)\right) + \overline{J}_i(S_i x, r)\right)^{q-1} d\mu(x) \\ &= \overline{p}_i^{q-1} \int_K \left(\mu\left(B\left(x, \frac{r}{r_i}\right)\right) + \frac{\overline{J}_i(S_i x, r)}{\overline{p}_i}\right)^{q-1} d\mu(x) = \overline{p}_i^{q-1} \overline{F}_i(q,r) \end{aligned}$$

and by using steps analogous to those above, we obtain $I_{\mu,\nu,C}(q,r) \leq p^{q-1}\overline{F}_C(q,r)$. Finally, applying Lemma 3.3, we have

$$I_\mu(q,r) \leq \sum_{i\in I} \overline{p}_i I_{\mu,\mu\circ S_i^{-1},S_iK}(q,r) + p\, I_{\mu,\nu,C}(q,r) \leq \sum_{i\in I} \overline{p}_i^q \overline{F}_i(q,r) + p^q \overline{F}_C(q,r). \qquad \square$$

To define the L^q spectra for $l, m \in \mathcal{M}_1(\mathbb{R}^d)$, $A \subseteq \operatorname{supp} m$ and $q \in \mathbb{R}$, we set

$$\overline{\tau}_{m|A}(q) = \limsup_{r\to 0} \frac{\log \int_A m(B(x,r))^{q-1}\,dm(x)}{-\log r}, \qquad \underline{\tau}_{m|A}(q) = \liminf_{r\to 0} \frac{\log \int_A m(B(x,r))^{q-1}\,dm(x)}{-\log r},$$

$$\overline{\tau}_{l,m,A}(q) = \limsup_{r\to 0} \frac{\log \int_A l(B(x,r))^{q-1}\,dm(x)}{-\log r}, \qquad \underline{\tau}_{l,m,A}(q) = \liminf_{r\to 0} \frac{\log \int_A l(B(x,r))^{q-1}\,dm(x)}{-\log r}.$$

In particular, for $l = m = \mu$ and $A = K$, we obtain the *upper* and *lower* L^q *spectrum* of the measure μ:

$$\overline{\tau}_{\mu}(q) = \limsup_{r\to 0} \frac{\log \int_K \mu(B(x,r))^{q-1}\,d\mu(x)}{-\log r}, \qquad \underline{\tau}_{\mu}(q) = \liminf_{r\to 0} \frac{\log \int_K \mu(B(x,r))^{q-1}\,d\mu(x)}{-\log r}.$$

Now we prove the result which plays a crucial role for the lower estimate of the L^q spectra of the measure (6).

Proposition 3.5.

Let $q \in \mathbb{R}$ and $n \in \mathbb{N}$, and let μ be a probability measure. Let $A \subseteq \operatorname{supp}\mu$ and assume that

(a) *for all $q \geq 1$, $I_{\mu|A}(q,r) \geq \sum_{i=1}^{n} \underline{p}_i^q I_{\mu|A}(q, r/r_i)$,*

(b) *for all $q < 1$, $I_{\mu|A}(q,r) \geq \sum_{i=1}^{n} \underline{p}_i \overline{p}_i^{q-1} I_{\mu|A}(q, r/r_i)$.*

(a.1) *If $q \geq 1$ then let t be such that $\underline{\beta}_n(q) > t$. Then, there exists a constant $c_0 > 0$ such that the function $G\colon (0,\infty) \to \mathbb{R}$ defined by the formula $G(r) = c_0 r^{-t}$ satisfies $\sum_{i=1}^n \underline{p}_i^q G(r/r_i) \geq G(r)$ for all $r > 0$ and $I_{\mu|A}(q,r) \geq G(r)$ for all $r \in [r_{\min}, 1]$.*

(a.2) *We have $\underline{\tau}_{\mu|A}(q) \geq \underline{\beta}_n(q)$.*

(b.1) *If $q < 1$ then let t be such that $\underline{\beta}_n(q-1) > t$. Then, there exists a constant $c_0 > 0$ such that the function $G\colon (0,\infty) \to \mathbb{R}$ defined by the formula $G(r) = c_0 r^{-t}$ satisfies $\sum_{i=1}^n \underline{p}_i \overline{p}_i^{q-1} G(r/r_i) \geq G(r)$ for all $r > 0$ and $I_{\mu|A}(q,r) \geq G(r)$ for all $r \in [r_{\min}, 1]$.*

(b.2) *We have $\underline{\tau}_{\mu|A}(q) \geq \underline{\beta}_n(q-1)$.*

The proof is quite similar to the proof of [18, Theorem 5.2] and therefore we omit it here. The key to obtaining the upper estimate of the L^q spectra of the measure (6) is the next proposition.

Proposition 3.6.

Let μ and ν be probability measures, and let $(\mu_m)_{m\in\mathbb{N}}$ and $(\nu_m)_{m\in\mathbb{N}}$ be sequences of probability measures. Let K_m and C_m denote the supports of μ_m, ν_m, respectively, and let $n \in \mathbb{N}$. Assume that, for each $m \in \mathbb{N}$,

(a) *for all $q \geq 1$, $I_{\mu,\mu_m,K_m}(q,r) \leq \sum_{i=1}^{n} \overline{p}_i^q I_{\mu,\mu_m,K_m}(q, r/r_i) + p^q I_{\nu,\nu_m,C_m}(q,r)$,*

(b) *for all $q < 1$, $I_{\mu,\mu_m,K_m}(q,r) \leq \sum_{i=1}^{n} \overline{p}_i \underline{p}_i^{q-1} I_{\mu,\mu_m,K_m}(q, r/r_i) + p^q I_{\nu,\nu_m,C_m}(q,r)$.*

Then,

(a.1) *for all $q \geq 1$, $\overline{\tau}_{\mu,\mu_m,K_m}(q) \leq \max\big(\overline{\beta}_n(q), \overline{\tau}_{\nu,\nu_m,C_m}(q)\big)$, $m \in \mathbb{N}$,*

(b.1) *for all $q < 1$, $\overline{\tau}_{\mu,\mu_m,K_m}(q) \leq \max\big(\overline{\beta}_n(q-1), \overline{\tau}_{\nu,\nu_m,C_m}(q)\big)$, $m \in \mathbb{N}$.*

The proof is analogous to the proof of [19, Proposition 4.2] when it is applied for each $m \in \mathbb{N}$.

We are now in a position to state the main theorem of the paper. This theorem generalizes our previous main result in [18] by providing estimates for the L^q spectra of the generalized form of inhomogeneous self-similar measures (6). As the methods used in the work [19] cannot be applied to the case of infinitely many transformations S_i, we will provide the proof for the case in which the set of indexes I is infinite. We will observe during the proof that it is applicable if I is finite. This observation will be formulated later as a corollary.

Theorem 3.7.
Assume that the IOSC is satisfied and that the set of indexes I is infinite.

(a) *For all $q \geq 1$, we have*

$$\max\left(\underline{\beta}(q), \overline{\tau}_\nu(q)\right) \leq \overline{\tau}_\mu(q) \leq \max\left(\overline{\beta}(q), \overline{\tau}_\nu(q)\right), \qquad \max\left(\underline{\beta}(q), \underline{\tau}_\nu(q)\right) \leq \underline{\tau}_\mu(q).$$

(b) *For all $q < 1$, we have*

$$\max\left(\underline{\beta}(q-1), \overline{\tau}_\nu(q)\right) \leq \overline{\tau}_\mu(q) \leq \max\left(\overline{\beta}(q-1), \overline{\tau}_\nu(q)\right), \qquad \max\left(\underline{\beta}(q-1), \underline{\tau}_\nu(q)\right) \leq \underline{\tau}_\mu(q).$$

Proof. The proof goes using the technique developed in the proof of [18, Theorem 5.4]. We present the proof for $q \geqslant 1$, the case $q < 1$ is analogous.

Fix any $q \geqslant 1$ and $n \in \mathbb{N}$. Let $(T_m)_{m=1}^\infty : \mathbb{R}^d \to \mathbb{R}^d$ be a sequence of contracting similarities of contracting ratios $(t_m)_{m=1}^\infty$ such that $\lim_{m\to\infty} t_m = 1$, and $T_m(S_iK) \subset \operatorname{int} S_iK$ and $T_m(C) \subset \operatorname{int} C$. Define

$$K_m = \bigcup_{i=1}^n T_m(S_i(K)) \cup T_m(C). \tag{7}$$

We start by showing the lower estimate in (a). First observe that Proposition 3.4 implies $I_\mu(q,r) \geq \sum_{i=1}^n p_i^q \underline{F}_i(q,r)$. Hence,

$$I_{\mu|K_m}(q,r) \geq \sum_{i=1}^n p_i^q \underline{F}_{i,\mu,K_m}(q,r).$$

From conditions (I3) and (I4) of the IOSC, we conclude that, for every $m \in \mathbb{N}$, the sets $(S_1K_m, \dots, S_nK_m, C)$ are pairwise disjoint. Let

$$r_m = \min\left\{\min_{i\in\{1,\dots,n\}} \inf_{j\neq i} \operatorname{dist}(S_iK_m, S_jK), \min_{i\in\{1,\dots,n\}} \operatorname{dist}(S_iK_m, C)\right\}.$$

Then, for all $0 < r < r_m$,

$$I_{\mu|K_m}(q,r) \geq \sum_{i=1}^n p_i^q I_{\mu|K_m}\left(q, \frac{r}{r_i}\right)$$

because $\underline{J}_{i,\mu,K_m}(S_ix, r) = 0$ for $0 < r < r_m$. Hence, from Proposition 3.5, it follows that $\underline{\tau}_{\mu|K_m}(q) \geqslant \underline{\beta}_n(q)$. As $\left(\underline{\tau}_{\mu|K_m}(q)\right)_{m\in\mathbb{N}}$ is monotonic and tends to $\underline{\tau}_{\mu|K_{|n}}(q)$, we have $\underline{\tau}_{\mu|K_{|n}}(q) \geq \underline{\beta}_n(q)$, $n \in \mathbb{N}$. Furthermore, $\underline{\tau}_{\mu|K_{|n}}(q)$ is monotonic and tends to $\underline{\tau}_\mu(q)$, so

$$\overline{\tau}_\mu(q) \geqslant \underline{\tau}_\mu(q) = \lim_{n\to\infty} \underline{\tau}_{\mu|K_{|n}}(q) \geq \lim_{n\to\infty} \underline{\beta}_n(q) = \underline{\beta}(q).$$

To show that $\overline{\tau}_\mu(q) \geqslant \overline{\tau}_\nu(q)$, $\underline{\tau}_\mu(q) \geqslant \underline{\tau}_\nu(q)$, from Proposition 3.4, observe that $I_\mu(q,r) \geq p^q \underline{F}_C(q,r)$. Define $C_m = T_m(C)$. By the above, $I_\mu(q,r) \geq p^q \underline{F}_{C,\nu,C_m}(q,r)$. From condition (I4), for every $m \in \mathbb{N}$, we have $r_m = \inf_{i\in\mathbb{N}} \operatorname{dist}(S_iK, C_m) > 0$. Thus, for all $0 < r < r_m$, $I_\mu(q,r) \geq p^q I_{\nu|C_m}(q,r)$, as $\underline{J}_{C,\mu,C_m}(x,r) = 0$ for $0 < r < r_m$. Hence, $\overline{\tau}_\mu(q) \geq \overline{\tau}_{\nu|C_m}(q)$ and $\underline{\tau}_\mu(q) \geq \underline{\tau}_{\nu|C_m}(q)$. The sequences $\left(\overline{\tau}_{\nu|C_m}(q)\right)_{m\in\mathbb{N}}$ and $\left(\underline{\tau}_{\nu|C_m}(q)\right)_{m\in\mathbb{N}}$ are monotonic and converge, respectively, to $\overline{\tau}_\mu(q)$ and $\underline{\tau}_\mu(q)$, so

$$\overline{\tau}_\mu(q) \geq \overline{\tau}_\nu(q), \qquad \underline{\tau}_\mu(q) \geq \underline{\tau}_\nu(q).$$

The proof of the lower estimate in (a) is finished.

To establish the upper estimate, let K_m denote the set (7). Define the sequences of measures

$$\mu_m(A) = \mu(A \cap K_m), \qquad \nu_m(A) = \nu(A \cap C_m), \qquad \mu_{|n}(A) = \mu(A \cap K_{|n}).$$

Then, from the IOSC and from (6), we deduce that

$$\mu_m(A) \le \sum_{i=1}^{n} \overline{p}_i \mu_m \circ S_i^{-1}(A) + p\,\nu_m(A).$$

Note that $\operatorname{supp} \mu_m = K_m$. From the proofs of Lemma 3.3 and Proposition 3.4,

$$I_{\mu,\mu_m,K_m}(q,r) \le \sum_{i=1}^{n} \overline{p}_i^{\,q} \overline{F}_{i,\mu,\mu_m,K_m}(q,r) + p^q \overline{F}_{C,\nu,\nu_m,C_m}(q,r).$$

By (I3) and (I4), for every $m \in \mathbb{N}$, the sets $(S_1K_m, \dots, S_nK_m, C)$ are pairwise disjoint. Let

$$r_m = \min\left\{ \min_{i \in \{1,\dots,n\}} \inf_{j \ne i} \operatorname{dist}(S_iK_m, S_jK), \min_{i \in \{1,\dots,n\}} \operatorname{dist}(S_iK_m, C), \inf_{i \in \mathbb{N}} \operatorname{dist}(S_iK, C_m) \right\}.$$

Then, for all $0 < r < r_m$,

$$I_{\mu,\mu_m,K_m}(q,r) \le \sum_{i=1}^{n} \overline{p}_i^{\,q} I_{\mu,\mu_m,K_m}\left(q, \frac{r}{r_i}\right) + p^q I_{\nu,\nu_m,C_m}(q,r),$$

as $\overline{J}_{i,\mu,K_m}(S_ix, r) = \overline{J}_{C,\mu,C_m}(x,r) = 0$ for $0 < r < r_m$. Hence, from Proposition 3.6, $\overline{\tau}_{\mu,\mu_m,K_m}(q) \le \max\left(\overline{\beta}_n(q), \overline{\tau}_{\nu,\nu_m,C_m}(q)\right)$. As the sequences $\left(\overline{\tau}_{\mu,\mu_m,K_m}(q)\right)_{m\in\mathbb{N}}$ and $\left(\overline{\tau}_{\nu,\nu_m,C_m}(q)\right)_{m\in\mathbb{N}}$ are monotonic and converge, respectively, to $\overline{\tau}_{\mu,\mu_{|n},K_{|n}}(q)$ and $\overline{\tau}_\nu(q)$, we have

$$\overline{\tau}_{\mu,\mu_{|n},K_{|n}}(q) \le \max\left(\overline{\beta}_n(q), \overline{\tau}_\nu(q)\right), \qquad n \in \mathbb{N}.$$

Furthermore, the sequence $\left(\overline{\tau}_{\mu,\mu_{|n},K_{|n}}(q)\right)_{n\in\mathbb{N}}$ is also monotonic and converges to $\overline{\tau}_\mu(q)$. Hence, $\overline{\tau}_\mu(q) \le \max\left(\overline{\beta}(q), \overline{\tau}_\nu(q)\right)$. □

From the proof of Theorem 3.7 we immediately obtain the following two corollaries which generalize [18, Corollary 5.1] and [18, Corollary 5.2] that were related to the particular form of inhomogeneous self-similar measures (2). Corollary 3.10 provides a necessary condition for existence of the dimension for all $q \in \mathbb{R}$.

Corollary 3.8.

Assume that the IOSC is satisfied and that the set of indexes I is finite. Then, the assertion of Theorem 3.7 is holds.

Corollary 3.9.

Assume that the set of indexes I is finite and let K be a unique, non-empty and compact set given by (5). Moreover, assume that the sets $(S_1K, \dots, S_NK, C)$ are pairwise disjoint. Then, the assertion of Theorem 3.7 is holds.

Corollary 3.10.

Let us take $p_i(x) = p_i$ for all $i \in I$ in (6). If $\overline{\tau}_\nu(q) = \underline{\tau}_\nu(q)$ then $\overline{\tau}_\mu(q) = \underline{\tau}_\mu(q)$ for all $q \in \mathbb{R}$.

An interesting result related to phase transitions of inhomogeneous self-similar measures follows from Corollary 3.10 and it generalizes [19, Proposition 2.4] by extending its assertion to all $q \in \mathbb{R}$ and relaxing the assumed separability condition there. However, due to the breadth of this topic we will not discuss it here. For a discussion on phase transitions of homogeneous and inhomogeneous self-similar measures we refer the reader to [19, Section 2 (3)] and [2, 7]. To satisfy interest of the reader, we will elaborate on the assumption $I^+ = I$ in the following remarks.

Remark.

If $I^+ \subsetneq I$ then from the considerations conducted in this section it is relatively easy to deduce that in this case the functions $\underline{\beta}_I(q)$ and $\underline{\beta}_I(q-1)$ depend on the cardinality of the set I^+. If $I^+ = \emptyset$ then in Theorem 3.7 we have the lower estimate by the inhomogeneous term, because $p > 0$. The problem occurs for all $q < 1$, as the case of $I^+ \subsetneq I$ means that we cannot make upper estimate using $\underline{p}_i$ and hence using the function $\overline{\beta}_I(q-1)$. This is a separate problem for consideration.

Remark.

The assumption $I^+ = I$ does not imply that the values $\overline{\beta}(q-1)$ or $\underline{\beta}(q-1)$ will be finite for all $q < 1$. For example, if I is infinite, it may happen that we will have to adopt $\underline{\beta}(q-1) = \infty$ for all $q < 0$ (cf. [18, Example 4.2]). As a result, from Theorem 3.7 it follows that in this case $\underline{\tau}_\mu(q) = \overline{\tau}_\mu(q) = \infty$ for all $q < 0$.

Remark.

In practice, it is very convenient to assume that the probabilities satisfy the inequalities $ar_i \leq p_i(x) \leq br_i$, $i \in I$, for some constants $a, b > 0$. This assumption results in the fact that $I^+ = I$, and in the case in which I is infinite it also asserts the finiteness of values $\overline{\beta}(q-1)$ and $\underline{\beta}(q-1)$ for all $q < 1$.

4. The Rényi dimension

The Rényi dimensions are closely related to the L^q spectra. For $m \in \mathcal{M}_1(\mathbb{R}^d)$ and $q \in \mathbb{R} \setminus \{1\}$, we define the *upper* and *lower q-Renyi dimensions* of m by

$$\overline{D}_m(q) = \limsup_{r\to 0} \frac{1}{q-1} \frac{\log \int_{\operatorname{supp} m} m(B(x,r))^{q-1} dm(x)}{\log r}, \qquad \underline{D}_m(q) = \liminf_{r\to 0} \frac{1}{q-1} \frac{\log \int_{\operatorname{supp} m} m(B(x,r))^{q-1} dm(x)}{\log r}.$$

For more information of the Rényi dimension and its applications as a tool for analyzing various problems in information theory, we refer the reader to [23].

We immediately obtain the following result from Theorem 3.7. This result generalizes [19, Theorem 2.8] and [19, Corollary 2.9] by providing estimates of the Rényi dimension for the generalized inhomogeneous self-similar measures (6) under the IOSC assumption. As a result, it gives a partial answer to [19, Question 2.13].

Theorem 4.1.

Assume that the IOSC is satisfied. For all $q > 1$,

$$\overline{D}_\mu(q) \leq \min\left(\frac{\underline{\beta}_I(q)}{1-q}, \overline{D}_\nu(q)\right), \qquad \min\left(\frac{\overline{\beta}_I(q)}{1-q}, \underline{D}_\nu(q)\right) \leq \underline{D}_\mu(q) \leq \min\left(\frac{\underline{\beta}_I(q)}{1-q}, \underline{D}_\nu(q)\right).$$

For all $q < 1$,

$$\max\left(\frac{\underline{\beta}_I(q-1)}{1-q}, \overline{D}_\nu(q)\right) \leq \overline{D}_\mu(q) \leq \max\left(\frac{\overline{\beta}_I(q-1)}{1-q}, \overline{D}_\nu(q)\right), \qquad \max\left(\frac{\underline{\beta}_I(q-1)}{1-q}, \underline{D}_\nu(q)\right) \leq \underline{D}_\mu(q).$$

The proof is immediate from Theorem 3.7 and from the fact that for all $q > 1$, we have $\overline{D}_\mu(q) = \underline{\tau}_\mu(q)/(1-q)$ and $\underline{D}_\mu(q) = \overline{\tau}_\mu(q)/(1-q)$.

Remark.

It would be of interest to investigate if by using, e.g., methods from [19] and the approach from the proof of Theorem 3.7, we are able to answer [19, Question 2.13] in relation to the two limiting cases of the Réyni dimension: $q = 1$ and $q = \pm\infty$.

5. Applications and examples

We begin with formulating the following theorem, which will be a starting point for further discussion in this section. In this theorem, we obtain some non-trivial estimates of the L^q spectra of an inhomogeneous self-similar measure satisfying (6) without any separation conditions; in particular, we are not assuming that the conditions (I3) and (I4) are satisfied.

Theorem 5.1.

Let μ be an inhomogeneous self-similar measure (6). Assume that only conditions (I1) and (I2) of the IOSC are satisfied.

- *For all $q \geqslant 1$, we have $\max\big(\underline{\beta}_{I^+}(q), \underline{\tau}_\nu(q)\big) \leqslant \underline{\tau}_\mu(q)$.*
- *If $I^+ = I$ then for all $q < 1$, we have $\overline{\tau}_\mu(q) \leqslant \max\big(\overline{\beta}(q-1), \overline{\tau}_\nu(q)\big)$.*

Proof. The proof follows from the idea of the proof of Theorem 3.7 and from the observations that for all $q \geqslant 1$ and $r > 0$,

$$\mu(B(x,r))^{q-1} \geqslant \begin{cases} \big(\underline{p}_i \mu\big(S_i^{-1}(B(x,r) \cap S_i K)\big) + \underline{J}_i(x,r)\big)^{q-1} & \text{for} \quad x \in S_i K, \\ \big(p\,\nu(B(x,r) \cap C) + \underline{J}_C(x,r)\big)^{q-1} & \text{for} \quad x \in C, \end{cases}$$
$$\geqslant \begin{cases} \left(\underline{p}_i \mu\left(B\left(S_i^{-1}x, \dfrac{r}{r_i}\right)\right)\right)^{q-1} & \text{for} \quad x \in S_i K, \\ \big(p\,\nu(B(x,r) \cap C)\big)^{q-1} & \text{for} \quad x \in C, \end{cases}$$

and for all $q < 1$ and $r > 0$,

$$\mu(B(x,r))^{q-1} \leqslant \begin{cases} \big(\underline{p}_i \mu\big(S_i^{-1}(B(x,r) \cap S_i K)\big) + \underline{J}_i(x,r)\big)^{q-1} & \text{for} \quad x \in S_i K, \\ \big(p\,\nu(B(x,r) \cap C) + \underline{J}_C(x,r)\big)^{q-1} & \text{for} \quad x \in C, \end{cases}$$
$$\leqslant \begin{cases} \left(\underline{p}_i \mu\left(B\left(S_i^{-1}x, \dfrac{r}{r_i}\right)\right)\right)^{q-1} & \text{for} \quad x \in S_i K, \\ \big(p\,\nu(B(x,r) \cap C)\big)^{q-1} & \text{for} \quad x \in C. \end{cases} \qquad \square$$

Arbeiter and Patzschke [1] in 1996 computed the L^q spectra of homogeneous self-similar measures (1) satisfying the OSC (see also [15]). Here, from Theorem 5.1 we obtain some non-trivial estimates for more general form of such measures (in this regard see also [14]) without assuming any separation conditions; in particular, we are not assuming the OSC.

Corollary 5.2.

Let μ be a homogeneous self-similar measure of the form

$$\mu(A) = \sum_{i \in I} \int_{S_i^{-1}(A)} p_i(x)\, d\mu(x),$$

where the set of indexes I is at most countable, $(S_i)_{i\in I}\colon \mathbb{R}^d \to \mathbb{R}^d$ are contracting similarities and $(p_i(x))_{i\in I}\colon \mathbb{R}^d \to [0,1]$ is a place dependent probability vector. Assume only that there exists a non-empty and bounded open set U such that $S_i(U) \subseteq U$ for all $i \in I$. For all $q \geqslant 1$, we have $\underline{\beta}_{I^+}(q) \leqslant \underline{\tau}_\mu(q)$. If $I^+ = I$ then for all $q < 1$, we have $\overline{\tau}_\mu(q) \leqslant \overline{\beta}(q-1)$.

In turn, the following theorem, together with its corollary, provide a systematic approach to obtaining non-trivial lower bounds for the L^q spectra of self-similar measures not satisfying the IOSC or OSC.

Theorem 5.3.

Let μ be an inhomogeneous self-similar measure of the form

$$\mu(A) = \sum_{i\in I} \int_{S_i^{-1}(A)} p_i(x)\, d\mu(x) + \sum_{j\in J} p_i \mu \circ S_j^{-1}(A) + p\, \nu(A),$$

where the sets of indexes I and J are at most countable, $(S_i)_{i\in I}, (S_j)_{j\in I} \colon \mathbb{R}^d \to \mathbb{R}^d$ are contracting similarities, $\big((p_i(x))_{i\in I}, (p_j)_{j\in J}, p\big) \colon \mathbb{R}^d \to [0,1]$ is a partially place dependent probability vector, and ν is a probability measure on $\mathbb{R}^d$ with compact support C. Assume that only the list $\big((S_i)_{i\in I}, C\big)$ satisfies the IOSC with U and

- *for all $j \in J$, we have $S_j(U) \subseteq U$,*
- *for all $i \in I$, we have $S_i(U) \cap \bigcup_{j\in J} S_j(U) = \emptyset$.*

Then:

- *For all $q \geqslant 1$, we have $\max\big(\underline{\beta}_{I^+}(q), \underline{\tau}_\nu(q)\big) \leqslant \underline{\tau}_\mu(q)$.*
- *For all $q < 1$, we have $\max\big(\underline{\beta}_{I^+}(q-1), \underline{\tau}_{\nu|\underline{C}}(q)\big) \leqslant \underline{\tau}_\mu(q)$, where $\underline{C} \subseteq C$ is such that $\underline{C} \cap \bigcup_{j\in J} S_j(U) = \emptyset$.*
- *In particular, if $C \cap \bigcup_{j\in J} S_j(U) = \emptyset$ then $\max\big(\underline{\beta}_{I^+}(q-1), \underline{\tau}_\nu(q)\big) \leqslant \underline{\tau}_\mu(q)$. If $C \subseteq \operatorname{cl} \bigcup_{j\in J} S_j(U)$ then $\underline{\beta}_{I^+}(q-1) \leqslant \underline{\tau}_\mu(q)$.*

Proof. Define the probability ρ and the probability measure η by

$$\rho = \sum_{j\in J} p_j + p, \qquad \eta = \frac{1}{\rho}\left(\sum_{j\in J} p_i \mu \circ S_j^{-1} + p\,\nu\right).$$

Then, we can write μ as the following inhomogeneous self-similar measure:

$$\mu(A) = \sum_{i\in I} \int_{S_i^{-1}(A)} p_i(x)\, d\mu(x) + \rho\eta(A).$$

Observe first that the inequality $\underline{\tau}_\nu(q) \leqslant \underline{\tau}_\eta(q)$ for all $q \geqslant 1$ is obvious. In turn, if $\underline{C}$ is a subset of C such that $\underline{C} \cap \bigcup_{j\in J} S_j(U) = \emptyset$ then using the same argument as in the proof of Theorem 3.7 we obtain the inequality $\underline{\tau}_{\nu|\underline{C}}(q) \leqslant \underline{\tau}_\eta(q)$ for all $q < 1$. Let us denote $D = \bigcup_{j\in J} \operatorname{supp} \mu \circ S_j^{-1} \cup C$. By assumptions the list $\big((S_i)_{i\in I}, D\big)$ satisfies the IOSC with U and the assertion follows at once from Theorem 3.7. □

From the proof of Theorem 5.3, we immediately obtain the following corollary.

Corollary 5.4.

Let μ be a homogeneous self-similar measure of the form

$$\mu(A) = \sum_{i\in I} \int_{S_i^{-1}(A)} p_i(x)\, d\mu(x) + \sum_{j\in J} p_i \mu \circ S_j^{-1}(A),$$

where the sets of indexes I and J are at most countable, $(S_i)_{i\in I}, (S_j)_{j\in I} \colon \mathbb{R}^d \to \mathbb{R}^d$ are contracting similarities and $\big((p_i(x))_{i\in I}, (p_j)_{j\in J}\big) \colon \mathbb{R}^d \to [0,1]$ is a partially place dependent probability vector. Assume that only the list $(S_i)_{i\in I}$ satisfies the OSC with U and

- *for all $j \in J$, we have $S_j(U) \subseteq U$,*
- *for all $i \in I$, we have $S_i(U) \cap \bigcup_{j\in J} S_j(U) = \emptyset$.*

Then for all $q \geqslant 1$, $\underline{\beta}_{I^+}(q) \leqslant \underline{\tau}_\mu(q)$ and for all $q < 1$, $\underline{\beta}_{I^+}(q-1) \leqslant \underline{\tau}_\mu(q)$.

Remark.
From Theorems 5.1 and 5.3, Corollaries 5.2 and 5.4, and from the facts that $\overline{D}_\mu(q) = \underline{\tau}_\mu(q)/(1-q)$, $\underline{D}_\mu(q) = \overline{\tau}_\mu(q)/(1-q)$ for all $q > 1$ and $\overline{D}_\mu(q) = \overline{\tau}_\mu(q)/(1-q)$, $\underline{D}_\mu(q) = \underline{\tau}_\mu(q)/(1-q)$ for all $q < 1$, we immediately obtain some non-trivial estimates for the Rényi dimension of μ.

We will illustrate now Corollary 5.2 and Theorem 5.3 by the following examples.

Example 5.5 (the so-called $(2,3)$-Bernoulli convolution, see [11]).
Let us consider the following homogeneous self-similar measure:

$$\mu(A) = \sum_{i=1}^{3} \int_{S_i^{-1}(A)} p_i(x)\; d\mu(x),$$

where the maps $S_1, S_2, S_3\colon [0,1] \to \mathbb{R}$ are defined by $S_i(x) = x/2 + (i-1)/4$ and $p_1(x)\colon [0,1] \to [0,1] = (x+1)/3$, $p_2(x)\colon [0,1] \to [0,1] = (x+1)/9$, $p_3(x)\colon [0,1] \to [0,1] = 1 - (4x+4)/9$ (the average value of $\overline{p}_i$ and $\underline{p}_i$ is 1/3). It is clear that this homogeneous self-similar measure does not satisfy the OSC and the overlaps are quite severe. However, from Corollary 5.2 we obtain the following estimates for the L^q spectra of μ:

$$-2,88 \lessapprox \underline{\tau}_\mu(2), \qquad \overline{\tau}_\mu(-2) \lessapprox 9,19.$$

Example 5.6.
Let us consider the following inhomogeneous self-similar measure:

$$\mu(A) = \sum_{i=1}^{2} \int_{S_i^{-1}(A)} p_i(x)\, d\mu(x) + \sum_{j=3}^{4} p_i \mu \circ S_j^{-1}(A) + p\,\nu(A),$$

where the maps $S_1, S_2, S_3, S_4 : [0,1] \to \mathbb{R}$ are defined by $S_1(x) = x/3 + 1/3$, $S_2(x) = x/3 + 2/3$, $S_3(x) = x/4$, $S_4(x) = x/4$ and $p_1(x)\colon [0,1] \to [0,1] = (x+1)/6$, $p_2(x)\colon [0,1] \to [0,1] = 5/12 - (x+1)/6$, and $(p_3, p_4, p) = (1/4, 1/4, 1/12)$. Let $\nu = l_{1|[0,1/3]}$. It is clear that this inhomogeneous self-similar measure does not satisfy the IOSC and, consequently, the L^q spectra and Rényi dimension of μ can not be calculated by using the methods developed in Sections 3 and 4. However, observe that for $I = \{1,2\}$, $J = \{3,4\}$ and $U = (0,1)$ the assumptions of Theorem 5.3 are satisfied and hence we can provide the following estimates for the L^q spectra and Rényi dimension of μ. Namely, for, e.g., $q = 2$, we have

$$-1 \le \underline{\tau}_\mu(2), \qquad \overline{D}_\mu(2) \le 1.$$

To conclude these examples, it is also worth mentioning that calculating the Hausdorff dimension of the attractor of, for instance, $S_1(x) = x/3 + 1/3$, $S_2(x) = x/3 + 2/3$, $S_3(x) = x/4$, $S_4(x) = x/4$ is far from being simple (see [3]). This stems from the fact that the OSC is not satisfied.

In the spirit of the results presented in this section, we will give now some further application.

5.1. Non-linear self-similar measures

Let us consider probability measures on $\mathbb{R}^d$ that satisfy a non-linear self-similar identity involving convolutions

$$\mu = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1} + \sum_{j=1}^{M} q_j (\mu * \mu) \circ T_j^{-1}, \tag{8}$$

where $S_i, T_j\colon \mathbb{R}^d \to \mathbb{R}^d$ are contractive similarities and the contraction ratios of T_j are less than 1/2 (in order to counteract the doubling of support in the convolution product $\mu * \mu$), and $(p_1, \dots, p_N, q_1, \dots, q_M)$ is a probability vector with positive

p_i and q_j. These measures were studied by Glickenstein and Strichartz [12] as a generalization of homogeneous self-similar measures and in more general form they were also a subject of investigation by Olsen and Snigireva [20]. We now consider an even more general form of these measures. Namely, let us consider the following form of (8):

$$\mu(A) = \sum_{i\in I}\int_{S_i^{-1}(A)} p_i(x)\, d\mu(x) + \sum_{j\in J} q_j(\underbrace{\mu * \cdots * \mu}_{k_j \text{ times}})\circ T_j^{-1}(A), \tag{9}$$

where $(k_j)_{j\in J}$ are positive integers with $k_j \geq 2$, I and J are at most countable set of indexes, and $\big((p_i(x))_{i\in I}, (q_j)_{j\in J}\big)\colon \mathbb{R}^d \to [0,1]$ is a partially place dependent probability vector, i.e., $\sum_{i\in I} p_i(x) = 1 - \sum_{j\in J} q_j$, $x \in \mathbb{R}^d$. If $\sup_i r_i \sum_{i\in I} \overline{p}_i + (1/2)\sum_{j\in J} q_j < 1$ then it is a simple exercise (following the proof in [13] or by an argument similar to the one in [20, Proposition 1.1]) to show that under these hypotheses there is a unique probability measure satisfying (9).

As an application of our results, we will show that μ satisfying (9) can be studied as a generalized inhomogeneous self-similar measure. Thus, we can apply Theorem 3.7 to obtain non-trivial lower bounds for the L^q spectra and as a consequence to obtain some non-trivial bounds for the Rényi dimension of μ. Namely, a probability measure μ on $\mathbb{R}^d$ satisfying the non-linear self-similar identity (9) can be viewed as a generalized inhomogeneous self-similar measure as follows: define $p \in (0,1)$ and the probability measure ν by

$$p = \sum_{j\in J} q_j, \qquad \nu = \frac{1}{p}\sum_{j\in J} q_j(\underbrace{\mu * \cdots * \mu}_{k_j \text{ times}})\circ T_j^{-1}.$$

Then, clearly, μ satisfying (9) can be written as

$$\mu(A) = \sum_{i\in I}\int_{S_i^{-1}(A)} p_i(x)\, d\mu(x) + p\,\nu(A),$$

i.e., μ is the generalized inhomogeneous self-similar measure associated with the list $\big((S_i)_{i\in I}, (p_i(x))_{i\in I}, p, \nu\big)$. Hence, we can now formulate the following result as a consequence of Theorem 3.7.

Theorem 5.7.

Let μ be a non-linear self-similar measure satisfying the identity (9). Assume that the list $(S_i)_{i\in I}$ satisfies the OSC with U and

- *for all $j \in J$, we have $T_j\,(\underbrace{\overline{U} + \cdots + \overline{U}}_{k_j\ \ times}) \subseteq \overline{U}$,*
- *for all $i \in I$, we have $S_iU \cap \bigcup_{j\in J} T_j(\underbrace{\overline{U} + \cdots + \overline{U}}_{k_j\ \ times}) = \emptyset$.*

Then for all $q \geq 1$, $\underline{\beta}_{I^+}(q) \leq \underline{\tau}_\mu(q)$ and for all $q < 1$, $\underline{\beta}_{I^+}(q-1) \leq \underline{\tau}_\mu(q)$.

Proof. It is sufficient to show that the list $\big((S_i)_{i\in I}, \operatorname{supp}\nu\big)$ satisfies the IOSC with U. Conditions (I2) and (I3) immediately follow by assumption. Conditions (I1) and (I4) can be shown in a similar way as in the proof of [21, Theorem 2.8] combined with the proof of [18, Theorem 3.1], therefore, are left to the reader. □

Remark.

From Theorem 5.7 and from the facts that $\overline{D}_\mu(q) = \underline{\tau}_\mu(q)/(1-q)$, $\underline{D}_\mu(q) = \overline{\tau}_\mu(q)/(1-q)$ for all $q > 1$ and $\overline{D}_\mu(q) = \overline{\tau}_\mu(q)/(1-q)$, $\underline{D}_\mu(q) = \underline{\tau}_\mu(q)/(1-q)$ for all $q < 1$, we immediately obtain some non-trivial estimates for the Rényi dimension of μ.

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