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Shape-invariant hypergeometric type operators with application to quantum mechanics

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Abstract: A hypergeometric type equation satisfying certain conditions defines either a finite or an infinite system of orthogonal polynomials. The associated special functions are eigenfunctions of some shape-invariant operators. These operators can be analysed together and the mathematical formalism we use can be extended in order to define other shape-invariant operators. All the shape-invariant operators considered are directly related to Schrödinger-type equations.

Keywords: Orthogonal polynomials, associated special functions, shape-invariant operators, raising and lowering operators, Schrödinger equation
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1 Introduction

Many problems in quantum mechanics and mathematical physics lead to equations of the type

$$\sigma(s)y''(s) + \tau(s)y'(s) + \lambda y(s) = 0 \tag{1}$$

where $\sigma(s)$ and $\tau(s)$ are polynomials of at most second and first degree, respectively, and λ is a constant. These equations are usually called *equations of hypergeometric type* [1], and each of them can be reduced to the self-adjoint form

$$[\sigma(s)\varrho(s)y'(s)]' + \lambda\varrho(s)y(s) = 0 \tag{2}$$

by choosing a function ϱ such that $(\sigma\varrho)' = \tau\varrho$.

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Equation (1) is usually considered on an interval (a,b) chosen such that

$$\begin{aligned} \sigma(s) > 0 \qquad & \text{for all} \quad s \in (a,b) \\ \varrho(s) > 0 \qquad & \text{for all} \quad s \in (a,b) \\ \lim_{s\to a} \sigma(s)\varrho(s) = \lim_{s\to b} \sigma(s)\varrho(s) = 0. & \end{aligned} \tag{3}$$

Since the form of Equation (1) is invariant under a change of variable $s \mapsto cs + d$, it is sufficient to analyse the cases presented in Table 1. Some restrictions are imposed on α, β in order that the interval (a,b) exist. Equation (1) defines an infinite sequence of orthogonal polynomials in the case $\sigma(s) \in \{1,\ s,\ 1-s^2\}$, and a finite one in the case $\sigma(s) \in \{s^2-1,\ s^2,\ s^2+1\}$.

Table 1 The main cases.

$\sigma(s)$	$\tau(s)$	$\varrho(s)$	α, β	(a,b)
1	$\alpha s+\beta$	$\mathrm{e}^{\alpha s^2/2+\beta s}$	$\alpha<0$	$\mathbb{R}$
s	$\alpha s+\beta$	$s^{\beta-1}\mathrm{e}^{\alpha s}$	$\alpha<0,\ \beta>0$	$(0,\infty)$
$1-s^2$	$\alpha s+\beta$	$(1+s)^{-(\alpha-\beta)/2-1}(1-s)^{-(\alpha+\beta)/2-1}$	$\alpha<\beta<-\alpha$	$(-1,1)$
s^2-1	$\alpha s+\beta$	$(s+1)^{(\alpha-\beta)/2-1}(s-1)^{(\alpha+\beta)/2-1}$	$-\beta<\alpha<0$	$(1,\infty)$
s^2	$\alpha s+\beta$	$s^{\alpha-2}\mathrm{e}^{-\beta/s}$	$\alpha<0,\ \beta>0$	$(0,\infty)$
s^2+1	$\alpha s+\beta$	$(1+s^2)^{\alpha/2-1}\mathrm{e}^{\beta \arctan s}$	$\alpha<0$	$\mathbb{R}$

The literature discussing special function theory and its application to mathematical and theoretical physics is vast, and there are a multitude of different conventions concerning the definition of functions. A unified approach is not possible without a unified definition for the associated special functions. In this paper, we define them as

$$\Phi_{l,m}(s) = \left(\sqrt{\sigma(s)}\right)^m \frac{\mathrm{d}^m}{\mathrm{d}s^m}\Phi_l(s), \tag{4}$$

where Φ_l are the orthogonal polynomials defined by Equation (1). Table 1 allows one to pass in each case from our parameters α, β to the parameters used in a different approach.

In [2, 3], we presented a systematic study of the Schrödinger equations which are exactly solvable in terms of associated special functions. In the present paper, based on the factorization method [4, 5] and certain results of Jafarizadeh and Fakhri [6], we extend our unified formalism by adding other shape-invariant operators.

2 Orthogonal polynomials

Let $\tau(s) = \alpha s + \beta$ be a fixed polynomial, and let

$$\lambda_l = -\frac{\sigma''(s)}{2}l(l-1) - \tau'(s)l = -\frac{\sigma''}{2}l(l-1) - \alpha\, l \tag{5}$$

for any $l \in \mathbb{N}$. It is well-known [1] that for $\lambda = \lambda_l$, Equation (1) admits a polynomial solution $\Phi_l = \Phi_l^{(\alpha,\beta)}$ of at most l degree:

$$\sigma(s)\Phi_l'' + \tau(s)\Phi_l' + \lambda_l\Phi_l = 0. \tag{6}$$

If the degree of the polynomial Φ_l is l, then it satisfies the Rodrigues formula [1]

$$\Phi_l(s) = \frac{B_l}{\varrho(s)}\frac{\mathrm{d}^l}{\mathrm{d}s^l}[\sigma^l(s)\varrho(s)] \tag{7}$$

where B_l is a constant. Based on the relation

$$\begin{aligned} &\{\ \delta \in \mathbb{R}\ |\ \lim_{s\to a}\sigma(s)\varrho(s)s^\delta = \lim_{s\to b}\sigma(s)\varrho(s)s^\delta = 0\ \} \\ &= \begin{cases} [0,\infty) & \text{if } \sigma(s) \in \{1,\ s,\ 1-s^2\} \\ [0,-\alpha) & \text{if } \sigma(s) \in \{s^2-1,\ s^2,\ s^2+1\} \end{cases} \end{aligned} \tag{8}$$

one can prove [1, 3] that the system of polynomials $\{\Phi_l \mid l < \Lambda\}$, where

$$\Lambda = \begin{cases} \infty & \text{for } \sigma(s) \in \{1,\ s,\ 1-s^2\} \\ \frac{1-\alpha}{2} & \text{for } \sigma(s) \in \{s^2-1,\ s^2,\ s^2+1\} \end{cases} \tag{9}$$

is orthogonal with weight function $\varrho(s)$ on (a,b). This means that Equation (1) defines an infinite sequence of orthogonal polynomials

$$\Phi_0,\quad \Phi_1,\quad \Phi_2,\ \ldots$$

in the case $\sigma(s) \in \{1,\ s,\ 1-s^2\}$, and a finite one

$$\Phi_0,\quad \Phi_1,\quad \ldots,\quad \Phi_L$$

with $L = \max\{l \in \mathbb{N} \mid l < (1-\alpha)/2\}$ in the case $\sigma(s) \in \{s^2-1,\ s^2,\ s^2+1\}$.

The polynomials $\Phi_l^{(\alpha,\beta)}$ can be expressed (up to a multiplicative constant) in terms of the classical orthogonal polynomials as

$$\Phi_l^{(\alpha,\beta)}(s) = \begin{cases} \mathbf{H}_l\left(\sqrt{\frac{-\alpha}{2}}\,s - \frac{\beta}{\sqrt{-2\alpha}}\right) & \text{in the case} \quad \sigma(s) = 1 \\ \mathbf{L}_l^{\beta-1}(-\alpha s) & \text{in the case} \quad \sigma(s) = s \\ \mathbf{P}_l^{(-(\alpha+\beta)/2-1,\ (-\alpha+\beta)/2-1)}(s) & \text{in the case} \quad \sigma(s) = 1-s^2 \\ \mathbf{P}_l^{((\alpha-\beta)/2-1,\ (\alpha+\beta)/2-1)}(-s) & \text{in the case} \quad \sigma(s) = s^2-1 \\ \left(\frac{s}{\beta}\right)^l \mathbf{L}_l^{1-\alpha-2l}\left(\frac{\beta}{s}\right) & \text{in the case} \quad \sigma(s) = s^2 \\ \mathrm{i}^l\mathbf{P}_l^{((\alpha+\mathrm{i}\beta)/2-1,\ (\alpha-\mathrm{i}\beta)/2-1)}(\mathrm{i}s) & \text{in the case} \quad \sigma(s) = s^2+1 \end{cases} \tag{10}$$

where $\mathbf{H}_l$, $\mathbf{L}_l^p$ and $\mathbf{P}_l^{(p,q)}$ are the Hermite, Laguerre and Jacobi polynomials, respectively. Relation (10) does not have a very simple form. In certain cases we have to consider the classical polynomials outside the interval where they are orthogonal or for complex values of parameters.

3 Associated special functions. Shape-invariant operators

Let $l \in \mathbb{N}$, $l < \Lambda$, and let $m \in \{0, 1, ..., l\}$. The functions

$$\Phi_{l,m}(s) = \kappa^m(s) \frac{\mathrm{d}^m}{\mathrm{d}s^m} \Phi_l(s) \qquad \text{where} \qquad \kappa(s) = \sqrt{\sigma(s)} \tag{11}$$

are called the *associated special functions*. If we differentiate (6) m times and then multiply the relation we obtain by $\kappa^m(s)$, we get the equation

$$H_m \Phi_{l,m} = \lambda_l \Phi_{l,m} \tag{12}$$

where H_m is the differential operator

$$\begin{aligned} H_m = & -\sigma(s)\tfrac{d^2}{ds^2} - \tau(s)\tfrac{d}{ds} + \tfrac{m(m-2)}{4}\tfrac{(\sigma'(s))^2}{\sigma(s)} \\ & + \tfrac{m\tau(s)}{2}\tfrac{\sigma'(s)}{\sigma(s)} - \tfrac{1}{2}m(m-2)\sigma''(s) - m\tau'(s). \end{aligned} \tag{13}$$

For each $m < \Lambda$, the special functions $\Phi_{l,m}$ with $m \le l < \Lambda$ are orthogonal with respect to the scalar product

$$\langle f, g \rangle = \int_a^b \overline{f(s)}\, g(s) \varrho(s) ds \tag{14}$$

and the functions corresponding to consecutive values of m are related through the raising/lowering operators [2, 3]

$$\begin{aligned} A_m &= \kappa(s)\tfrac{d}{ds} - m\kappa'(s) \\ A_m^+ &= -\kappa(s)\tfrac{d}{ds} - \tfrac{\tau(s)}{\kappa(s)} - (m-1)\kappa'(s) \end{aligned} \tag{15}$$

namely,

$$\begin{aligned} A_m \Phi_{l,m} &= \begin{cases} 0 & \text{for } l = m \\ \Phi_{l,m+1} & \text{for } m < l < \Lambda \end{cases} \\ A_m^+ \Phi_{l,m+1} &= (\lambda_l - \lambda_m)\Phi_{l,m} \quad \text{for} \quad 0 \le m < l < \Lambda. \end{aligned} \tag{16}$$

Up to a multiplicative constant

$$\Phi_{l,m}(s) = \begin{cases} \kappa^l(s) & \text{for} \quad m = l \\ \frac{A_m^+}{\lambda_l - \lambda_m} \frac{A_{m+1}^+}{\lambda_l - \lambda_{m+1}} \cdots \frac{A_{l-1}^+}{\lambda_l - \lambda_{l-1}} \kappa^l(s) & \text{for} \quad m < l \end{cases} \tag{17}$$

and the operators H_m are shape-invariant [2, 3]

$$\begin{array}{ll} H_m - \lambda_m = A_m^+ A_m & A_m H_m = H_{m+1} A_m \\ H_{m+1} - \lambda_m = A_m A_m^+ & H_m A_m^+ = A_m^+ H_{m+1}. \end{array} \tag{18}$$

The functions

$$\phi_{l,m} = \Phi_{l,m}/||\Phi_{l,m}|| \tag{19}$$

where $||f|| = \sqrt{\langle f, f\rangle}$ are the *normalized associated special functions*. Since [2, 3]

$$||\Phi_{l,m+1}|| = \sqrt{\lambda_l - \lambda_m}\, ||\Phi_{l,m}||, \tag{20}$$

they satisfy the relations

$$\begin{array}{l} A_m\ \phi_{l,m} = \begin{cases} 0 & \text{for } l = m \\ \sqrt{\lambda_l - \lambda_m}\ \phi_{l,m+1} & \text{for } m < l < \Lambda \end{cases} \\ A_m^+\ \phi_{l,m+1} = \sqrt{\lambda_l - \lambda_m}\ \phi_{l,m} \quad \text{for } 0 \le m < l < \Lambda \\ \phi_{l,m} = \frac{A_m^+}{\sqrt{\lambda_l - \lambda_m}} \frac{A_{m+1}^+}{\sqrt{\lambda_l - \lambda_{m+1}}} \ldots \frac{A_{l-1}^+}{\sqrt{\lambda_l - \lambda_{l-1}}} \phi_{l,l}. \end{array} \tag{21}$$

4 Application to Schrödinger-type equations

It is well-known [5] that the equations $H_m \Phi_{l,m} = \lambda_l \Phi_{l,m}$ are directly related to certain Schrödinger-type equations. If $\eta : (a,b) \longrightarrow \mathbb{R}$ is a twice differentiable function with $\eta(s) \neq 0$ for any $s \in (a,b)$ and if $(a,b) \longrightarrow (a',b') : s \mapsto x = x(s)$ is a differentiable bijective mapping, then

$$\begin{array}{r} \{[\eta(s)]^{-1} H_m\, \eta(s)\}|_{s=s(x)} = -\sigma(s(x))\, \xi^2(s(x)) \frac{d^2}{dx^2} - [\sigma(s(x))\xi'(s(x)) \\ + 2\sigma(s(x))\, \xi(s(x))\, \frac{\eta'(s(x))}{\eta(s(x))} + \tau(s(x))\xi(s(x))] \frac{d}{dx} + V_m(s(x)) \end{array}$$

where $\xi(s) = dx/ds$, the function $s(x)$ is the inverse of $(a,b) \longrightarrow (a',b') : s \mapsto x(s)$ and

$$\begin{array}{r} V_m(s) = \frac{m(m-2)}{4} \frac{(\sigma'(s))^2}{\sigma(s)} + \frac{m\tau(s)}{2} \frac{\sigma'(s)}{\sigma(s)} - \frac{1}{2} m(m-2)\sigma''(s) \\ -m\tau'(s) - \sigma(s) \frac{\eta''(s)}{\eta(s)} - \tau(s) \frac{\eta'(s)}{\eta(s)}. \end{array} \tag{22}$$

The operator

$$\mathcal{H}_m = \left\{[\eta(s)]^{-1} H_m\, \eta(s)\right\}\Big|_{s=s(x)} \tag{23}$$

becomes a Schrödinger-type operator when $\xi(s)$ and $\eta(s)$ satisfy the conditions

$$\sigma(s)\, \xi^2(s) = 1 \qquad \sigma(s)\xi'(s) + 2\sigma(s)\, \xi(s)\, \frac{\eta'(s)}{\eta(s)} + \tau(s)\xi(s) = 0 \tag{24}$$

which lead to

$$\xi(s) = \pm\frac{1}{\kappa(s)} \qquad \eta(s) = \frac{1}{\sqrt{\kappa(s)\,\varrho(s)}} \tag{25}$$

up to a multiplicative constant. In this case (the only considered in what follows)

$$\mathcal{H}_m = \left\{[\kappa(s)\varrho(s)]^{1/2} H_m\ [\kappa(s)\varrho(s)]^{-1/2}\right\}\Big|_{s=s(x)} = -\frac{d^2}{dx^2} + \mathcal{V}_m(x) \tag{26}$$

where

$$\mathcal{V}_m(x) = V(s(x)). \tag{27}$$

The functions

$$\Psi_{l,m}(x) = \sqrt{\kappa(s(x))\,\varrho(s(x))}\,\Phi_{l,m}(s(x)). \tag{28}$$

with $m \leq l < \Lambda$ are orthogonal eigenfunctions of $\mathcal{H}_m$:

$$\begin{aligned}(\mathcal{H}_m\Psi_{l,m})(x) &= \sqrt{\kappa(s(x))\,\varrho(s(x))}\,(H_m|_{s=s(x)})\,\Phi_{l,m}(s(x))\\ &= \sqrt{\kappa(s(x))\,\varrho(s(x))}\,(H_m\Phi_{l,m})(s(x)) = \lambda_l\Psi_{l,m}(x)\end{aligned}$$

$$\begin{aligned}\int_{a'}^{b'}\overline{\Psi}_{l,m}(x)\Psi_{k,m}(x)dx &= \int_a^b\overline{\Phi}_{l,m}(s(x))\Phi_{k,m}(s(x))\varrho(s(x))\ \left|\tfrac{ds}{dx}\right|\ dx\\ &= \int_a^b\overline{\Phi}_{l,m}(s)\Phi_{k,m}(s)\varrho(s)ds = 0\end{aligned}$$

for $k \neq l$. They satisfy the relations

$$\begin{aligned}\mathcal{A}_m\Psi_{l,m}(x) &= \begin{cases}0 & \text{for } l=m\\ \Psi_{l,m+1} & \text{for } m<l<\Lambda\end{cases}\\ \mathcal{A}_m^+\Psi_{l,m+1}(x) &= (\lambda_l-\lambda_m)\Psi_{l,m}(x)\end{aligned} \tag{29}$$

where

$$\begin{aligned}\mathcal{A}_m &= \left\{[\kappa(s)\varrho(s)]^{1/2}A_m\ [\kappa(s)\varrho(s)]^{-1/2}\right\}\Big|_{s=s(x)}\\ \mathcal{A}_m^+ &= \left\{[\kappa(s)\varrho(s)]^{1/2}A_m^+\ [\kappa(s)\varrho(s)]^{-1/2}\right\}\Big|_{s=s(x)}\end{aligned} \tag{30}$$

are the operators corresponding to A_m and A_m^+. The first relation follows from (16) and

$$\begin{aligned}(\mathcal{A}_m\Psi_{l,m})(x) &= \sqrt{\kappa(s(x))\,\varrho(s(x))}\,(A_m|_{s=s(x)})\,\Phi_{l,m}(s(x))\\ &= \sqrt{\kappa(s(x))\,\varrho(s(x))}\,(A_m\Phi_{l,m})(s(x)).\end{aligned}$$

Up to a multiplicative constant,

$$\Psi_{l,m}(x) = \begin{cases}\sqrt{\kappa(s(x))\,\varrho(s(x))}\,\kappa^l(s(x)) & \text{for} \quad m=l\\ \frac{\mathcal{A}_m^+}{\lambda_l-\lambda_m}\frac{\mathcal{A}_{m+1}^+}{\lambda_l-\lambda_{m+1}}\cdots\frac{\mathcal{A}_{l-2}^+}{\lambda_l-\lambda_{l-2}}\frac{\mathcal{A}_{l-1}^+}{\lambda_l-\lambda_{l-1}}\Psi_{l,l}(x) & \text{for} \quad m<l,\end{cases} \tag{31}$$

and the operators $\mathcal{H}_m$ are shape-invariant

$$\begin{array}{ll} \mathcal{H}_m - \lambda_m = \mathcal{A}_m^+ \mathcal{A}_m & \mathcal{A}_m \mathcal{H}_m = \mathcal{H}_{m+1} \mathcal{A}_m \\ \mathcal{H}_{m+1} - \lambda_m = \mathcal{A}_m \mathcal{A}_m^+ & \mathcal{H}_m \mathcal{A}_m^+ = \mathcal{A}_m^+ \mathcal{H}_{m+1}. \end{array} \tag{32}$$

These formulae follow from (17), (18), (26) and (30).

Theorem 4.1. *If the change of variable $x = x(s)$ is such that $dx/ds = \pm 1/\kappa(s)$, then*

$$\mathcal{A}_m = \pm \frac{d}{dx} + W_m(x) \qquad \mathcal{A}_m^+ = \mp \frac{d}{dx} + W_m(x) \tag{33}$$

and

$$\mathcal{V}_m(x) = W_m^2(x) \mp \dot{W}_m(x) + \lambda_m = \frac{\ddot{\Psi}_{m,m}(x)}{\Psi_{m,m}(x)} + \lambda_m \tag{34}$$

where $W_m(x)$ is the superpotential [6]

$$W_m(x) = -\frac{\tau(s(x))}{2\kappa(s(x))} - \left(m - \frac{1}{2}\right) \frac{d\kappa}{ds}(s(x)) = \mp \frac{\dot{\Psi}_{m,m}(x)}{\Psi_{m,m}(x)}. \tag{35}$$

Proof. From $(\sigma\varrho)' = \tau\varrho$ and $dx/ds = \pm 1/\kappa(s)$, we get

$$\frac{\varrho'}{\varrho} = \frac{\tau}{\kappa^2} - 2\frac{\kappa'}{\kappa} \qquad \frac{d}{ds} = \pm \frac{1}{\kappa(s(x))} \frac{d}{dx}$$

and

$$[\kappa(s)\varrho(s)]^{1/2} A_m \, [\kappa(s)\varrho(s)]^{-1/2} = [\kappa(s)\varrho(s)]^{1/2} \left(\kappa(s)\frac{d}{ds} - m\kappa'(s)\right) [\kappa(s)\varrho(s)]^{-1/2}$$

$$= \kappa(s)\frac{d}{ds} - \frac{1}{2}\kappa'(s) - \frac{1}{2}\kappa(s)\frac{\varrho'(s)}{\varrho(s)} - m\kappa'(s) = \kappa(s)\frac{d}{ds} - \frac{\tau(s)}{2\kappa(s)} - \left(m - \frac{1}{2}\right)\kappa'(s)$$

whence the first part of (33) and (35). Since $\mathcal{A}_m \Psi_{m,m} = 0$, from (26), (32) and (33) we obtain

$$\pm\dot{\Psi}_{m,m} + W_m(x)\Psi_{m,m} = 0 \qquad -\ddot{\Psi}_{m,m} + (V_m(x) - \lambda_m)\Psi_{m,m} = 0.$$

The first part of (34) is a direct consequence of (26), (32) and (33).

The functions

$$\psi_{l,m}(x) = \sqrt{\kappa(s(x))\,\varrho(s(x))}\,\phi_{l,m}(s(x)) \tag{36}$$

corresponding to $\phi_{l,m}$ are normalized

$$\int_{a'}^{b'} |\psi_{k,m}(x)|^2 dx = \int_a^b |\phi_{k,m}(s(x))|^2 \varrho(s(x)) \left|\frac{ds}{dx}\right| dx = \int_a^b |\phi_{k,m}(s)|^2 \varrho(s) ds = 1$$

and satisfy the relations

$$\mathcal{A}_m\ \psi_{l,m} = \begin{cases} 0 & \text{for } l = m \\ \sqrt{\lambda_l - \lambda_m}\ \psi_{l,m+1} & \text{for } m < l < \Lambda \end{cases}$$
$$\mathcal{A}_m^+\ \psi_{l,m+1} = \sqrt{\lambda_l - \lambda_m}\ \psi_{l,m} \quad \text{for } 0 \leq m < l < \Lambda \tag{37}$$

$$\psi_{l,m} = \frac{\mathcal{A}_m^+}{\sqrt{\lambda_l - \lambda_m}} \frac{\mathcal{A}_{m+1}^+}{\sqrt{\lambda_l - \lambda_{m+1}}} \cdots \frac{\mathcal{A}_{l-1}^+}{\sqrt{\lambda_l - \lambda_{l-1}}} \psi_{l,l}.$$

Particular cases [4, 6, 7]. Let $\alpha_m = -(2m + \alpha - 1)/2$, $\alpha'_m = (2m - \alpha - 1)/2$.

(1) *Shifted oscillator*
In the case $\sigma(s) = 1$, the change of variable $\mathbb{R} \longrightarrow \mathbb{R} : x \mapsto s(x) = x$ leads by substitution into (34) and (35) to

$$W_m(x) = -\frac{\alpha x + \beta}{2}$$
$$\mathcal{V}_m(x) = \frac{(\alpha x + \beta)^2}{4} + \frac{\alpha}{2} + \lambda_m \tag{38}$$

where $\lambda_m = -\alpha m$.

(2) *Three-dimensional oscillator*
In the case $\sigma(s) = s$, the change of variable $(0, \infty) \longrightarrow (0, \infty) : x \mapsto s(x) = x^2/4$ leads by substitution into (34) and (35) to

$$W_m(x) = -\frac{\alpha}{4}x - \left(\beta + m - \frac{1}{2}\right)\frac{1}{x}$$
$$\mathcal{V}_m(x) = \frac{\alpha^2}{16}x^2 + \left(\beta + m - \frac{1}{2}\right)\left(\beta + m - \frac{3}{2}\right)\frac{1}{x^2} + \frac{\alpha}{2}(\beta + m) + \lambda_m \tag{39}$$

where $\lambda_m = -\alpha m$.

(3) *Pöschl-Teller-type potential*
In the case $\sigma(s) = 1 - s^2$, the change of variable $(0, \pi) \longrightarrow (-1, 1) : x \mapsto s(x) = \cos x$ leads by substitution into (34) and (35) to

$$W_m(x) = \alpha'_m \text{ cotan}\, x - \frac{\beta}{2} \text{ cosec}\, x = \frac{\alpha'_m + \beta}{2} \text{ cotan}\, \frac{x}{2} - \frac{\alpha'_m - \beta}{2} \tan\frac{x}{2}$$
$$\mathcal{V}_m(x) = \left(\alpha_m'^2 - \alpha'_m + \frac{\beta^2}{4}\right) \text{cosec}^2 x - (2\alpha'_m - 1)\frac{\beta}{2} \text{ cotan}\, x \text{ cosec}\, x - \alpha_m'^2 + \lambda_m \tag{40}$$

where $\lambda_m = m(m - \alpha - 1)$.

(4) *Generalized Pöschl-Teller potential*
In the case $\sigma(s) = s^2 - 1$, the change of variable $(0, \infty) \longrightarrow (1, \infty) : x \mapsto s(x) = \cosh x$ leads by substitution into (34) and (35) to

$$W_m(x) = \alpha_m \text{ cotanh}\, x - \frac{\beta}{2} \text{ cosech}\, x$$
$$\mathcal{V}_m(x) = \left(\alpha_m^2 + \alpha_m + \frac{\beta^2}{4}\right) \text{cosech}^2 x - (2\alpha_m + 1)\frac{\beta}{2} \text{ cotanh}\, x \text{ cosech}\, x + \alpha_m^2 + \lambda_m \tag{41}$$

where $\lambda_m = -m(m+\alpha-1)$.

(5) *Morse-type potential*

In the case $\sigma(s) = s^2$, the change of variable $\mathbb{R} \longrightarrow (0,\infty) : x \mapsto s(x) = e^x$ leads by substitution into (34) and (35) to

$$\begin{aligned} W_m(x) &= -\tfrac{\beta}{2}e^{-x} + \alpha_m \\ \mathcal{V}_m(x) &= \tfrac{\beta^2}{4}e^{-2x} - (2\alpha_m+1)\tfrac{\beta}{2}e^{-x} + \alpha_m^2 + \lambda_m \end{aligned} \tag{42}$$

where $\lambda_m = -m(m+\alpha-1)$.

(6) *Scarf hyperbolic-type potential*

In the case $\sigma(s)=s^2+1$, the change of variable $\mathbb{R} \longrightarrow \mathbb{R} : x \mapsto s(x) = \sinh x$ leads by substitution into (34) and (35) to

$$\begin{aligned} W_m(x) &= \alpha_m \tanh x - \tfrac{\beta}{2}\,\mathrm{sech}\, x \\ \mathcal{V}_m(x) &= \left(-\alpha_m^2 - \alpha_m + \tfrac{\beta^2}{4}\right)\,\mathrm{sech}^2 x - (2\alpha_m+1)\tfrac{\beta}{2}\tanh x\,\mathrm{sech}\, x + \alpha_m^2 + \lambda_m. \end{aligned} \tag{43}$$

where $\lambda_m = -m(m+\alpha-1)$.

5 Other shape-invariant operators

In this section we restrict ourselves [6] to the particular non-trivial cases when α and β are such that there exists $k \in \mathbb{R}$ with $\varrho(s) = \sigma^k(s)$ (see Table 2).

Table 2 The cases when $\varrho(s) = \sigma^k(s)$.

$\sigma(s)$	$\tau(s)$	$\varrho(s)$	k	(a,b)
s	β	$s^{\beta-1}$	$\beta-1$	$(0,\infty)$
$1-s^2$	αs	$(1-s^2)^{-\alpha/2-1}$	$-\frac{\alpha}{2}-1$	$(-1,1)$
s^2-1	αs	$(s^2-1)^{\alpha/2-1}$	$\frac{\alpha}{2}-1$	$(1,\infty)$
s^2	αs	$s^{\alpha/2-1}$	$\frac{\alpha}{2}-1$	$(0,\infty)$
s^2+1	αs	$(s^2+1)^{\alpha/2-1}$	$\frac{\alpha}{2}-1$	$(-\infty,\infty)$

From $(\sigma\varrho)' = \tau\varrho$ we get $\tau(s) = (k+1)\sigma'(s) = 2(k+1)\kappa(s)\,\kappa'(s)$, and

$$\begin{aligned} &A_m = \kappa(s)\tfrac{d}{ds} - m\kappa'(s) \qquad A_m^+ = -\kappa(s)\tfrac{d}{ds} - (2k+m+1)\kappa'(s) \\ &H_m = -\kappa^2(s)\tfrac{d^2}{ds^2} - 2(k+1)\kappa(s)\,\kappa'(s)\tfrac{d}{ds} - m(m+2k)\kappa(s)\,\kappa''(s) \\ &\lambda_m = -m(2k+m+1)\tfrac{\sigma''(s)}{2} = -m(2k+m+1)[\kappa'^2(s) + \kappa(s)\,\kappa''(s))]. \end{aligned} \tag{44}$$

Theorem 5.1. *If α and β are such that $\varrho(s)=\sigma^k(s)$, then for any $\gamma\in\mathbb{R}$ the operators*

$$\tilde{A}_m = A_m + \frac{\gamma}{2m+2k+1} \qquad \tilde{A}_m^+ = A_m^+ + \frac{\gamma}{2m+2k+1} \tag{45}$$

satisfy for $m<\Lambda-1$ with $2m+2k+1\neq 0$ the relations

$$\begin{array}{ll} \tilde{A}_m^+\tilde{A}_m = \tilde{H}_m - \tilde{\lambda}_m & \tilde{A}_m\tilde{H}_m = \tilde{H}_{m+1}\tilde{A}_m \\ \tilde{A}_m\tilde{A}_m^+ = \tilde{H}_{m+1} - \tilde{\lambda}_m & \tilde{H}_m\tilde{A}_m^+ = \tilde{A}_m^+\tilde{H}_{m+1} \end{array} \tag{46}$$

where

$$\tilde{H}_m = H_m - \gamma\frac{d\kappa}{ds} \qquad \tilde{\lambda}_m = \lambda_m - \frac{\gamma^2}{(2m+2k+1)^2}. \tag{47}$$

Proof. Since $A_m^+A_m = H_m - \lambda_m$ and $A_mA_m^+ = H_{m+1}-\lambda_m$, we obtain

$$(A_m^+ + \varepsilon)(A_m+\varepsilon) = H_m - \lambda_m - \varepsilon(2m+2k+1)\kappa'(s) + \varepsilon^2$$
$$(A_m + \varepsilon)(A_m^+ +\varepsilon) = H_{m+1} - \lambda_m - \varepsilon(2m+2k+1)\kappa'(s) + \varepsilon^2$$

for any constant ε. If we choose $\varepsilon = 1/(2m+2k+1)$, then we get (46)

$$\tilde{H}_m\tilde{A}_m^+ = (\tilde{A}_m^+\tilde{A}_m + \tilde{\lambda}_m)\tilde{A}_m^+ = \tilde{A}_m^+(\tilde{A}_m\tilde{A}_m^+ + \tilde{\lambda}_m) = \tilde{A}_m^+\tilde{H}_{m+1}$$
$$\tilde{A}_m\tilde{H}_m = \tilde{A}_m(\tilde{A}_m^+\tilde{A}_m + \tilde{\lambda}_m) = (\tilde{A}_m\tilde{A}_m^+ + \tilde{\lambda}_m)\tilde{A}_m = \tilde{H}_{m+1}\tilde{A}_m.$$

Theorem 5.2. *If $0\leq m\leq l<\Lambda$ and if $\tilde{\Phi}_{l,l}$ satisfies the relation $\tilde{A}_l\tilde{\Phi}_{l,l}=0$, then*

$$\tilde{\Phi}_{l,m} = \frac{\tilde{A}_m^+}{\tilde{\lambda}_l-\tilde{\lambda}_m}\,\frac{\tilde{A}_{m+1}^+}{\tilde{\lambda}_l-\tilde{\lambda}_{m+1}}\cdots\frac{\tilde{A}_{l-2}^+}{\tilde{\lambda}_l-\tilde{\lambda}_{l-2}}\,\frac{\tilde{A}_{l-1}^+}{\tilde{\lambda}_l-\tilde{\lambda}_{l-1}}\,\tilde{\Phi}_{l,l} \tag{48}$$

is an eigenfunction of $\tilde{H}_m$,

$$\tilde{H}_m\tilde{\Phi}_{l,m} = \tilde{\lambda}_l\tilde{\Phi}_{l,m}, \tag{49}$$

and

$$\tilde{A}_m\tilde{\Phi}_{l,m} = \begin{cases} 0 & \text{if } m=l \\ \tilde{\Phi}_{l,m+1} & \text{if } m<l \end{cases} \tag{50}$$
$$\tilde{A}_m^+\tilde{\Phi}_{l,m+1} = (\tilde{\lambda}_l - \tilde{\lambda}_m)\tilde{\Phi}_{l,m}.$$

Proof. The definition (48) of $\tilde{\Phi}_{l,m}$ can be re-written as

$$\tilde{\Phi}_{l,m} = \frac{\tilde{A}_m^+}{\tilde{\lambda}_l-\tilde{\lambda}_m}\,\tilde{\Phi}_{l,m+1} \tag{51}$$

and $\tilde{H}_l\tilde{\Phi}_{l,l} = (\tilde{A}_l^+\tilde{A}_l + \tilde{\lambda}_l)\tilde{\Phi}_{l,l} = \tilde{\lambda}_l\tilde{\Phi}_{l,l}$. The relation $\tilde{H}_m\tilde{\Phi}_{l,m}=\tilde{\lambda}_l\tilde{\Phi}_{l,m}$ follows by recurrence:

$$\tilde{H}_{m+1}\tilde{\Phi}_{l,m+1} = \tilde{\lambda}_l\tilde{\Phi}_{l,m+1} \quad\Longrightarrow\quad \tilde{H}_m\tilde{\Phi}_{l,m} = \frac{\tilde{H}_m\tilde{A}_m^+}{\tilde{\lambda}_l-\tilde{\lambda}_m}\,\tilde{\Phi}_{l,m+1} = \frac{\tilde{A}_m^+\tilde{H}_{m+1}}{\tilde{\lambda}_l-\tilde{\lambda}_m}\,\tilde{\Phi}_{l,m+1} = \tilde{\lambda}_l\tilde{\Phi}_{l,m}.$$

From Relation (51), we get

$$\tilde{A}_m\tilde{\Phi}_{l,m} = \frac{\tilde{A}_m\tilde{A}_m^+}{\tilde{\lambda}_l - \tilde{\lambda}_m}\,\tilde{\Phi}_{l,m+1} = \frac{\tilde{H}_{m+1} - \tilde{\lambda}_m}{\tilde{\lambda}_l - \tilde{\lambda}_m}\,\tilde{\Phi}_{l,m+1} = \tilde{\Phi}_{l,m+1}.$$

If in Equation (49) we pass to a new variable $x = x(s)$ such that $dx/ds = \pm 1/\kappa(s)$ and to the new functions

$$\tilde{\Psi}_{l,m}(x) = \sqrt{\kappa(s(x))\,\varrho(s(x))}\,\tilde{\Phi}_{l,m}(s(x)), \tag{52}$$

we get the Schrödinger-type equation

$$\tilde{\mathcal{H}}_m\tilde{\Psi}_{l,m} = \tilde{\lambda}_l\tilde{\Psi}_{l,m} \tag{53}$$

where

$$\tilde{\mathcal{H}}_m = -\frac{d^2}{dx^2} + \tilde{\mathcal{V}}_m(x) \qquad \text{and} \qquad \tilde{\mathcal{V}}_m(x) = \mathcal{V}_m(x) - \gamma\frac{d\kappa}{ds}(s(x)). \tag{54}$$

The operators

$$\begin{aligned} \tilde{\mathcal{A}}_m &= \left\{[\kappa(s)\varrho(s)]^{1/2}\tilde{A}_m[\kappa(s)\varrho(s)]^{-1/2}\right\}\Big|_{s=s(x)} \\ \tilde{\mathcal{A}}_m^+ &= \left\{[\kappa(s)\varrho(s)]^{1/2}\tilde{A}_m^+[\kappa(s)\varrho(s)]^{-1/2}\right\}\Big|_{s=s(x)} \end{aligned} \tag{55}$$

corresponding to $\tilde{A}_m$ and $\tilde{A}_m^+$ satisfy the relations

$$\begin{aligned} \tilde{\mathcal{A}}_m\tilde{\Psi}_{l,m}(x) &= \begin{cases} 0 & \text{if } m = l \\ \tilde{\Psi}_{l,m+1} & \text{if } m < l \end{cases} \\ \tilde{\mathcal{A}}_m^+\tilde{\Psi}_{l,m+1}(x) &= (\tilde{\lambda}_l - \tilde{\lambda}_m)\tilde{\Psi}_{l,m}(x) \end{aligned} \tag{56}$$

and

$$\begin{aligned} \tilde{\mathcal{H}}_m - \tilde{\lambda}_m &= \tilde{\mathcal{A}}_m^+\tilde{\mathcal{A}}_m \qquad & \tilde{\mathcal{A}}_m\tilde{\mathcal{H}}_m &= \tilde{\mathcal{H}}_{m+1}\tilde{\mathcal{A}}_m \\ \tilde{\mathcal{H}}_{m+1} - \tilde{\lambda}_m &= \tilde{\mathcal{A}}_m\tilde{\mathcal{A}}_m^+ \qquad & \tilde{\mathcal{H}}_m\tilde{\mathcal{A}}_m^+ &= \tilde{\mathcal{A}}_m^+\tilde{\mathcal{H}}_{m+1}. \end{aligned} \tag{57}$$

If the change of variable $x = x(s)$ is such that $dx/ds = \pm 1/\kappa(s)$, then

$$\tilde{\mathcal{A}}_m = \pm\frac{d}{dx} + \tilde{W}_m(x) \qquad\qquad \tilde{\mathcal{A}}_m^+ = \mp\frac{d}{dx} + \tilde{W}_m(x) \tag{58}$$

$$\tilde{\mathcal{V}}_m(x) = \tilde{W}_m^2(x) \mp \dot{\tilde{W}}_m(x) + \tilde{\lambda}_m = \frac{\ddot{\tilde{\Psi}}_{m,m}(x)}{\tilde{\Psi}_{m,m}(x)} + \tilde{\lambda}_m \tag{59}$$

where $\tilde{W}_m(x)$ is the superpotential [6]

$$\tilde{W}_m(x) = -\frac{\tau(s(x))}{2\kappa(s(x))} - \left(m - \frac{1}{2}\right)\frac{d\kappa}{ds}(s(x)) + \frac{\gamma}{2m + 2k + 1} = \mp\frac{\dot{\tilde{\Psi}}_{m,m}(x)}{\tilde{\Psi}_{m,m}(x)}. \tag{60}$$

Particular cases [4, 6, 7]. Let $\alpha_m = -(2m+\alpha-1)/2$, $\alpha'_m = (2m-\alpha-1)/2$.

(1) *Coulomb-type potential*

In the case $\sigma(s) = s$, the change of variable $(0,\infty) \longrightarrow (0,\infty) : x \mapsto s(x) = x^2/4$ leads by substitution into (47), (59) and (60) to

$$\begin{aligned} \tilde{W}_m(x) &= -\left(\beta+m-\tfrac{1}{2}\right)\tfrac{1}{x} + \tfrac{\gamma}{2m+2\beta-1} \\ \tilde{\mathcal{V}}_m(x) &= \left(\beta+m-\tfrac{1}{2}\right)\left(\beta+m-\tfrac{3}{2}\right)\tfrac{1}{x^2} - \gamma\tfrac{1}{x} \\ \tilde{\lambda}_m &= -\tfrac{\gamma^2}{(2m+2\beta-1)^2}. \end{aligned} \tag{61}$$

(2) *Trigonometric Rosen-Morse-type potential*

In the case $\sigma(s) = 1-s^2$, the change of variable $(0,\pi) \longrightarrow (-1,1) : x \mapsto s(x) = \cos x$ leads by substitution into (47), (59) and (60) to

$$\begin{aligned} \tilde{W}_m(x) &= \alpha'_m \text{ cotan } x + \tfrac{\gamma}{2m-\alpha-1} \\ \tilde{\mathcal{V}}_m(x) &= \left({\alpha'_m}^2 - \alpha'_m\right) \text{ cosec}^2 x + \gamma \text{ cotan}\, x - {\alpha'_m}^2 + m(m-\alpha-1) \\ \tilde{\lambda}_m &= m(m-\alpha-1) - \tfrac{\gamma^2}{(2m-\alpha-1)^2}. \end{aligned} \tag{62}$$

(3) *Eckart-type potential*

In the case $\sigma(s) = s^2 - 1$, the change of variable $(0,\infty) \longrightarrow (1,\infty) : x \mapsto s(x) = \cosh x$ leads by substitution into (47), (59) and (60) to

$$\begin{aligned} \tilde{W}_m(x) &= \alpha_m \text{ cotanh } x + \tfrac{\gamma}{2m+\alpha-1} \\ \tilde{\mathcal{V}}_m(x) &= (\alpha_m^2 + \alpha_m) \text{ cosech}^2 x - \gamma \text{ cotanh}\, x + \alpha_m^2 - m(m-\alpha-1) \\ \tilde{\lambda}_m &= -m(m-\alpha-1) - \tfrac{\gamma^2}{(2m+\alpha-1)^2}. \end{aligned} \tag{63}$$

(4) *Hyperbolic Rosen-Morse-type potential*

In the case $\sigma(s) = s^2+1$, the change of variable $\mathbb{R} \longrightarrow \mathbb{R} : x \mapsto s(x) = \sinh x$ leads by substitution into (47), (59) and (60) to

$$\begin{aligned} \tilde{W}_m(x) &= \alpha_m \tanh x + \tfrac{\gamma}{2m+\alpha-1} \\ \tilde{\mathcal{V}}_m(x) &= -(\alpha_m^2 + \alpha_m) \text{ sech}^2 x - \gamma \tanh x + \alpha_m^2 - m(m-\alpha-1) \\ \tilde{\lambda}_m &= -m(m-\alpha-1) - \tfrac{\gamma^2}{(2m+\alpha-1)^2}. \end{aligned} \tag{64}$$

6 Concluding remarks

Most of the known exactly solvable Schrödinger equations are directly related to some shape-invariant operators, and most of the formulae occurring in the study of these quantum systems follow from a small number of mathematical results concerning the hypergeometric type operators. It is simpler to study these shape-invariant operators than the corresponding operators occurring in various applications to quantum mechanics. Our systematic study recovers known results in a natural, unified way, and allows one to extend certain results known in particular cases.

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