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The asymptotic iteration method applied to certain quasinormal modes and non Hermitian systems

Research Article

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Abstract: We study the Schrödinger equation with potentials admitting quasinormal modes using the asymptotic iteration method (AIM). We also study non-Hermitian $\mathcal{PT}$ symmetric potentials using AIM. The spectra, in all cases, are found to be in excellent agreement with exact results.

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1. Introduction

Quantum mechanics often requires exact or approximate solutions of the Schrödinger equation. Unfortunately there are not many potentials which admit exact solutions, so many methods are used to obtain approximate solutions. Recently a technique, called the Asymptotic Iteration Method (AIM) has been introduced [1, 2] to obtain eigenvalues of second order homogeneous differential equations. In the case of the Schrödinger equation, it has been found that AIM exactly reproduces the energy spectrum for most exactly solvable potentials [2–4] and for non-exactly solvable potentials it produces very good results [5–7].

On the other hand there is a class of quantum mechanical potentials which admit discrete complex eigenvalues (or eigenfrequencies). The solutions corresponding to these eigenvalues exhibit outgoing wave features and are called quasinormal modes (QNM) [8]. QNM's play an important role in the study of black holes [8] and recently been studied widely [9]. Here, our objective is to study the application of AIM to potentials admitting QNM's and to another class of non-Hermitian potentials, namely the $\mathcal{PT}$ symmetric potentials. The paper is organized as follows. In section 2, we briefly review the asymptotic iteration method. Section 3 is devoted to obtaining the QNM eigenvalues by AIM. We consider a $\mathcal{PT}$ symmetric model in section 4 and finally section 5 is devoted to a conclusion.

2. AIM

Although a complete description of the AIM can be found elsewhere [1, 2], we shall, for the sake of com-

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pleteness, outline the method. To begin, we note that the Schrödinger equation for one-dimensional and radial models can be written as

$$-\psi''(x) + [V(x) - E]\psi(x) = 0. \tag{1}$$

It is not, however, convenient to apply AIM directly to Eq. (1). Instead, we perform a change of wave function in the form

$$\psi(x) = g(x)y(x) \tag{2}$$

(together with a possible change of coordinates) to transform Eq. (1) to an equation of the form [1]

$$y'' = \lambda_0(x)y' + s_0(x)y, \tag{3}$$

where $\lambda_0(x) \neq 0$ and $s_0(x)$, $\lambda_0(x)$ are functions in $C_\infty(a,b)$. It is important to note that in most applications the asymptotic behaviour[1] is contained in the function $g(x)$ and is factored out. On the other hand one tries to determine $y(x)$ in a polynomial form so as not to disturb the asymptotic behaviour irrespective of whether it is convergent or otherwise. The general solution of Eq. (3) can be written as (see [1] for details)

$$y(x) = \exp\left(-\int^x \alpha(x_1)dx_1\right)\left[C_2 + C_1\int^x \exp\left(\int^{x_1}(\lambda_0(x_2) + 2\alpha(x_2))dx_2\right)dx_1\right]. \tag{4}$$

For sufficiently many k iterations, $\alpha(x)$ can be obtained from the relations

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x), \quad k = 1, 2, 3, \ldots, \tag{5}$$

where

$$\begin{aligned} \lambda_k(x) &= \lambda'_{k-1}(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \\ s_k(x) &= s'_{k-1}(x) + s_0(x)\lambda_{k-1}(x). \end{aligned} \tag{6}$$

The energy eigenvalues are obtained from the quantization condition, given by the termination condition in Eq. (5). The quantization condition combined with Eq. (6) can also be written as

$$\delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) = 0. \tag{7}$$

If the Schrödinger equation can be transformed into the form of Eq. (3), then one can find the energy spectrum and wave function of the given potential. Thus, using Eq. (3) $s_0(x)$ and $\lambda_0(x)$ are determined and then $s_k(x)$ and $\lambda_k(x)$ parameters can be calculated iteratively. The energy eigenvalues of the potential in interest are figured out by the quantization condition given by Eq. (7). The wave functions are determined by using the following wave function generator

$$y_n(x) = C_2 \exp\left(-\int^x \frac{s_k(x')}{\lambda_k(x')}dx'\right). \tag{8}$$

The reason for choosing $C_1 = 0$ is that the part associated with C_2 gives polynomial solutions and consequently the solution (4) satisfies appropriate boundary conditions. Therefore, the corresponding eigenfunctions can be derived from the wave function generator given in (8).

3. AIM approach to Quasinormal modes

Generally quantum mechanical potentials admitting bound states can be classified as solvable, non-solvable and quasi-exactly solvable (QES). In problems belonging to the last category, part of their spectrum can be found analytically while the remaining part is unknown. Recently some QES potentials have also been found to yield quasinormal modes [10]. Here we shall use AIM to determine QNM's of several potentials, including some belonging to the QES category.

3.1. Scarf II potential

As an example of a potential admitting QNM's we first consider the Scarf II potential. It is given by [10]

$$V(x) = -\frac{cd}{2\nu}\tanh\left(\sqrt{\nu}x\right)\operatorname{sech}\left(\sqrt{\nu}x\right) + \frac{1}{4\nu}\left(\nu^2 + c^2 - d^2\right)\operatorname{sech}^2\left(\sqrt{\nu}x\right), \quad -\infty < x < \infty, \tag{9}$$

where $(\nu^2 + c^2 - d^2) > 0$. It may be noted that the potential (9) can be derived (apart from a constant) from a superpotential $W(x)$ given by

$$W(x) = A\tanh\left(\sqrt{\nu}x\right) + B\operatorname{sech}\left(\sqrt{\nu}x\right), \tag{10}$$

[1] *For usual bound states the function $g(x)$ vanishes at the boundaries. However for QNM's, $g(x)$ behaves as outgoing waves.*

where

$$A = -\frac{ic}{2\sqrt{v}} - \frac{\sqrt{v}}{2}, \quad B = -\frac{id}{2\sqrt{v}}.$$

The ground state wave function is given by

$$\begin{aligned}\psi_0(x) &\approx \exp\left[-\int W(x)dx\right] \\ &= \left(\cosh\sqrt{v}x\right)^{(v+ic)/2v} \exp\left(id\tan^{-1}\left(\sinh\sqrt{v}x\right)/2v\right).\end{aligned} \tag{11}$$

It can be seen that the ground state wave function (11) satisfies outgoing wave boundary conditions appropriate for QNM's. Now to bring the Schrödinger equation for (9) we must consider a transformation of the wave function similar to (2). To this end a suitable choice is to consider a transformation of the type

$$\begin{aligned}\psi(x) = \psi_0(x)f(x) = &\left(\cosh\sqrt{v}x\right)^{(v+ic)/2v} \\ &\exp\left(id\tan^{-1}\left(\sinh\sqrt{v}x\right)/2v\right)f(x),\end{aligned} \tag{12}$$

accompanied by a change of coordinates

$$y = \sinh\left(\sqrt{v}x\right). \tag{13}$$

To maintain correct asymptotic behaviour, we shall now constrain $f(x)$ to be of a polynomial type.
Using (12) and (13) the Schrödinger equation corresponding to 9) becomes

$$\begin{aligned}f''(y) = &-\frac{i[d+y(c-2iv)]}{(y^2+1)\,v}f'(y) \\ &+\frac{c^2-2ivc-v(4E+v)}{4\,(y^2+1)\,v^2}f(y).\end{aligned} \tag{14}$$

Now, comparing (3) and (14), we obtain

$$\begin{aligned}\lambda_0(y) &= -\frac{i(d+y(c-2iv))}{(y^2+1)\,v}, \\ s_0(y) &= \frac{c^2-2ivc-v(4E+v)}{4\,(y^2+1)\,v^2},\end{aligned} \tag{15}$$

and by means of Eq. (6), we may also calculate $\lambda_k(y)$ and $s_k(y)$. Finally, comparing the results with the quantization condition given by Eq. (7), we find

$$\begin{aligned}\frac{s_0(y)}{\lambda_0(y)} &= \frac{s_1(y)}{\lambda_1(y)} \Longrightarrow E_0 = \frac{c^2}{4v} - \frac{ic}{2} - \frac{v}{4}, \\ \frac{s_1(y)}{\lambda_1(y)} &= \frac{s_2(y)}{\lambda_2(y)} \Longrightarrow E_1 = \frac{c^2}{4v} - 3\frac{ic}{2} - 9\frac{v}{4}, \\ \frac{s_2(y)}{\lambda_2(y)} &= \frac{s_3(y)}{\lambda_3(y)} \Longrightarrow E_2 = \frac{c^2}{4v} - 5\frac{ic}{2} - 25\frac{v}{4}, \\ &\dots .\end{aligned} \tag{16}$$

Generalizing the above equations, the energy spectrum is found to be

$$E_n = \left[\frac{c}{2\sqrt{v}} - \frac{i(2n+1)\sqrt{v}}{2}\right]^2. \tag{17}$$

This is exactly the spectrum found in [10] using Lie algebraic techniques. Let us now determine the wave functions. Using (17) it can be seen that (14) corresponds to the differential equation for Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [11]

$$\begin{aligned}\left(1-x^2\right)\frac{d^2f}{dx^2} + [\beta-\alpha-(\alpha+\beta+2)x]\frac{df}{dx} \\ +n(\alpha+\beta+n+1)f = 0,\end{aligned} \tag{18}$$

where α, β are given by

$$\alpha = \frac{d}{2\sqrt{v}} - \frac{ic}{2\sqrt{v}} - 1, \quad \beta = -\frac{d}{2\sqrt{v}} - \frac{ic}{2\sqrt{v}} - 1. \tag{19}$$

Thus the wave functions are given by

$$\begin{aligned}\psi_n(x) \approx &\left(\cosh\sqrt{v}x\right)^{(v+ic)/2v} \\ &\exp\left(id\tan^{-1}\left(\sinh\sqrt{v}x\right)/2v\right)P_n^{(\alpha,\beta)}\left(i\sinh\left(\sqrt{v}x\right)\right).\end{aligned} \tag{20}$$

It may be noted that for $d = 0$, we find the potential which is commonly used as an approximation to the describe QNM's of Schwarzschild black hole [12].

3.2. Pöschl-Teller potential

As a second example, we consider the Pöschl-Teller potential given by [10]

$$\begin{aligned}V(x) = &-\frac{1}{4v}\left(v^2+c^2+d^2\right)\text{cosech}^2\left(\sqrt{v}x\right) \\ &-\frac{cd}{2v}\coth\left(\sqrt{v}x\right)\text{cosech}\left(\sqrt{v}x\right), \quad 0 < x < \infty.\end{aligned} \tag{21}$$

In this case the appropriate superpotential is

$$W(x) = A\,\coth\left(\sqrt{v}x\right) - B\,\text{cosech}\left(\sqrt{v}x\right), \tag{22}$$

where

$$A = -\frac{ic}{2\sqrt{v}} - \frac{\sqrt{v}}{2}, \quad B = \frac{id}{2\sqrt{v}}.$$

Now using the Ansatz wavefunction

$$\psi(x) = \exp\left[-\int W(x)dx\right] f(x)$$
$$= \left(\sinh\left(\sqrt{\nu}x\right)\right)^{(ic+\nu)/2\nu} \left(\tanh\left(\sqrt{\nu}x/2\right)\right)^{id/2\nu} f(x), \quad (23)$$

in Eq. (1) and then transforming the coordinate by $y = \cosh(\sqrt{\nu}x)$, we obtain a second-order differential equation in the form of (3):

$$f''(y) = \frac{i(d+cy)+2\nu y}{\nu(1-y^2)} f'(y) + \frac{c^2 - 2ic\nu - \nu(4E+\nu)}{4\nu^2(y^2-1)} f(y). \quad (24)$$

From (24), one gets $\lambda_0(y)$ and $s_0(y)$ as

$$\begin{aligned} \lambda_0(y) &= \frac{i(d+cy)+2\nu y}{\nu(1-y^2)}, \\ s_0(y) &= \frac{c^2 - 2ic\nu - \nu(4E+\nu)}{4\nu^2(y^2-1)}, \end{aligned} \quad (25)$$

and using the quantization condition given in Eq. (7), we obtain

$$\begin{aligned} \frac{s_0(y)}{\lambda_0(y)} &= \frac{s_1(y)}{\lambda_1(y)} \Longrightarrow E_0 = \frac{c^2}{4\nu} - \frac{ic}{2} - \frac{\nu}{4}, \\ \frac{s_1(y)}{\lambda_1(y)} &= \frac{s_2(y)}{\lambda_2(y)} \Longrightarrow E_1 = \frac{c^2}{4\nu} - 3\frac{ic}{2} - 9\frac{\nu}{4}, \\ \frac{s_2(y)}{\lambda_2(y)} &= \frac{s_3(y)}{\lambda_3(y)} \Longrightarrow E_2 = \frac{c^2}{4\nu} - 5\frac{ic}{2} - 25\frac{\nu}{4}, \\ &\ldots . \end{aligned} \quad (26)$$

Generalizing the above equations, the energy spectrum is

$$E_n = \left[\frac{c}{2\sqrt{\nu}} - \frac{i(2n+1)\sqrt{\nu}}{2}\right]^2. \quad (27)$$

3.3. Morse potential

Here we consider a Morse potential given by [10, 13]:

$$V(x) = Ae^{2\sqrt{\nu}x} + Be^{\sqrt{\nu}x} + Ce^{-\sqrt{\nu}x} + De^{-2\sqrt{\nu}x}, \quad (28)$$

where $x \,\epsilon\, (-\infty, \infty)$ and ν is a positive scale factor. Note that $V(x)$ reduces to the exactly solvable Morse potentials when $A = B = 0$, or $C = D = 0$. As pointed out in [13], the potential is QES when the parameters satisfy the following relations:

$$\begin{aligned} A &= \frac{\hat{b}^2}{4\nu}, & B &= \frac{\hat{c} + (j+1)\nu}{2\nu}\hat{b}, \\ C &= \frac{\hat{c} - (j+1)\nu}{2\nu}\hat{d}, & D &= \frac{\hat{d}^2}{4\nu}, \\ j &= 0, 1, 2, 3 \ldots, \end{aligned} \quad (29)$$

where $\hat{b}, \hat{c}, \hat{d}$ are arbitrary real constants and j is an arbitrary integer number. To use AIM, we choose an eigenfunction of the form

$$\psi(x) = \exp\left[\frac{\hat{b}}{2\nu}e^{\sqrt{\nu}x} + \frac{\hat{c} - j\nu}{2\sqrt{\nu}}x - \frac{\hat{d}}{2\nu}e^{-\sqrt{\nu}x}\right] f(x) \quad (30)$$

and a new coordinate y as

$$y = e^{x\sqrt{\nu}}. \quad (31)$$

Now using Eqs. (30) and (31), and making one possible choice of values of $\hat{b} = ib$, $\hat{c} = -(j+1)\nu$, $\hat{d} = d$ (where b and d are real constants), the Schrödinger equation for the potential (28) becomes

$$f''(y) = -\frac{d + y(c + by - j\nu + \nu)}{y^2\nu} f'(y) - \frac{c^2 - 2j\nu c + 2bd + 4E\nu + j\nu(j\nu - 4by)}{4y^2\nu^2} f(y). \quad (32)$$

From Eq. (32), one can easily obtain $\lambda_0(y)$ and $s_0(y)$:

$$\begin{aligned} \lambda_0(y) &= -\frac{d + y(c + by - j\nu + \nu)}{y^2\nu}, \\ s_0(y) &= -\frac{c^2 - 2j\nu c + 2bd + 4E\nu + j\nu(j\nu - 4by)}{4y^2\nu^2}, \end{aligned} \quad (33)$$

and by means of Eq. (6), $\lambda_k(y)$ and $s_k(y)$ can also be calculated. Then, using the quantization condition of the method given by Eq. (7), the energy eigenvalues are obtained where we have put $x = 0$ at the end of the iterations. We have compared our results with earlier studies in Table 1. It is seen that the results obtained by the AIM agree with those in reference [10].

Table 1. Values of QNM energy E for the Morse type potential with parameters $b = d = \nu = 1$ by WKB, QES and AIM. Note that for each j, there are $j+1$ values of E.

j	E (WKB)	E (QES)	E (AIM)
0	–	$-0.25 - 0.5i$	$-0.25 - 0.5i$
1	$-2.313 - 0.0588i$	$(-2.349 - 0.0449i)$	$-2.349 - 0.0449i$
	–	$-0.151 - 0.955i$	$-0.151 - 0.955i$
2	$-6.267 - 0.000147i$	$(-6.271 - 0.000140i)$	$-6.271 - 0.000140i$
	$-2.439 - 0.156i$	$(-2.447 - 0.135i)$	$-2.447 - 0.135i$
	–	$-0.0317 - 1.365i$	$-0.03166 - 1.365i$
3	$-12.257 - 4.813 \times 10^{-8}i$	$(-12.258 - 4.786 \times 10^{-8}i)$	$-12.258 - 4.7864 \times 10^{-8}i$
	$-6.284 - 0.000902i$	$(-6.293 - 0.000817i)$	$-6.2934 - 0.0008168i$
	–	$-2.542 - 0.259i$	$-2.5421 - 0.2589i$
	–	$0.0939 - 1.740i$	$0.09386 - 1.7403i$
4	$-20.254 - 4.328 \times 10^{-12}i$	$(-20.255 - 4.310 \times 10^{-12}i)$	$-20.255 - 4.3808 \times 10^{-12}i$
	$-12.263 - 4.010 \times 10^{-7}i$	$(-12.265 - 3.810 \times 10^{-7}i)$	$-12.265 - 3.810 \times 10^{-7}i$
	$-6.306 - 0.00311i$	$(-6.323 - 0.002742i)$	$-6.323 - 0.002742i$
	–	$-2.628 - 0.406i$	$-2.628 - 0.4063i$
	–	$0.220 - 2.091i$	$0.220 - 2.0909i$

We have observed that the energy eigenvalues for $j = 0$, $j = 1$ and $j = 2$ can be found just by $k = 2$ iterations, but it requires $k = 3$ and $k = 4$ iterations for $j = 3$ and $j = 4$, respectively.

4. Complex non-Hermitian potentials

Apart from the potentials admitting QNM's, there is a another class of non-Hermitian potentials generally known as $\mathcal{PT}$-symmetric or η-pseudo-Hermitian potentials. The Schrödinger equations corresponding to such potentials admit real eigenvalues even though the potentials are complex. Here we would like to point out that AIM can as well be applied to $\mathcal{PT}$ symmetric potentials. As an example we again consider the Pöschl-Teller potential. The $\mathcal{PT}$-symmetric Pöschl-Teller potential is given by [2] [14, 15]

$$V(x) = \left(\frac{\alpha^2+\beta^2}{2} - \frac{1}{4}\right)\operatorname{csch}^2(x+i\epsilon) + \left(\frac{\alpha^2-\beta^2}{2}\right)\coth(x+i\epsilon)\operatorname{csch}(x+i\epsilon), \quad (34)$$

where α, β and ϵ are real parameters. It can be easily seen that $V(x) = V^*(-x)$ so that (34) is indeed $\mathcal{PT}$ symmetric. Now, introducing the Ansatz wavefunction and the new coordinate as

$$\psi(x) = (1-\cosh(x+i\epsilon))^{\frac{\alpha}{2}+\frac{1}{4}}(1+\cosh(x+i\epsilon))^{\frac{\beta}{2}+\frac{1}{4}} f(x), \quad y = \cosh(x+i\epsilon)\,, \quad (35)$$

we obtain a second-order differential equation in the form of (3):

$$f''(y) = \frac{\beta-\alpha-y(\alpha+\beta+2)}{y^2-1} f'(y) - \frac{(\alpha+\beta+1)^2+4E}{4\,(y^2-1)} f(y). \quad (36)$$

In this case, we find

$$\lambda_0(y) = \frac{\beta-\alpha-y(\alpha+\beta+2)}{y^2-1}, \quad s_0(y) = -\frac{(\alpha+\beta+1)^2+4E}{4\,(y^2-1)}. \quad (37)$$

[2] *The potential (34) can be obtained from (21) by reparametrization.*

Now, proceeding as before, it can be easily shown that

$$E_n = -\left(n + \frac{\alpha + \beta + 1}{2}\right)^2. \qquad (38)$$

5. Conclusion

In this paper, we have applied the AIM to obtain eigenvalues of some potentials admitting QNM's as well as $\mathcal{PT}$-symmetric potentials. The AIM results agree with the exact ones whenever they exist and in other cases they agree remarkably well with WKB/QES results. We also determined the wave functions in the case of Scarf II potential. In other cases, the wave functions can be determined in a similar fashion. Because AIM works well with some elementary QNM potentials, we feel it worth exploring the possibility of applying AIM to more realistic Zerilli or the Regge-Wheeler potentials, *etc*, [8, 9, 12].

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