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Theoretical construction of stable traversable wormholes

Research Article

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Abstract: It is shown in this paper that it is possible, at least in principle, to construct a traversable wormhole that is stable to linearized radial perturbations by specifying relatively simple conditions on the shape and redshift functions.

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1. Introduction

Wormholes may be defined as handles or tunnels in the spacetime topology linking widely separated regions of our Universe or of different universes altogether. That such wormholes may be traversable by humanoid travelers was first conjectured by Morris and Thorne [1] in 1988. To hold a wormhole open, violations of certain energy conditions must be tolerated.

Another frequently discussed topic is stability, that is, determining whether a wormhole is stable when subjected to linearized perturbations around a static solution. Much of the earlier work concentrated on thin-shell Schwarzschild wormholes using the cut-and-paste technique [2]. In this paper we are more interested in constructing wormhole solutions by matching an interior traversable wormhole geometry with an exterior Schwarzschild vacuum solution and examining the junction surface. (For further discussion of this approach, see Refs. [3–8].) A linearized stability analysis of thin-shell wormholes with a cosmological constant can be found in Ref. [9], while Ref. [10] discusses the stability of phantom wormholes. In other, related, studies, the Ellis drainhole was found to be unstable to non-linear perturbations [11] but stable to linear perturbations [12]. According to Refs. [13, 14], however, such wormholes are actually unstable to both types of perturbations

A rather different approach to stability analysis is presented in Ref. [15]. In Ref. [16], an example of a stable traversable wormhole connecting two branes in the Randall-Sundrum model is considered, while Ref. [17] discusses the instability of scalar wormholes in a cosmological setting.

The purpose of this paper is to show that it is in principle possible to construct a traversable wormhole that is stable to linearized radial perturbations. The conditions on the redshift and shape functions at the junction surface are surprisingly simple.

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2. Traversable wormholes

Using units in which $c = G = 1$, the interior wormhole geometry is given by the following metric [1]:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1-b(r)/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (1)$$

The motivation for this idea is the Schwarzschild line element

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-2M/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \quad (2)$$

In Eq. (1), $\Phi = \Phi(r)$ is referred to as the *redshift function*, which must be everywhere finite to prevent an event horizon. The function $b = b(r)$ is usually referred to as the *shape function*. The minimum radius $r = r_0$ is the *throat* of the wormhole, where $b(r_0) = r_0$. To hold a wormhole open, the weak energy condition (WEC) must be violated. (The WEC requires the stress-energy tensor $T_{\alpha\beta}$ to obey $T_{\alpha\beta}\mu^\alpha\mu^\beta \geq 0$ for all time-like vectors and, by continuity, all null vectors.) As a result, the shape function must obey the additional flare-out condition $b'(r_0) < 1$ [1]. For $r > r_0$, we must have $b(r) < r$, while $\lim_{r\to\infty} b(r)/r = 0$ (asymptotic flatness). Well away from the throat both Φ and b need to be adjusted, as we will see.

The need to violate the WEC was first noted in Ref. [1]. A well-known mechanism for this violation is the Casimir effect. Other possibilities are phantom energy [18] and Chaplygin traversable wormholes [19].

Since the interior wormhole solution is to be matched with an exterior Schwarzschild solution at the junction surface $r = a$, denoted by S, our starting point is the Darmois-Israel formalism [20, 21]: if K_{ij} is the extrinsic curvature across S (also known as the second fundamental form), then the stress-energy tensor $S^i{}_j$ is given by the Lanczos equations

$$S^i{}_j = -\frac{1}{8\pi}\left([K^i{}_j] - \delta^i{}_j[K]\right), \quad (3)$$

where

$$[X] = \lim_{r\to a+} X - \lim_{r\to a-} X = X^+ - X^-.$$

So $[K_{ij}] = K^+_{ij} - K^-_{ij}$, which expresses the discontinuity in the second fundamental form, and $[K]$ is the trace of $[K^i{}_j]$. In terms of the energy-density σ and the surface pressure P, $S^i{}_j = \text{diag}(-\sigma, P, P)$. The Lanczos equations now yield

$$\sigma = -\frac{1}{4\pi}[K^\theta{}_\theta] \quad (4)$$

and

$$P = \frac{1}{8\pi}\left([K^\tau{}_\tau] + [K^\theta{}_\theta]\right). \quad (5)$$

A dynamic analysis can be obtained by letting the radius $r = a$ be a function of time, as in Ref. [2]. According to Lobo [10], the components of the extrinsic curvature are given by

$$K^{\tau}{}_{\tau}{}^{+} = \frac{\frac{M}{a^2} + \ddot{a}}{\sqrt{1 - \frac{2M}{a} + \dot{a}^2}}, \quad (6)$$

$$K^{\tau}{}_{\tau}{}^{-} = \frac{\Phi'\left(1 - \frac{b(a)}{a} + \dot{a}^2\right) + \ddot{a} - \frac{\dot{a}^2[b(a) - ab'(a)]}{2a[a-b(a)]}}{\sqrt{1 - \frac{b(a)}{a} + \dot{a}^2}} \quad (7)$$

and

$$K^{\theta}{}_{\theta}{}^{+} = \frac{1}{a}\sqrt{1 - \frac{2M}{a} + \dot{a}^2}, \quad (8)$$

$$K^{\theta}{}_{\theta}{}^{-} = \frac{1}{a}\sqrt{1 - \frac{b(a)}{a} + \dot{a}^2}. \quad (9)$$

For future use let us also obtain σ': from

$$\sigma = -\frac{1}{4\pi}(K^\theta{}_\theta{}^+ - K^\theta{}_\theta{}^-) = -\frac{1}{4\pi a}\left(\sqrt{1 - \frac{2M}{a} + \dot{a}^2} - \sqrt{1 - \frac{b(a)}{a} + \dot{a}^2}\right), \quad (10)$$

one can calculate

$$\sigma' = \frac{\dot{\sigma}}{\dot{a}} = \frac{1}{4\pi a^2}\left(\frac{1 - \frac{3M}{a} + \dot{a}^2 - a\ddot{a}}{\sqrt{1 - \frac{2M}{a} + \dot{a}^2}} - \frac{1 - \frac{3b(a)}{2a} + \frac{b'(a)}{2} + \dot{a}^2 - a\ddot{a}}{\sqrt{1 - \frac{b(a)}{a} + \dot{a}^2}}\right). \quad (11)$$

Again following Lobo [10], rewriting Eq. (10) in the form

$$\sqrt{1 - \frac{2M}{a} + \dot{a}^2} = \sqrt{1 - \frac{b(a)}{a} + \dot{a}^2} - 4\pi\sigma a \quad (12)$$

will yield the following equation of motion:

$$\dot{a}^2 + V(a) = 0. \quad (13)$$

Here $V(a)$ is the potential, which can be put into the following convenient form:

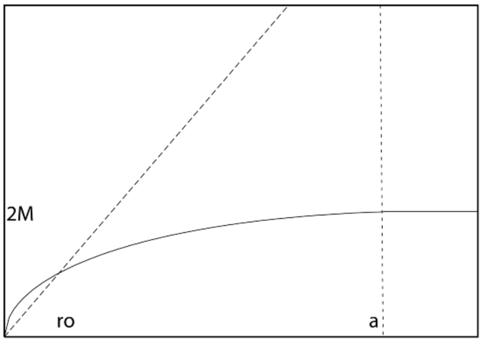

Figure 1. The interior shape function attains a maximum value at $r = a$.

$$V(a) = 1 - \frac{\frac{1}{2}b(a) + M}{a} - \frac{m_s^2}{4a^2} - \left(\frac{M - \frac{1}{2}b(a)}{m_s}\right)^2, \quad (14)$$

where $m_s = 4\pi a^2\sigma$ is the mass of the junction surface, which is a thin shell in Ref. [10].

When linearized around a static solution at $a = a_0$, the solution is stable if, and only if, $V(a)$ has a local minimum value of zero at $a = a_0$, that is, $V(a_0) = 0$ and $V'(a_0) = 0$, and its graph is concave up: $V''(a_0) > 0$. For $V(a)$ in Eq. (14), these conditions are met [6].

Since the junction surface S is understood to be well away from the throat, we expect σ to be positive. Eq. (10) then implies that $b(a) < 2M$, rather than $b(a) = 2M$, which the Schwarzschild line element (2) might suggest. (One can also say that the interior and exterior regions may be separated by a thin shell. The reason for this is that in its most general form, the junction formalism joins two distinct spacetime manifolds M_+ and M_- with metrics given in terms of independently defined coordinate systems x^μ_+ and x^μ_- [6].) What needs to be emphasized is that even if $b(a) < 2M$, $b(a)$ can be arbitrarily close to $2M$ without affecting the above analysis. In particular, $V(a_0) = 0$ and $V'(a_0) = 0$ even if $\lim_{a\to a_0-} b(a) = 2M$, since, by the definition of left-hand limit, $b(a) < 2M$. The condition $V''(a_0) > 0$ should now be written $V''(a_0-) > 0$.

3. The line element

Given our aim, the construction of a stable wormhole, our main requirement can now be stated as follows: apart from the usual conditions at the throat, we require that $b = b(r)$ be an increasing function of r having a continuous second derivative and reaching a maximum value at some $r = a$. In other words, we require that $b'(r)$ approach zero continuously as $r \to a$ (Fig. 1). Keeping in mind the Schwarzschild line element (2), we let $b(r) = 2M$ for $r > a$ since $M = \frac{1}{2}b(a)$. In this manner, both $b(r)$ and $b'(r)$ remain continuous across the junction surface $r = a$. It follows directly from Eq. (10) that $\sigma = 0$ at $r = a$. It is also desirable to have $P = 0$ at $r = a$, thereby making $S^i{}_j = 0$. To this end we choose $\Phi(r)$ so that

$$\Phi'(a-) = \frac{M}{a(a-2M)}. \quad (15)$$

[Of course, $\Phi(r)$ must still be finite at the throat, while $\Phi(a-) = \Phi(a+)$.] For $r > a$, $\Phi(r) = \frac{1}{2}\ln(1 - \frac{2M}{r})$, so that $\Phi'(a-) = \Phi'(a+)$. We now have $K^\tau{}_\tau{}^+ - K^\tau{}_\tau{}^- = 0$ and $K^\theta{}_\theta{}^+ - K^\theta{}_\theta{}^- = 0$ at $r = a$. So $P = 0$ at $r = a$ by Eqs. (5)-(9), as desired.

For the above choice of $\Phi(r)$, the resulting line element is

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(r)/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad r \le a,$$

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - b(a)/r} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad r > a. \quad (16)$$

Note especially that

$$\frac{d}{dr}g_{tt}(a-) = \frac{d}{dr}g_{tt}(a+) \quad \text{and} \quad \frac{d}{dr}g_{rr}(a-) = \frac{d}{dr}g_{rr}(a+). \quad (17)$$

Since the components of the stress-energy tensor are equal to zero at S, the junction is a boundary surface, rather than a thin shell [10], and K_{ij} is continuous across S.

4. Stability

As noted at the end of Sec. 2, $V(a_0-) = 0$ and $V'(a_0-) = 0$ even if $\lim_{a\to a_0-} b(a) = 2M$, since $b(a) < 2M$. In Sec. 3 we saw that in the absence of surface stresses our junction is a boundary surface, rather than a thin shell: since $b'(r)$ goes to zero continuously as $r \to a-$, $b(r)$ continues smoothly at $r = a$ to become $2M$ to the right of a (Fig. 1). This implies that the usual thin-shell formalism using the δ-function is not directly applicable. To show this, suppose we write the derivatives in Eq. (17) in the following form:

$$\frac{d}{dr}g_{\mu\nu} = \Theta(r-a)\frac{d}{dr}g^{+}_{\mu\nu}(r) + \Theta[-(r-a)]g^{-}_{\mu\nu}(r),$$

where Θ is the Heaviside step function. Then by the product rule,

$$\frac{d^2}{dr^2}g_{\mu\nu}(a\pm) = \Theta(r-a)\frac{d^2}{dr^2}g_{\mu\nu}(a+) + \Theta[-(r-a)]\frac{d^2}{dr^2}g_{\mu\nu}(a-) + \delta(r-a)\left\{\frac{d}{dr}g_{\mu\nu}(a+) - \frac{d}{dr}g_{\mu\nu}(a-)\right\}.$$

So by Eq. (17),

$$\frac{d^2}{dr^2}g_{\mu\nu}(a\pm) = \Theta(r-a)\frac{d^2}{dr^2}g_{\mu\nu}(a+) + \Theta[-(r-a)]\frac{d^2}{dr^2}g_{\mu\nu}(a-).$$

Up to the second derivatives, then, the δ-function does not appear, in agreement with Visser [21]: by adopting Gaussian normal coordinates, the total stress-energy tensor, which also depends on the second derivatives of $g_{\mu\nu}$, may be written in the form $T_{\mu\nu} = \delta(\eta)S_{\mu\nu} + \Theta(\eta)T^{+}_{\mu\nu} + \Theta(-\eta)T^{-}_{\mu\nu}$, thereby showing the δ-function contribution at the location of the thin shell; here σ is necessarily greater than zero. In our situation, however, $S_{ij} = 0$ at the boundary surface, so that, once again, the δ-function does not appear.

Even more critical in the stability analysis is the need to study the second derivative of $V(a)$ in Eq. (14). Since $V''(a)$ involves $m_s'' = (4\pi a^2\sigma)''$, let us first use Eq. (4) to write σ' in the following form:

$$\sigma' = -\frac{1}{4\pi}\left(\Theta(r-a)\frac{d}{dr}K^{\theta}{}_{\theta}{}^{+} + \Theta[-(r-a)]\frac{d}{dr}K^{\theta}{}_{\theta}{}^{-}\right).$$

As long as $b(r)$ is an increasing function without the assumed maximum value at $r = a$, σ' will have a jump discontinuity at $r = a$. So σ'' is equal to $\delta(r-a)$ times the magnitude of the jump [22]. If $b'(a_0) = 0$, on the other hand, the calculations leading to Eq. (11) show that σ' is continuous at $a = a_0$. It follows that there is no δ-function in the expression for $V''(a_0)$.

Without the δ-function, one cannot simply declare $4\pi a^2\sigma$ to be the (finite) mass of the spherical surface $r = a$, since the thickness of an ordinary surface is undefined. (It is quite another matter to assert that $dm = 4\pi\sigma a^2 da$, which can indeed be integrated over a *finite* interval.)

Returning to Eq. (11), when σ' is evaluated at the static solution, then $b'(a_0) = 0$ implies that $\sigma'(a_0) = 0$. So σ approaches zero, its minimum value, continuously as $a \to a_0-$, and, as a consequence, $\sigma > 0$ in the open interval $(a_0 - \varepsilon, a_0)$; here ε is arbitrarily small, but finite (as opposed to infinitesimal). As a result, σ is approximately constant, but nonzero, in the boundary layer extending from $r = a_0 - \varepsilon$ to $r = a_0$. So for $\bar{a}\ \epsilon\ (a_0 - \varepsilon, a_0)$, $m_s = 4\pi\bar{a}^2\sigma$ is a positive constant, but one that can be made as small as we please. Referring back to Eq. (14), we now find the second derivative of V, making use of the condition $b'(a_0) = 0$. Since m_s is fixed, we get

$$V''(a_0-) = -\frac{\frac{1}{2}b''(a_0-)}{a_0-} - \frac{b(a_0-)+2M}{(a_0-)^3} - \frac{3m_s^2}{2(a_0-)^4} + \frac{b''(a_0-)[M - \frac{1}{2}b(a_0-)]}{m_s^2}. \quad (18)$$

Since m_s is arbitrarily small, but nonzero, the third term on the right-hand side is arbitrarily close to zero, while the last term is equal to zero. From $V''(a_0-) > 0$, we obtain

$$b''(a_0-) < -\frac{2[b(a_0-)+2M]}{(a_0-)^2}.$$

Using our arbitrary ε, we can also say that

$$b''(a_0-\varepsilon) < -\frac{2[b(a_0-\varepsilon)+2M]}{(a_0-\varepsilon)^2}.$$

The continuity of $b(r)$ and a^2 now implies that

$$b''(a_0) < -\frac{8M}{a_0^2}. \quad (19)$$

This is the stability criterion.

5. Discussion

This paper discusses the stability of Morris-Thorne and other traversable wormholes, each having the metric given by Eq. (1), where $\Phi(r)$ and $b(r)$ are the redshift and shape functions, respectively. The shape function is assumed to satisfy the usual flare-out conditions at the throat, while the redshift function is assumed to be finite. The interior traversable wormhole solution is joined to an exterior Schwarzschild solution at the junction surface $r = a$, where Φ and b must meet the conditions discussed in Sec. 3. Our main conclusion is that the wormhole is stable to linearized radial perturbations if $b = b(r)$ satisfies the following condition at the static solution $a = a_0$: $b''(a_0) < -8M/a_0^2$, where M is the total mass of the wormhole in one asymptotic region.

Since the curve $b = b(r)$ is concave down, $b''(a_0) < 0$, but its curvature has to be sufficiently large in absolute value to overtake $8M/a_0^2 = 4b(a_0)/a_0^2$. This condition is simple enough to suggest that the form of $b(r)$ can be easily adjusted "by hand."

A function that meets the condition locally can also be obtained by converting the above inequality to the differential equation

$$b''(r) + \frac{4b(r)}{r^2} = -\lambda,$$

where λ is a small positive constant. Confining ourselves to the interval $(a_1, a_0]$, a solution is

$$\bar{b}(r) = c\sqrt{r}\sin\left(\frac{1}{2}\sqrt{15}\ \ln r\right) - \lambda r^2.$$

To the left of a_1, $\bar{b}(r)$ can be joined smoothly to a function that meets the required conditions at the throat, thereby completing the construction.

References

[1] M. S. Morris, K. S. Thorne, Am. J. Phys. 56, 395 (1988)
[2] E. Poisson, M. Visser, Phys. Rev. D 52, 7318 (1995)
[3] J. P. S. Lemos, F. S. N. Lobo, S. Q. de Oliveira, Phys. Rev. D 68, 064004, (2003)
[4] F. S. N. Lobo, Classical Quant. Grav. 21, 4811 (2004)
[5] F. S. N. Lobo, Classical Quant. Grav. 23, 1525 (2006)
[6] F. S. N. Lobo, P. Crawford, Classical Quant. Grav. 22, 4869 (2005)
[7] J. P. S. Lemos, F. S. N. Lobo, Phys. Rev. D 78, 044030 (2008)
[8] M. Ishak, K. Lake, Phys. Rev. D 65, 044011 (2002)
[9] F. S. N. Lobo, P. Crawford, Classical Quant. Grav. 21, 391 (2004)
[10] F. S. N. Lobo, Phys. Rev. D 71, 124022 (2005)
[11] H. Shinkai, S. A. Hayward, Phys. Rev. D 66, 044005 (2002)
[12] K. A. Bronnikov, G. Clement, C. P. Constantinidis, J. C. Fabris, Phys. Lett. A 243, 121 (1998)
[13] J. A. Gonzales, F. S. Guzman, O. Sarbach, Classical Quant. Grav. 26, 015010 (2009)
[14] J. A. Gonzales, F. S. Guzman, O. Sarbach, Classical Quant. Grav. 26, 015011 (2009)
[15] E. F. Eiroa, C. Simeone, Phys. Rev. D 76, 024021 (2007)
[16] R. A. Konoplya, C. Molina, Phys. Rev. D 71, 124009 (2005)
[17] S. V. Sushkov, Y.-Z. Zhang, Phys. Rev. D 77, 024042 (2008)
[18] S. Sushkov, Phys. Rev. D 71, 043520 (2005)
[19] F. S. N. Lobo, Phys. Rev. D 73, 064028 (2006)
[20] W. Israel, Nuovo Cimento 44B, 1 (1966) (corrections in 48B)
[21] M. Visser, Lorentzian Wormholes: From Einstein to Hawking (American Institute of Physics, New York, 1995) chapter 14
[22] L. Schwartz, Théorie des distributions (Hermann & Cie, Paris, 1950)