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Generalized and potential symmetries of the Rudenko–Robsman equation

Research Article

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Abstract: The paper presents a concrete study of the existence of generalized and potential symmetries for the 1+1 dimensional version of the Rudenko–Robsman equation, an interesting fourth-order partial differential equation that describes the evolution of nonlinear waves in a dispersive medium. As the main results, the existence of a two-parameter algebra of generalized symmetries and of an infinite-dimensional algebra when potential symmetries are taken into account is proven.

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1. Introduction

Symmetry-group methods and their recent generalizations proved to be useful in establishing complete integrability of certain systems of differential equations [1–3]: They allow us to understand the dynamics of concrete physical systems, to compute their conservation laws, as well as to construct exact solutions. For physicists, these results are very important in the study of concrete nonlinear dynamical systems with a finite or an infinite number of degrees of freedom. In recent years considerable attention has been devoted to applications of symmetry-group methods to a large variety of physical phenomena described by second- or third-order nonlinear partial differential equations, but relatively few complete results have been obtained for fourth-order evolution equations. For example, the Calabi flow equation has been studied, which appears in general relativity for describing spherical gravitational waves in vacuum. Another fourth-order equation with important physical applications is the Bretherton equation, which describes the propagation of an air finger into a channel with a rigid wall [4].

Recently, we initiated the study of symmetry properties and invariant quantities for some fourth-order equations arising in physics [5], using the general methodology presented in [6] and [7]. This paper intends to give a description of all arbitrary-order generalized symmetries for the Rudenko–Robsman (RR) equation

$$w_t = \alpha w w_x - \beta w_{xxxx}\,, \qquad (1)$$

as an alternative approach to study special classes of solutions and possible nontrivial linearizations of this model. The RR equation was introduced in 2002 for describing the nonlinear wave propagation in scattering media that

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are characterized by weak sound-signal attenuation proportional to the fourth power of frequency in a coordinate system moving at the speed of sound [8]. The first physical interpretation of the RR equation was that it describes shock waves in liquids containing gas bubbles, where the coefficient for sound damping has the form $\beta\omega^4$ and the dispersion equation for the corresponding infinitesimal perturbations takes the form

$$k = \frac{\omega}{c} + i\beta\omega^4 . \tag{2}$$

Here, c is the speed of sound and ω is the frequency.

Signal attenuation proportional to the frequency to the fourth power was also observed in media containing small inhomogeneities, for example, in rocks, in spongy cranial bones, or in any media with small-scale parameter fluctuations for which the Rayleigh scattering law is valid [8, 9]. It is interesting to note that the RR equation presents some similarities with the second-order Burgers equation. Indeed, both equations allow stationary self-similar solutions, travelling-wave solutions, and rational solutions (see Chapter 10 in [4]). On the other hand, a general study of the RR equation [10] shows that there are stationary solutions of this equation in the form of a shock wave that exhibit unusual oscillations around the shock front. This is a distinct feature of the classical Burgers equation. The explicit analysis of the profile of initially sinusoidal-wave solutions and their attenuation was given in [11]. The mechanisms responsible for anomalously strong acoustic nonlinearities in the case of the RR equation was presented in some recent papers of Rudenko [9, 12].

This paper has the following structure. After this introductory part, a short presentation of the general method that can be used for finding generalized symmetries will be given in the second section. The third section is dedicated to the concrete study of the existence of generalized symmetries for the Eq. (1), and the fourth section presents a complementary study of the potential nonclassical symmetries for the same equation. In the case of the generalized symmetries, we have shown that the system accepts only two independent symmetry operators, which have the form of classical, point-like symmetries. By contrast, there is an infinite set of generators for the potential symmetries of the equation, generators given by two arbitrary functions $a(t)$ and $b(t)$. Comments on these results are presented in the last section of the paper.

2. Generalized symmetries of evolution equations

The usual symmetries encountered in the study of differential equations are commonly referred to as point, or classical, symmetries. A point symmetry of a system of differential equations is a 1-parameter group of transformations of independent and dependent variables that carries any solution of the equations to another solution. For differential equations derived from a variational principle, the point symmetries that preserve the action lead to conservation laws. However, not all conservation laws are given by point symmetries. To account for all conservation laws in Lagrangian field theory one must enlarge the notion of symmetry to include generalized symmetries.
A *generalized symmetry* is an infinitesimal transformation, constructed locally from the independent variables, the dependent variables, and the *derivatives* of the dependent variables, that carries solutions of the differential equations to nearby solutions. The importance of generalized symmetries is underlined by their role in deciding on the complete integrability of non-linear differential equations. In particular, when a system of differential equations is integrable, it generally admits "high-orders" generalized symmetries [3].
Consider the nth order evolution equation:

$$\Delta = w_t - K\left(t, x, w, w_x, \cdots, w_x^{(n)}\right) = 0 , \tag{3}$$

where w_t, w_x means the time, respectively space, derivative of the dependent variable $w = w(t, x)$, and $w_x^{(n)}$ is the nth-order derivative.
The *point-symmetry* analysis considers the one-parameter Lie group of infinitesimal transformations in (x, t, w) given by

$$\begin{aligned} x^* &= x + \varepsilon\xi(x, t, w) + O\left(\varepsilon^2\right) , \\ t^* &= t + \varepsilon\tau(x, t, w) + O\left(\varepsilon^2\right) , \\ w^* &= w + \varepsilon\phi(x, t, w) + O\left(\varepsilon^2\right) , \end{aligned} \tag{4}$$

where ε is the group parameter. One requires that these transformations leave invariant the set of solutions of (3) $S_\Delta \equiv \{w(x, t) : \Delta = 0\}$. The associated Lie algebra is realized by vector fields of the form

$$\mathrm{U} = \xi(x, t, w)\partial_x + \tau(x, t, w)\partial_t + \phi(x, t, w)\partial_w . \tag{5}$$

The set S_Δ is invariant under the transformation (4) provided that

$$\mathrm{pr}^{(n)}\mathrm{U}(\Delta)|_{\Delta\equiv 0} = 0 , \tag{6}$$

where $\mathrm{pr}^{(n)}\mathrm{U}$ is the nth prolongation of the vector field (5), which is given explicitly in terms of ξ, τ, and ϕ in [3]. To look for *generalized symmetries* one means to considering all the solutions for ξ, τ and ϕ of the Eq. (6) depending on t, x, w, but on the derivatives of w too:

$$\mathbf{U} = \xi(x,t,w,w_x,\cdots)\partial_x + \tau(x,t,w,w_x,\cdots)\partial_t + \phi(x,t,w,w_x,\cdots)\partial_w\,. \quad (7)$$

The maximum order of derivatives of w gives the so-called *order* of the generalized (infinitesimal) symmetry.
Having determined the infinitesimals, the invariants I_α, $\alpha = 1,2$, are found by integrating the characteristic equations

$$\frac{dx}{\xi(x,t,w)} = \frac{dt}{\tau(x,t,w)} = \frac{dw}{\phi(x,t,w)}\,, \quad (8)$$

and they have the form $I_\alpha(x,t,w) = C_\alpha$, where C_α is a constant. The special invariant solutions of the initial equation corresponding to a generalized symmetry of the form (7) have the form $I_2 = \Theta(I_1)$, where Θ is an unknown function, and are obtained by substitution of the dependent variable $w(x,t)$ with $\Theta(I_1)$ in the initial equation. The new ODE obtained by this substitution is called *reduced equation*.
In practice, the Eq. (6) is very difficult to solve for a dependence on the derivatives of order higher than two, due to the large number of terms involved in the expression of the prolongation. An alternative version can be obtained by changing the form of the infinitesimal symmetry (7) to the equivalent *evolutionary* form

$$\mathbf{U}_Q = Q(t,x,w,w_x,\cdots)\partial_w\,, \quad (9)$$

where $Q = \phi - \xi w_x - \tau w_t$ is the characteristic of the generalized vector field $\mathbf{U}$.
The symmetry Eq. (6) for the initial Eq. (3) can be rewritten as:

$$\left(D_t - K'\right)\, Q = 0\,, \quad (10)$$

where D_t is the total time derivative (the evolutionary derivative)

$$D_t Q = \partial_t Q + \sum_{i=0}^{\infty} \frac{\partial Q^\alpha}{\partial w_x^{(i)}} D_x^i(w_t) = \partial_t Q + Q'(K)\,,$$

and K' denotes the Frechet derivative of K:

$$K' = \sum_{i=0}^{\infty} \frac{\partial K}{\partial w_x^{(i)}} D_x^i\,.$$

The simplest form of the symmetry condition (10) is:

$$\partial_t Q + Q'(K) = K'(Q)\,. \quad (11)$$

Usually, one chooses a particular order m for the characteristic

$$Q = Q\left(t,x,w,w_x,...,w_x^{(m)}\right)\,,$$

and one searches for solutions Q of the Eq. (11) by identifying the coefficients of all the corresponding monomials expressed in w and its x-derivatives.

3. Generalized symmetries of the RR equation

We are looking now for generalized symmetries of the form (9) for the RR Eq. (1) in the particular case $\alpha = \beta = 1$ [4]. By an appropriate scaling of the independent variables, the original equation can be reduced to the form:

$$w_t - ww_x + w_{xxxx} \equiv w_t - K\left(t,x,w,w_x,\cdots,w_x^{(4)}\right) = 0\,. \quad (12)$$

The symmetry Eq. (11) then becomes:

$$\frac{\partial Q}{\partial t} + \sum_{i=0}^{m} \frac{\partial Q}{\partial w_x^{(i)}} D_x^i\left(K\left(t,x,w,\cdots,w_x^{(4)}\right)\right) = \sum_{i=0}^{4} \frac{\partial K}{\partial w_x^{(i)}} D_x^i\left(Q\left(t,x,w,\cdots,w_x^{(m)}\right)\right)\,. \quad (13)$$

We have to note that it is not at all compulsory that the maximum order m of derivatives that appear in Q should be the same as the order of the studied equation (in our case 4). We will search for solutions of (13) supposing that the order m of $Q[w]$ is successively 0, 1, 2,..., and splitting Eq. (13) into a system of coefficients of the highest-order derivatives of w.
Let us start with the 0th order, where we have to consider $Q = Q(t,x,w)$. Eq. (13) holds if and only if

$$\begin{aligned} \frac{\partial Q}{\partial t} &= 0, \\ \frac{\partial^2 Q}{\partial w^2} &= 0, \\ D_x \frac{\partial Q(t,x,w(t,x))}{\partial w} &= 0, \\ \frac{\partial Q}{\partial t} + Qw_x &= 0, \end{aligned} \quad (14)$$

for all the solutions w of Eq. (12). The second and the third equations imply that Q is linear in w and Q_w does not depend on x, that is $Q = f(t)w + g(t,x)$. From the last equation it is simple to conclude that $f = g = 0$.

At order 1, consider $Q = Q(t, x, w, w_x)$. Eq. (13) implies a system of 9 partial differential equations in Q, which can be reduced to

$$\frac{\partial^2 Q}{\partial w_x^2} = \frac{\partial^2 Q}{\partial w^2} = \frac{\partial^2 Q}{\partial w \partial w_x} = \frac{\partial^2 Q}{\partial w \partial x} = \frac{\partial^2 Q}{\partial w_x \partial x} = 0 \qquad (15)$$

and

$$\frac{\partial Q}{\partial t} + Q_x = 0 \qquad (16)$$

for all the solutions w of Eq. (12). The first and second equations give $Q \equiv f(t, x, w) w_x + g(t, x, w)$ under the supplementary conditions (derived from the others equations):

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial w} = \frac{\partial g}{\partial x} = \frac{\partial g}{\partial w} = \frac{\partial g}{\partial t} = 0, \quad \frac{\partial f}{\partial t} = -g. \qquad (17)$$

Eqs. (17) admits the solution $f(t) = -bt + a$ and $g = b$, where a and b are arbitrary constants. One obtains the characteristic $Q_1 = a(1 - tw_x) + bw_x$ and the generalized evolutionary symmetries

$$\mathrm{U}_{Q_1} = [a(1 - tw_x) + bw_x]\partial_w\,. \qquad (18)$$

It is a two-parameters symmetry operator spanned by two infinitesimal symmetries:

$$\mathrm{U}_1 = \partial_x, \quad \mathrm{U}_2 = t\partial_x + \partial_w\,. \qquad (19)$$

The first one of this operators represents a classical translation along the x-axis, and it is useful to determine some special solutions of the RR equation. The characteristic Eq. (8) for U_2 is

$$\frac{\mathrm{d}x}{t} = \frac{\mathrm{d}t}{0} = \frac{\mathrm{d}w}{1} \qquad (20)$$

and determines two invariants $I_1 = t$ and $I_2 = x/t - w$. Now, if we look for some solution of the RR Eq. (12) of the form $w(x, t) = x/t - \Theta_{[x]}(t)$, where the variable x is considered here as a simple parameter, one obtains the reduced equation

$$\Theta'_{[x]}(t) = \Theta_{[x]}(t) - \frac{2x}{t^2}\,. \qquad (21)$$

The solution of (21), obtained with MAPLE, is

$$\Theta_{[x]}(t) = \frac{2x}{t} - 2\mathrm{e}^t x \int_{-\infty}^{t} \frac{\mathrm{e}^{st}}{s}\mathrm{d}s + A\mathrm{e}^t\,, \qquad (22)$$

where A is an arbitrary number.

For order 2, if we consider $Q = Q(t, x, w, w_x, w_{xx})$, Eq. (13) implies an over-determined system of differential equations for Q. We find the following conditions among these equations:

$$\frac{\partial^2 Q}{\partial w_{xx}^2} = \frac{\partial^2 Q}{\partial w_{xx} \partial w_x} = \frac{\partial^2 Q}{\partial w_{xx} \partial w} = \frac{\partial^2 Q}{\partial w_{xx} \partial x} = 0\,. \qquad (23)$$

So we have to consider $Q = h(t) w_{xx} + k(t, x, w, w_x)$. Substituting this form of Q in (13), the term containing the higher-order derivative in w leads to

$$h w w_{xx} w_{xxxxx} = 0\,. \qquad (24)$$

As (24) must be valid for any w, it is compulsory to set $h \equiv 0$. So Q cannot depend on second-order derivatives of w.

In general, if we consider $Q = Q\left(t, x, w, w_x, ..., w_x^{(n)}\right)$ with $n > 2$, the symmetry-determining Eq. (13) is

$$\frac{\partial Q}{\partial t} + \sum_{i=0}^{n} \frac{\partial Q}{\partial w_x^{(i)}} D_x^i (w w_x - w_{xxxx}) = Q_t + w_x Q + w D_x Q - D_x^4 Q = 0. \qquad (25)$$

This equation can be split by a procedure similar to that in the case $n = 2$ into an over-determined system that contains all the equation of the form:

$$\frac{\partial^2 Q}{\partial w_x^{(i)} \partial w_x^{(j)}} = 0 \qquad (26)$$

(for any $i = 2, \ldots, n$ and $j = 0, \ldots, n$, resulting in the explicit expression of the right-hand side of Eq. (25)). We conclude that Q must be linear in $w_x^{(n)}$ and of the form:

$$Q = f(t, x) w_x^{(n)} + g(t, x, w, w_x)\,. \qquad (27)$$

Inserting this expression of Q into (25), one can isolate the term containing the higher-order derivative in w:

$$4 f_x w_x^{(n+3)}\,. \qquad (28)$$

As (25) must be valid for any w, it is compulsory to take $f = f(t)$.

Using the particular expression $Q = f(t) w_x^{(n)} + g(t, x, w, w_x)$ in (25) and splitting the obtained equation into monomials expressed in the derivatives of w, one obtains a system that contains

$$f w w_x^{(n+1)} = 0\,. \qquad (29)$$

Then $f \equiv 0$, and the RR equation cannot have generalized evolutionary symmetries of order $n > 1$. All the generalized symmetries of Eq. (12) are spanned by two very simple symmetry operators: $\mathrm{U}_1 = \partial_x$ and $\mathrm{U}_2 = -t\partial_x + \partial_w$.

4. Potential symmetries of the RR equation

In [1], a method for finding new classes of symmetries for a PDE has been described. By writing a given PDE, denoted by $\Delta[x,t,w] = 0$, in a conserved form, a related system denoted by $S[x,t,w,v] = 0$ with potential v as an additional dependent variable is obtained. If a set $\{w(x,t), v(x,t)\}$ satisfies $S[x,t,w,v] = 0$, then $w(x,t)$ solves $\Delta[x,t,w] = 0$, and $v(x,t)$ solves the *integrated related equation* $T[x,t,v] = 0$. Any Lie group of point transformations admitted by $S[x,t,w,v] = 0$ induces a symmetry for $\Delta[x,t,w] = 0$. When at least one of the generators of the group depends explicitly on the potential v, then the corresponding symmetry is neither a point nor a Lie–Backlund symmetry. These symmetries of $\Delta[x,t,w] = 0$ are called potential symmetries.
We will now apply this method to the case of the RR equation. We will follow the steps in [13], where the potential symmetries of the Burgers equation

$$u_t - uu_x + u_{xx} = 0$$

have been studied. In order to find the potential symmetries of the RR Eq. (12), we will rewrite the equation in a conserved form:

$$w_t - \left(\frac{1}{2}w^2 - w_{xxx}\right)_x = 0\,. \qquad (30)$$

From this conserved form, the associated auxiliary system $S[x,t,w,v] = 0$ is given by:

$$\left\{\begin{array}{c} v_x = w, \\ v_t = \frac{1}{2}w^2 - w_{xxx}. \end{array}\right. \qquad (31)$$

The integrated related equation is obtained by eliminating the w variable from (31):

$$v_t - \frac{1}{2}v_x^2 + v_{xxxx} = v_t - K_1[v] = 0\,. \qquad (32)$$

We are looking for the generalized symmetries of the integrated related Eq. (32). The symmetry Eq. (11) for the mth order generalized symmetry U_Q is written as:

$$\frac{\partial Q}{\partial_t} + \sum_{i=0}^{m} \frac{\partial Q}{\partial v_x^{(i)}} D_x^i(K_1[v]) = \sum_{i=0}^{4} \frac{\partial K_1}{\partial v_x^{(i)}} D_x^i(Q[v]) = 0\,. \qquad (33)$$

The analysis of the existence of the generalized symmetries for (32) is similar to the case of the symmetries of (12) described in the previous section.

At order 0, if we consider $Q = Q(t,x,v)$, Eq. (33) holds if and only if $Q = 0$. In the case when $Q = Q(t,x,v,v_x)$, Eq. (33) gives rise to a system of determining equations for the characteristic Q of the evolutionary symmetry $Q\frac{\partial}{\partial v}$; the first of these are:

$$\frac{\partial^2 Q}{\partial v_x^2} = \frac{\partial^2 Q}{\partial v_x \partial v} = \frac{\partial^2 Q}{\partial v_x \partial x} = \frac{\partial^2 Q}{\partial v^2} = 0\,. \qquad (34)$$

These equations give $Q \equiv f(t,x)v_x + g(t,x,v)$ under the supplementary conditions derived from Eq. (33):

$$\frac{\partial^2 g}{\partial x^2} = 0, \quad \frac{\partial f}{\partial t} = -\frac{\partial g}{\partial x}, \quad \frac{\partial f}{\partial x} = 0\,. \qquad (35)$$

The general solution of the system (35) is

$$f(t) = a(t) + c, \quad g(x,t) = -a'(t)x + b(t)$$

for arbitrary regular functions $a(t)$ and $b(t)$, and a constant c.
Consequently, the evolutionary generalized symmetry is

$$\mathrm{U} = [(a(t) + c)v_x + (b(t) - a'(t)x)]\partial_v\,. \qquad (36)$$

Then the integrated Eq. (32) admits an infinite-parameter Lie group of point symmetries corresponding to the infinitesimal generators:

$$\left\{\begin{array}{rcl} \mathrm{U}_1 &=& \frac{\partial}{\partial x}, \\ \mathrm{U}_3^{(a(t))} &=& -a(t)\frac{\partial}{\partial x} - xa'(t)\frac{\partial}{\partial v}, \\ \mathrm{U}_4^{(b(t))} &=& b(t)\frac{\partial}{\partial v}. \end{array}\right. \qquad (37)$$

The first generator correspond to a general translation along the x-axis, and it was already identified in the previous section. The other two operators, $\mathrm{U}_3^{(a(t))}$ and $\mathrm{U}_4^{(b(t))}$, define an infinite algebra and can be identified as new, potential-type symmetries for the RR Eq. (12).
With an approach similar to that for the case of generalized symmetries of RR equation, we can prove that Eq. (32) does not posses any other higher-order potential symmetries.

5. Concluding remarks

We investigated the problem of the existence of generalized and potential symmetries of the RR equation using the classical Lie approach and the Bluman complementary method. Important in itself, the RR equation also represents at the same time a good toy model of a fourth-order differential equation that can be investigated with these techniques. The symmetries of other fourth-order equations, as for example Calabi flow equation, have been investigated in [7]. The main results we obtained for the RR equation can be summarized as follows:

(i) The group of classical Lie symmetries for the RR Eq. (12) is generated by two operators $\mathbf{U}_1 = \partial_x$, which is a simple translation along the x-axis, and $\mathbf{U}_2 = t\partial_x + \partial_w$, which is a non-trivial symmetry. The class of solution invariants for the group of symmetries generated by $\mathbf{U}_2$ have the form (see (22)):

$$w(x,t) = -\frac{x}{t} + 2e^t x \int_{-\infty}^{t} \frac{e^{st}}{s} ds - Ae^t , \qquad (38)$$

where A is an arbitrary parameter.

(ii) The RR equation admits an infinite-parameter Lie group of potential symmetries, generated by the infinitesimal generators $\mathbf{U}_3^{(a(t))}$ and $\mathbf{U}_4^{(b(t))}$ from (37). We have to note that in our approach we have used a form of the generalized symmetry operator $\mathbf{U}$ in terms of its characteristic Q, as in (9). The determination of explicit conservation laws and the search for solutions associated with the symmetries we found will be done in a forthcoming paper.

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