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On the analytic properties of the S-matrix for the unknown interactions surrounded by centrifugal and rapidly decreasing potentials

Review Article

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Abstract: The analytic structure of the non-relativistic unitary and non-unitary S-matrix is investigated for the cases of the unknown interactions with the unknown motion equations inside a sphere of radius a, surrounded by the centrifugal and rapidly decreasing (exponentially or by the Yukawian law or by the more rapidly decreasing) potentials. The one-channel case and special examples of many-channel cases are considered. Some kinds of symmetry conditions are imposed. The Schroedinger equation for $r > a$ for the particle motion and the condition of the completeness of the correspondent wave functions are assumed. The connection of the obtained results with the usual (temporal) causality is examined. Finally a scientific program is presented as a clear continuation and extension of the obtained results.

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1. Introduction

A lot of papers and monographs on non-relativistic quantum collision theory are dedicated to the analysis of the solutions of the Schroedinger equation and of the analytical properties of the correspondent S-matrix for various potentials of different forms, extended in all the three-dimensional space with a radial coordinate along the axis $(0, \infty)$. Only a rather small number of papers are concentrated on the study of the analytical properties of the S-matrix with the minimal number of assumptions on the interaction properties at small distances (practically none, with the exception of very general physical and mathematical principles, such as certain symmetry properties, causality or the condition of the completeness of the wave functions at the external interaction range, and also the possibility of the S-matrix analytic continuation at the complex plane of the kinetic energies or of the wave numbers). This approach arises from the old idea of Heisenberg [1, 2] (see also [3, 6–8] and precedent references therein) on the unique fundamental quantity (the S-matrix) which will be sufficient for the predictions of many observable quantities based only on general physical and mathematical principles. However, up to now there is no manual, or practically even any paper, for arbitrary or unknown interactions inside a sphere of small

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radius, with reliably established properties of the S-matrix or scattering amplitude. There are though a lot of papers and manuals for various known potentials along the whole radial axis, beginning from 0, for the Shroedinger equation, strong-interaction, great-unification, or super-great-unification interaction dynamical models, with various submicroscopic assumptions for the π-meson condensate or quark-gluon plasma etc. One can hope that accumulations and reviews of all possible results of rare publications on the general analytic properties of the S-matrix for various kinds of arbitrary or unknown interactions at small distances (central, non-central, P-violating, CP– or T-violating interactions and so on), up to the violations of the unitarity of the S-matrix, will be very useful for forthcoming research in quantum mechanics (both non-relativistic and relativistic), quantum field theory, quantum gravity etc.

Now we shall outline the main results of [7, part II] for the unitary S-matrix, since this will be an initial base for the further review of the results of papers [9–16]. Namely in [7, part II] the analytical expression of the function $S_l(k)$ was obtained, which defines the relation between the amplitudes of ingoing and outgoing l-waves for the elastic scattering of non-relativistic particles without spin (with $l = 0$) for arbitrary interaction, localized inside the sphere of radius a, starting from the unitary condition

$$S_l(k)S_l^*(k^*) = 1, \tag{1}$$

the symmetry condition

$$S_l(k)S_l(-k) = 1 \tag{2}$$

or

$$S_l^*(k)S_l(-k^*) = 1 \tag{3}$$

and the particular "causality" condition (*if the ingoing wave packet is normalized so, that at $t = -\infty$ it represents one particle, then the total probability of finding the particle in any successive time moment (for instance, $t = 0$) outside the interaction sphere cannot be more than* 1). Strictly speaking, this condition is not the usual causality but the conservation of the total probability. In [17, 18] it was shown that it does directly follow from the orthogonality of the eigen functions of a self-adjoint operator, describing the motion and interaction of the colliding particles.

Then, the following had also been assumed: the existence of the analytic continuation of $S_l(k)$ into the complex plane of k and the condition of the quadratic integrability of the weight functions of the wave packets, which in turn ensured the uniform convergence (at the range $r > a$) of the integrals over momentum in the Fourier-expansions of the wave packets. Finally the following expression for $S_0(k)$ was obtained:

$$S_0(k) = \exp(-2ik\alpha)\prod_{\lambda}\frac{k_\lambda - k}{k_\lambda + k}\prod_{s}\frac{(k_s - k)(k_s^* + k)}{(k_s^* - k)(k_s + k)}, \tag{4}$$

where $\alpha \leqslant a, k_\lambda$ are zeros on the imaginary axis (which are simple on the lower semi-axis), k_s are the zeros in the upper half-plane D^+, the products $\prod_\lambda$ and $\prod_s$ converge on the real axis k. In [19] it was shown that zeros k_s on the lower and upper imaginary semi-axes and zeros ks correspond to bound, virtual (anti-bound) and resonance states, respectively.

If the interaction is described by a local central potential $V(r)$, independent from k, and the conditions

$$\int_0^\infty drr^n\,|V(r)| < \infty, \quad n = 1, 2, \tag{5}$$

and

$$V(r) \equiv 0 \quad \text{for } r > a, \tag{6}$$

are fulfilled, the expression (4) is also valid for arbitrary values of l, with $\alpha = a$ and the product over λ contains a finite number of poles on the upper imaginary semi-axis. But if only condition (5) is fulfilled, then expression (4) is, generally speaking, invalid and one does often use the following expression

$$S_l(k) = \frac{f_{l-}(k)}{f_{l+}(k)}, \tag{7}$$

where $f_{0\pm}(k) = f_{0\pm}(k, 0)$ for $l = 0$ and

$$f_{l\pm}(k) = \frac{k^l\exp(\pm il\pi/2)}{(2l-1)!!}\lim_{r\to 0} r^l f_{l\pm}(k, r)$$

for $l > 0$, $f_{l\pm}(k, r)$ is the solution of the radial Schroedinger equation or of the equivalent integral equation

$$\begin{aligned} f_{l\pm}(k, r) &= \pm i\exp(\pm il\pi/2)krh_l^{(1,2)}(kr) \\ &- \frac{2\mu}{\hbar^2 k}\int_0^\infty dr' g_l(k; r, r')V(r')f_{l\pm}(k, r'), \end{aligned} \tag{8}$$

with the boundary condition

$$\lim_{r\to\infty} f_{l\pm}(k, r)\exp(\mp ikr) = 1, \tag{9}$$

where

$$g_l(k;r,r') = \frac{ikrr'}{2}\left[h_l^{(1)}(kr')h_l^{(2)}(kr) - h_l^{(1)}(kr)h_l^{(2)}(kr')\right],$$

and $h_l^{(1,2)}(kr) = j_l(kr) \pm in_l(kr)$ are the Hankel spherical functions of the first and the second kind, respectively ($j_l(kr), n_l(kr)$ are the Bessel and the Neiman spherical function, respectively). At such conditions the function $S_l(k)$ can have, besides the singularities described by (4), additional singularities, corresponding to the singularities of $f_{l\pm}(k,r)$.

The author's (partly with his collaborators) papers [9–16] present the results of that approach, published gradually during 1961- 2006 (mainly in the Russia and Ukraine), and can evidently be continued in the future. In the final section of this paper a scientific program is presented which connects the remaining tasks, problems and also the gradually revealed perspective, unexpected previously - of how the rigorous mathematical method or approach can help to reveal quite concrete and sometimes paradoxical physical phenomena.

2. The properties of the non-unitary one-channel S-matrix for the unknown interactions surrounded by the centrifugal barrier and a potential, which is decreasing more rapidly then any exponential function

Now, following [10], we consider a generalized case when the interaction and motion equation inside the sphere of radius a are unknown as before, but at $r > a$ contains the centrifugal barrier $\hbar^2 l(l+1)/r^2$ and a potential $V(r)$, and there is not only a scattering but also a partial particle absorption or generation. For convenience let us introduce new interaction characteristics - a complex "interaction constant" γ. We agree conventionally that its real part Re γ will characterize that interaction part which causes by itself only scattering without particle absorption or generating. And we agree to set up the negative (positive) value of Im γ in correspondence with that interaction part, the absence of which causes the absence of the particle absorption (generating). If we further connect the particle absorption and generating with the simple decreasing or increasing of the flux of the scattered particles in comparison with the flux of bombarding particles, assuming the conservation of their impulse and other characteristics, then it will be natural to impose the following conditions:

$$0 < |S_l(\gamma,k)|^2 \leqslant 1, \qquad (10a)$$

$$1 \leqslant |S_l(\gamma^*,k)|^2 < \infty, \qquad (10b)$$

with $\Im\gamma < 0$, for real positive k. Since the conditions (10a) and (10b) are evidently insufficient for the study of the analytic properties of $S_l(\gamma,k)$, let us introduce, generalizing (1)-(3), the new symmetry properties (typical for central interactions)

$$S_l(\gamma,k)S_l(\gamma,-k) = 1, \qquad (2a)$$

$$S_l(\gamma^*,k)S_l(\gamma^*,-k) = 1, \qquad (3a)$$

and the generalized "unitarity" condition

$$S_l(\gamma,k)S_l^*(\gamma^*,k^*) = 1, \qquad (1a)$$

thus selecting for any interaction with constant $\gamma(\Im\gamma,0)$ the "conjugate" interaction with the complex conjugate constant γ^*.

One can easily check that the conditions (1a), (2a), (3a) and (10a), (10b) are automatically fulfilled in the case when the interaction can be described by the complex potential which satisfies the condition (5) [19, 20]. In that case the values γ and γ^* are not only conventional but also factual parameters of the potential $V(\gamma,r) = \Re[\gamma V_1(r)] + i\Im[\gamma V_2(r)]$.

Instead of the "causality" condition from [7, part II], we shall use the condition of completeness for the wave functions outside the sphere of unknown interaction, factually assuming in this region (i.e. for $r \geqslant a$) the possibility of describing the colliding particles by the Schroedinger equation with a self-adjoint Hamiltonian:

$$\frac{2}{\pi}\int_0^\infty k^2 dk R_l^{(+)}(\gamma,k,r)R_l^{(+)*}(\gamma,k,r') + \sum_n R_{nl}(\gamma,k_{nl},r)R_{nl}(\gamma,k_{nl},r') = \frac{\delta(r-r')}{r^2}, \qquad (11)$$

where

$$R_l^{(+)}(\gamma,k,r) = \frac{i}{2kr}\left[f_{l-}(k,r)\exp(il\pi/2) - S_l(\gamma,k)f_{l+}(k,r)\exp(-il\pi/2)\right],$$

$$R_{nl} = \frac{1}{\sqrt{2\pi}}B_{nl}(\gamma,k_{nl})f_{l+}(k_{nl},k)/r,$$

functions $f_{l\pm}(k,r)$ are defined by Eq. (8); $\Im k_{nl} > 0$ and consequently the functions R_{nl} are integrable together with their squares (at least, at the range $a \leqslant r < \infty$); all the information on the interaction inside the sphere with radius $r < a$ is contained in the functions $S_l(\gamma,k)$

and the constants $B_{nl}(\gamma, k_{nl})$. Let us note that we (tacitly) assume that $R_{nl}(\gamma, k_{nl}, r) = R^{\bullet}_{nl}(\gamma^{\bullet}, k^{\bullet}_{nl}, r)$ in (11).

Eq. (11) represents a generalization of the completeness relation for the eigen functions of the most simple classes of the non-Hermitian Hamiltonians [19] for the cases when all the eigen values k_{nl} are simple (non-multiple) and are situated outside the real axis k. When $\gamma = \Re\gamma$ the functions R_{nl} simply describe the bounded states of the system. For the complex values of γ they have the same boundary conditions as the bound states, and their properties for the non-singular potentials with the negative imaginary part are partially described in [21].

In order to be sure that $S_l(\gamma, k)$ can have the analytic continuation into the complex plane of k, one has to impose some limitations on the potential tails at the range of $r > a$. In correspondence with the study of the potential scattering in [19–22], we can try, at least, to limit ourselves by the cases when at the range $r > a$, besides the centrifugal barrier, there is present a potential which satisfies the following condition

$$\int\limits_0^\infty dr r \, |V(r)| \exp(br) < \infty, \tag{12}$$

at least, for sufficiently large b.

Using the properties

$$f^*_{l+}(k^{\bullet}, r) = f_{l+}(-k, r) = f_{l-}(k, r) \tag{13}$$

for real k and relations (1a), (2a) and (3a) for $S_l(\gamma, k)$, one can transform (11) into the form

$$\begin{aligned} &\frac{1}{rr'} \int\limits_C dk f_{l+}(k, r) f_{l-}(k, r') \\ &- \frac{(-1)^l}{rr'} \int\limits_C dk S_l(\gamma, k) f_{l+}(k, r) f_{l+}(k, r') \\ &+ \frac{1}{rr'} \sum_n (B_{nl})^2 f_{l+}(k_{nl}, r) f_{l+}(k_{nl}, r') = \frac{2\pi\delta(r - r')}{r^2}, \end{aligned} \tag{14}$$

where the integration trajectory C goes along the real axis k from $-\infty$ to ∞, bypassing the point $k = 0$ where $f_{l\pm}$ have the pole of the l—th order by a semi-circle of the infinitesimal small radius, located in the upper semi-space.

We shall limit ourselves by the case when $f_{l\pm}(k, r)$ behaves as $\exp(\pm ikr)$ in all the complex plane at $|k| \to \infty$. Then, shifting the integration contour into D^+, enclosing all the singularities and utilizing equalities

$$\begin{aligned} \int\limits_{\Gamma+} dk f_{l+}(k, r) f_{l-}(k, r) &= \int\limits_{\Gamma+} dk e^{ik(r-r')} \\ &= \int\limits_{-\infty}^{\infty} dk e^{ik(r-r')} = 2\pi\delta(r - r'), \end{aligned} \tag{15}$$

$$\int\limits_{\Gamma+} dk S_l(\gamma, k) f_{l+}(k, r) f_{l+}(k, r') = \int\limits_{\Gamma+} dk S_l(\gamma, k) e^{ik(r+r')}, \tag{16}$$

we obtain

$$\begin{aligned} &\sum_n \oint\limits_{k_m} dk f_{l+}(k, r) f_{l-}(k, r') + \\ &\sum_p \oint\limits_{\gamma_p} dk f_{l+}(k, r) f_{l-}(k, r') - \\ (-1)^l &\sum_j \oint\limits_{k_j} dk S_l(\gamma, k) f_{l+}(k, r) f_{l+}(k, r') - \\ (-1)^l &\sum_q \oint\limits_{\gamma_q} dk S_l(\gamma, k) f_{l+}(k, r) f_{l+}(k, r') - \\ (-1)^l &\sum_n (B_{nl})^2 f_{l+}(k_{nl}, r) f_{nl}(k_{nl}, r') = 0, \end{aligned} \tag{17}$$

where $\int\limits_{\Gamma+}$ is the integral over the infinitely large semi-circle above the real axis, $\oint\limits_{k_n}$ is the integral over an infinitesimal circle around an isolated singular point, $\oint\limits_{\gamma_p}$ is the integral over a contour which envelops a non-isolated singularity (for instance, over the edges of the cut conducted for a branch point). Since all these contours are independent, equality (17) is equivalent to the following system of equalities:

$$\begin{aligned} (-1)^l &\oint\limits_{k_n} dk S_l(\gamma, k) f_{l+}(k, r) f_{l+}(k, r') \\ &= (B_{nl})^2 f_{l+}(k_{nl}, r) f_{l+}(k_{nl}, r'), \end{aligned} \tag{18}$$

$$\begin{aligned} (-1)^l &\oint\limits_{k_n} dk S_l(\gamma, k) f_{l+}(k, r) f_{l+}(k, r') \\ &= \oint\limits_{k_n} dk f_{l+}(k, r) f_{l-}(k, r'), \end{aligned} \tag{19}$$

$$\begin{aligned} (-1)^l &\oint\limits_{\gamma_p} dk S_l(\gamma, k) f_{l+}(k, r) f_{l+}(k, r') \\ &= \oint\limits_{\gamma_p} dk f_{l+}(k, r) f_{l-}(k, r'), \end{aligned} \tag{20}$$

$$\int_{\Gamma} dk S_l(\gamma,k) e^{ik(r+r')} = 0. \tag{21}$$

A simple analysis of eq. (18) shows that $S_l(\gamma,k)$ has the poles of the first order on the positive imaginary semi-axis (which at $\gamma = \Re\gamma$ correspond to bound states) with residues $(-1)^{l+1} i \frac{(B_{nl})^2}{2\pi}$. Re-writing eq. (19) in the form

$$\oint_{k_n} dk f_{l+}(k,r) f_{l+}(k,r') \left[\frac{f_{l-}(k,r')}{f_{l+}(k,r')} - (-1)^l S_l(\gamma,k) \right] = 0, \tag{19a}$$

after simple reasoning one can easily conclude that $S_l(\gamma,k)$ has to have additional isolated singularities D^+, coincident with those isolated singularities of $f_{l-}(k,r)$ in the upper semi-space near which

$$\lim_{k\to k_{mn}} f_{l-}(k,r) = \lim_{k\to k_{mn}} D_m(k) f_{l+}(k,r), \tag{22}$$

where the function $D_m(k)$ does not depend on r and has an isolated singular point k_m. Similarly one can study non-isolated singularities of $S_l(\gamma,k)$ coming from analysis of Eq. (20).

In the simplest case when outside the sphere of radius a there is only a centrifugal potential, functions $f_{l\pm}(k,r)$ have the form

$$f_{l\pm}(k,r) = (\pm i) \exp(\pm i l\pi/2) k r h_l^{(1,2)}(kr). \tag{8a}$$

Since functions $h_l^{(1,2)}(kr)$ are analytical in the whole complex plane k, with the exception of points $k = 0$ and ∞, then we can choose at $|k| \to \infty$

$$(\pm i) \exp(\pm i l\pi/2) k r h_l^{(1,2)}(kr) \underset{|k|\to\infty}{\to} \exp(\pm ikr) \tag{9a}$$

in the whole complex plane k, and so, in correspondence with Eq. (18)-(21), the function $\exp(2ik\alpha)S_l(\gamma,k), \alpha \leqslant a$, is regular everywhere in the whole D^+, except for the isolated singularities k_{nl} which for $\gamma = \Re\gamma$ are localized on the positive imaginary semi-axis. In this last case, we can find the product expansion of type (4), where points k_λ are the zeros k_{nl} on the lower imaginary semi-axis, corresponding to bound states, and also the zeros on the upper imaginary semi-axis, which define virtual (anti-bound) states and correspond to the poles situated between the poles k_{nl} and k_{n+1}, following an approach that was outlined in [12, 13, 17, 18].

For the complex values of γ the final result for $S_l(\gamma,k)$ can be also represented in analytical form. Considering that the zeros (poles) of $S_l(\gamma,k)$ in the first and the second quadrants do in consequence of the symmetry conditions (2a) correspond to the poles (zeros) in the third and the forth quadrants, mirror-like symmetrical to them relative to the direct lines $\Im k = \Re k$ and $\Im k = -\Re k$, respectively, we can find the product expansion of $S_l(\gamma,k)$, following an approach that was outlined in [14, 16]. Its derivation is performed in Appendix A, and the final forms obtained are:

$$S_l(\gamma,k) = \exp(-2i\alpha k) \prod_n \frac{k_{nl}+k}{k_{nl}-k} \prod_\lambda \frac{k_\lambda - k}{k_\lambda + k} \prod_s \frac{k_s - k}{k_s + k} \prod_{s'} \frac{k_{s'} - k}{k_{s'} + k}, \tag{23a}$$

$$S_l(\gamma^*,k) = \exp(-2i\alpha k) \prod_n \frac{k_{nl}^\bullet+k}{k_{nl}^\bullet-k} \prod_\lambda \frac{k_\lambda^\bullet - k}{k_\lambda + k} \prod_s \frac{k_s^\bullet - k}{k_s^\bullet + k} \prod_{s'} \frac{k_{s'}^\bullet - k}{k_{s'}^\bullet + k}, \tag{23b}$$

which generalizes (4), taking conditions (1a)-(3a) into account. Here k_{nl} are the poles in the lower half-space D^-, k_λ are the zeros in D^+, k_s and $k_{s'}$ are the zeros in the first and the second quadrants, respectively. The results (23a), (23b) had first been explicitly obtained in [10] and had not been analyzed before even for the simple interactions described by the complex potentials.

The simplified assumptions written above on the eigen values k_{nl} in the completeness condition (11) factually bring to an insignificant limitation of the interaction class. The absence of values k_{nl} on the real axis k, i.e. the absence of poles and zeros (spectral points) of $S_l(\gamma,k)$ and $S_l(\gamma^*,k)$ corresponding to them (as well as the absence of values of k_s and $k_{s'}$) does simply signify the rejection cases of the total absorption of bombarding particles and also the rejection of the infinite increasing of the new-particle birth for the physical values of $k \geqslant 0$. The condition of the absence of the eigen values k_{nl} with the multiplicity of more than 1 does not apparently bring to the essential limitation of the interaction class: Really, if one naturally assumes that a smooth change of the interaction parameter γ can bring to the smooth shift of the values k_{nl}, and so the arbitrarily small change of the parameter γ will bring to a certain small divergence of the various trajectories $k_{nl}(\gamma)$ from the point of the k_{nl} coincidence with the increasing of the multiplicity. In [12, 13] it was shown (with the help of another method) that expressions (23a), (23b) are valid for local potentials inside $r \leqslant a$ with a hard (infinite) core of radius $r_0 < a$, for non-local separable potentials of the type $v(r)v(r')$ with $0 < r, r' < a$, for non-local separable potentials with a hard(infinite) core of radius $r_0 < a$. And expressions (23a), (23b) were generalized for local complex potentials with multiple zeros - k_{nl}, k_λ and k_s. In the last case of (23a), (23b) there will be present factors of the type $\left(\frac{k_{nl}+k}{k_{nl}-k}\right)^{\alpha_{nl}} \left(\frac{k_\lambda-k}{k_\lambda+k}\right)^{\alpha_\lambda} \left(\frac{k_s-k}{k_s+k}\right)^{\alpha_s}$, where $\alpha_{nl}, \alpha_\lambda$ and α_s are the multiplicities of zeros - k_{nl}, k_λ and k_s, respectively.

If at the external region, when $r \geqslant a$, there are the centrifugal barrier and a potential, which is decreasing more rapidly then any exponential function, then the results (23a), (23b) remain valid since in this case the functions $f_{l\pm}(k, r)$ are analytical everywhere (see Appendix C), besides points $k = 0$ and ∞, and at the limit $|k| \to \infty$ they tend to $\exp(\pm ikr)$.

3. The properties of the non-unitary one-channel S-matrix for the unknown interactions surrounded by the centrifugal barrier and a potential, the tail of which is decreasing like an exponential function or an exponential function multiplied by a polynomial

If at the external region, where $r \geqslant a$, there are the centrifugal barrier and an exponential potential of the type $V = V_0 \exp(-br)$, $V_0, b > 0$, then the functions $f_{l\pm}(k, r)$ have the simple poles in points $k = \mp i\frac{b}{2}m \quad (m = 1, 2, \ldots)$ and at the limit $|k| \to \infty$ they tend to $\exp(\pm ikr)$. Similar results can be obtained for the Eckart, Hulthén and Woods-Saxon potentials [17, 18]. And if at the external region (with $r \geqslant a$) there are the centrifugal barrier and a potential of the type $V = V_0 P_n(r) \exp(-br)$, where $P_n(r)$ is an nth-order polynomial and $b > 0$, then the function $f_{l\pm}(-k, r)$ has poles of an order not higher than $n+1$ at the points $\mp ib/2, \mp ib, \mp 3ib/2, \ldots$, and it is analytic at all other points of the complex plane.

Moreover, in Appendix B the following theorem is proved: For $f_0(-k, r)$ to have poles of order not higher than (n_1+1) at the points $ib_1/2, ib_1, 3ib_1/2, \ldots$, not higher than (n_2+1) at the points $ib_2/2, ib_2, 3ib_2/2, \ldots$, not higher than (n_m+1) at the points $ib_m/2, ib_m, 3ib_m/2, \ldots$, it is necessary and sufficient that the corresponding potential would have a term $\sum_{n_m} P_{n_m}(r) \exp(-b_m r)$. (This theorem had first been proved in [9]). Then, in Appendix C, adopting from [16] and [19] the appropriate integral equation, which allows the computation of $f_l(-k, r)$ from $f_0(-k, r)$, it is shown that in this case $f_l(-k, r)$ has the same isolate singular points as $f_0(-k, r)$.

Thus, coming from [23–26], one will also in this case obtain the results (23a), (23b), where in $\prod_m$ one must include the factors, corresponding to “redundant” poles $\frac{i}{2}bm' \quad (m' = 1, 2, \ldots)$ of the first order at the presence of an exponential potential tail $V = V_0 \exp(-br)$, and the factors of the type $\prod_{m''} \left(\frac{k_{m''}-k}{k_{m''}+k}\right)^n$ corresponding to multiple “redundant” poles $\frac{i}{2}bm'' \quad (m'' = 1, 2, \ldots)$ at the presence of the potential tail of the type $V_0 \sum_n P_n(r) \exp(-br)$.

4. The properties of the non-unitary one-channel S-matrix for the unknown interactions surrounded by the centrifugal barrier and a potential, which is decreasing like a Yukawa potential

If at the external region, where $r \geqslant a$, there are the centrifugal barrier and a central Yukawa potential of the type $V = V_0[(br)^{-1} \exp(-br)]$, $V_0, b^{-1} \sim a$, we can consider the analytic properties of $f_{l\pm}(k, r)$, following [16]. Firstly, according to Ref. [19], for the case $l = 0$, one has

$$f_{0\pm}(k, r) = \left[1 + \int_b^\infty db' S_\pm(b', k) \exp(-b'r)\right] \exp(\pm ikr), \tag{24}$$

where $S_\pm(b', k)$ is the solution of the equation

$$b(b \mp 2ik) S_\pm(b, k) = \rho + \int_b^\infty db' \rho S \pm (b', k), \tag{25}$$

where $\rho = 2\mu V_0/\hbar^2 b^3 < 0$ and μ is the reduced mass. The solution of eq. (25) can be written as

$$S_\pm(b, k) = \rho \left\{ b(b \mp 2ik) \left[1 \mp \frac{i\rho}{2k} \ln(1 \mp \frac{2ik}{b})\right]^{-1} \right\} \tag{26}$$

and from Eq. (24) one obtains

$$f_{0\pm}(k,r) = \left\{1 + \rho\left[1 \mp \frac{i\rho}{2k}\ln\left(1 \mp \frac{2ik}{b}\right)\right]^{-1}\int_b^\infty db'\frac{\exp(-b'r)}{b'(b' \mp 2ik)}\right\}\exp(\pm ikr)\ . \tag{24a}$$

In Eq. (26) there is a logarithmic singularity of $f_{0\pm}\ (k,r)$ at the point $k = k_\gamma = ib/2$. We consider the analytic properties of the factor

$$A_- = \left[1 + \frac{i\rho}{2k}\ln\left(1 + \frac{2ik}{b}\right)\right]^{-1}. \tag{27}$$

It is easy to verify that for complex values of k both Re k and Im k are nonzero, the factor A_-^{-1} has no zeros, and A_- therefore has no poles. The same result is obtained for real k. Further, we set $k = ix$, where x is real, and rewrite Eq. (27) in the form

$$A_- = \left[1 + \frac{\rho}{2x}\ln\left(1 - \frac{2x}{b}\right)\right]^{-1}.$$

In the case $2x/b > 1$, the factor A_-^{-1} has no zeros because the logarithm is complex and A_- therefore has no poles. In the case $0 \leqslant 2x/b < 1$ for $\rho < 0$, as in the case of the long-range part of the nuclear forces, A_- again has no poles. Finally, in the case $2x/b \leqslant 0$, poles can exist, but they must be located in the lower half of the complex k plane D^-.

In conclusion, we note that the factor A_- contains no additional singularities in D^+ except a branch point at $k_\gamma = ib/2$.

The treatment can be extended to higher angular momenta $l > 0$. The same logarithmic singularity at the point $k = k_\gamma = ib/2$ also appears in $f_{l-}(k,r)$. To show this, the following integral equation, which allows the computation of $f_{l-}(k,r)$ from $f_{0-}(k,r)$, can be used:

$$f_{l-}(k,r) = f_{0-}(k,r) + l(l+1)\int_r^\infty dr' G(k;r,r')(r')^{-2} f_{l-}(k,r'), \tag{28}$$

where $r > a$ and the Green's function $G(k;r,r')$ has the form [21]

$$G(k;r,r') = (2ik)^{-1}[f_{0-}(k,r)f_{0+}(k,r') - f_{0-}(k,r')f_{0+}(k,r)]. \tag{29}$$

Because the function $\Phi(k,r)$ is regular everywhere, the Green's function in Eq. (29) has no singularity at the point k_γ. The solution $f_{l-}(k,r)$ of Eq. (28) for any value of l contains the same logarithmic singularity as the function $f_{0-}(k,r)$ at the point $k_\gamma = ib/2$.

We now consider Eq. (20) near the logarithmic singularities of $f_{l-}(k,r)$ in D^+. The contour γ_p can be chosen in the form shown in Fig. 1. It consists of the almost closed circle $\gamma_{\rm acc}$ around $k_\gamma = ib/2$, with the small radius $\varepsilon \equiv (\varepsilon/b)b$ and the two infinite lines $\gamma_{\rm edge}$ along the edges of the cut with a much smaller distance between them given by $(\varepsilon/b)^\delta b, \delta > 2$, i.e. $\gamma_p = \gamma_{\rm acc} + \gamma_{\rm edge}$. We let γ_{12} denote the segment joining the lowest points 1 and 2 of the two lines, according to Fig. 1. Letting γ_c denote the closed contour formed by the almost closed circle and the segment, we can write the identity $\gamma_{\rm acc}$

$$\int_{\gamma_{\rm acc}} = \oint_{\gamma_c} - \int_{\gamma_{12}} \underset{\varepsilon\to 0}{\to} \oint_{\gamma_c}. \tag{30}$$

for the integrals in Eq. (20).

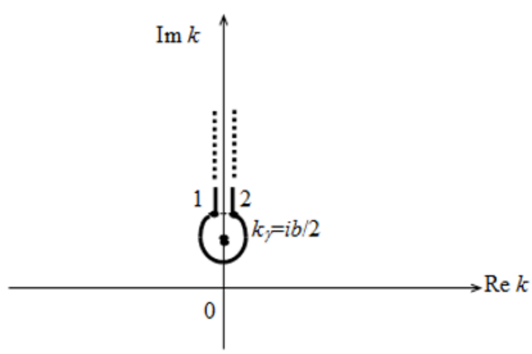

Figure 1. A form of contour γ_q.

Because the length of γ_{12} is $(\varepsilon/b)^\delta b$, the integrals over γ_{12} and over γ_{edge} vanish as $\mathcal{O}(\varepsilon^{\delta-2})$ for $\varepsilon \to 0$. Therefore, only the contour integral over the closed circle γ_c centered at the point k_γ remains.

We consider the integral over γ_c in detail. The value of the integral is determined by the behavior of the integrand as the radius of the circle tends to zero. Therefore, we consider the limit

$$\lim_{k\to k_\gamma}\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}\int_b^\infty \frac{\exp(ik-b')r}{b'(b'+2ik)}db' \tag{31}$$

the small radius $\varepsilon \equiv (\varepsilon/b)b$ and the two infinite lines γ_{edge} along the edges of the cut with a much smaller distance between them given by $(\varepsilon/b)^\delta b, \delta > 2$, i.e. $\gamma_p = \gamma_{\text{acc}} + \gamma_{\text{edge}}$. We let γ_{12} denote the segment joining the lowest points 1 and 2 of the two lines, according to Fig. 1.

It is easy to show that the integral over the variable b' in eq. (31) has a logarithmic divergence at k_γ that cancels when the corresponding factor vanishes, and the function $f_{0-}(k,r)$ therefore has no pole at k_γ. Explicitly evaluating the limit of (30) shows that in the vicinity of k_γ, the function $f_{0-}(k,r)$ can be written as (see Appendix D)

$$f_{0-}(k,r) \to W(k,r) + \left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1} U(k,r), \tag{32}$$

where the functions W and U are analytic functions of k at the point k_γ and inside the small closed circle γ_c. Eq. (20) can therefore be rewritten as

$$\begin{aligned}\oint_{\gamma_c} dk S_0(k) f_{0+}(k,r) f_{0+}(k,r') &= \oint_{\gamma_c} dk f_{0+}(k,r) f_{0-}(k,r') \\ = \oint_{\gamma_c} dk W(k,r) f_{0+}(k,r') &+ \left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}\oint_{\gamma_c} dk U(k,r) f_{0+}(k,r').\end{aligned} \tag{33}$$

Because each integral on the right-hand side vanishes, we can conclude that

$$\oint_{\gamma_c} dk S_0(k) f_{0+}(k,r) f_{0+}(k,r') = 0. \tag{34}$$

It hence follows that $S_0(k)$ can contain at most a singular factor of the type

$$F = \frac{\left[1-\frac{i\rho}{2k}\ln\left(1-\frac{2ik}{b}\right)\right]}{\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]} \tag{35}$$

connected with the analogous logarithmic branch points at $k = k_\gamma$. Of course, there may be no such factor, or $S_0(k)$ may contain other factors which have a logarithmic branch point but vanish at k_γ.

We consider a few special cases where this factor actually occurs.

If the interaction inside the sphere $r \leqslant a$ is such that the scattering wave function in the external part ($r > a$) can be written in the form

$$\Psi_{\text{ext}} = f_{0-}(k,r) - S_0(k) f_{0+}(k,r) \tag{36}$$

and vanishes at some point $r = r_0 > a$, then

$$S_0(k) = \frac{f_{0-}(k,r_0)}{f_{0+}(k,r_0)}. \tag{37}$$

It follows that S_0 *must contain* at most a factor F given by Eq. (35).

Another possibility occurs for a wide class of potentials [1, 2], namely when the interaction inside the sphere $r \leqslant a$ is such that the following continuity relations hold true

$$\begin{aligned}\Psi_{\text{int}} &\equiv \text{const}\cdot\Phi(k,a) = f_{0-}(k,a) - S_0(k) f_{0+}(k,a),\\ \left.\frac{d\Psi_{\text{int}}}{dr}\right|_{r=a} &\equiv \text{const}\left.\frac{d\Phi(k,r)}{dr}\right|_{r=a} = \left.\frac{df_{0-}(k,r)}{dr}\right|_{r=a} - S_0\left.\frac{df_{0+}(k,r)}{dr}\right|_{r=a}.\end{aligned} \tag{38}$$

In Eq. (38) the function $\Phi(k,r)$ is the regular solution of the radial Schroedinger equation inside the sphere $r \leqslant a$ with the boundary condition $\Phi(k,0) = 0$. This function is determined only by the interaction inside the sphere

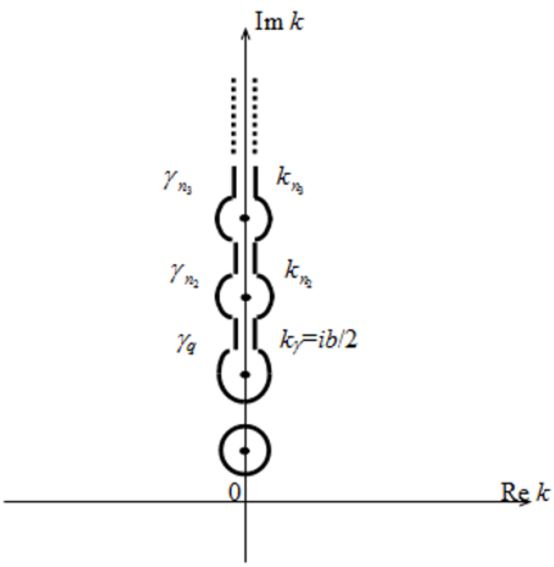

Figure 2. A disposition of the poles and the cut.

$r \leqslant a$. Eq. (38) determine the const (constant) and the corresponding S-matrix

$$S_0(k) = \frac{\varphi(k,a)\frac{df_{0-}(k,a)}{da} - f_{0-}(k,a)\frac{d\varphi(k,a)}{da}}{f_{0+}(k,a)\frac{d\varphi(k,a)}{da} - \varphi(k,a)\frac{df_{0+}(k,a)}{da}}, \qquad (39)$$

i.e. then $S_0(k)$ must also then contain factor (35).

According to (C, 1)-(C, 2), the same result is also obtained for $S_l(k)$ with $l > 0$.

Using the same approach, we can study a more general case where there is a covering of the cut (see Fig. 2) by the poles (corresponding to bound states and/or "redundant" poles that appear when the potentials decrease exponentially). In this case it sufficient to use equalities like (30):

$$\int_{\gamma_{accc}(k_n)} = \oint_{k_n} - \int_{\gamma_{12}} \lim_{\varepsilon\to 0} \oint_{k_n}$$

and simply repeat the reasoning proceeding (36). It is then easy to prove that for all these singularities, Eq. (22) continues to hold, and the results obtained previously concerning the singularities of $S_l(k)$ in D^+ also continue to hold.

Finally, the analytic continuation of the functions $S_l(k)$ to the lower half-plane D^- can be found as usual based on symmetry condition (2a), (3a) and the known general theorem about the analytic continuation. Thus, considering (23a), (23b) and (35), we finally obtain

$$S_l(\gamma,k) = \exp(-2i\alpha k)F(k)\prod_n \frac{k_{nl}+k}{k_{nl}-k}\prod_\lambda \frac{k_\lambda - k}{k_\lambda + k}\prod_s \frac{k_s - k}{k_s + k}\prod_{s'} \frac{k_{s'} - k}{k_{s'} + k}, \qquad (40a)$$

$$S_l(\gamma^*,k) = \exp(-2i\alpha k)F(k)\prod_n \frac{k^\bullet_{nl}+k}{k^\bullet_{nl}-k}\prod_\lambda \frac{k^\bullet_\lambda - k}{k_\lambda + k}\prod_s \frac{k^\bullet_s - k}{k^\bullet_s + k}\prod_{s'} \frac{k^\bullet_{s'} - k}{k^\bullet_{s'} + k}. \qquad (40b)$$

Conditions (10a, 10b) impose certain limitations on the distribution of zeros in D^+. We consider the following example as an illustration. Let in expression (23a) or (40a) for $S_l(\gamma,k)$ the factor $\frac{(\nu-k)(\nu'-k)}{(\nu+k)(\nu'+k)}$ essentially dominate and, hence,

$$S_l(\gamma,k) \approx \frac{(\nu-k)(\nu'-k)}{(\nu+k)(\nu'+k)}, \qquad (41)$$

according to (10a), $\nu = \alpha + i\beta, \nu' = -\alpha + \beta', \alpha > 0, \beta > 0, \beta' > \beta$. Then the partial cross sections of scattering, absorption and both processes together are respectively equal to

$$\sigma^{(l)}_{scatt} = \frac{\pi}{k^2}(2l+1)\left|1-S_l\right|^2 \approx \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma_{scatt}/2)^2}{(E-E_r)^2+\Gamma^2/4}, \qquad (42)$$

$$\sigma^{(l)}_{absorp} = \frac{\pi}{k^2}(2l+1)\left[1-\left|S_l\right|^2\right] \approx \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma_{scatt}/2)(\Gamma_{absorp}/2)}{(E-E_r)^2+\Gamma^2/4}, \qquad (43)$$

$$\sigma^{(l)}_{tot} = \sigma^{(l)}_{scatt} + \sigma^{(l)}_{absorp} \approx \frac{4\pi}{k^2}(2l+1)\frac{(\Gamma_{scatt}/2)(\Gamma/2)}{(E-E_r)^2+\Gamma^2/4}, \qquad (44)$$

where $\Gamma_{scatt} = \frac{4\mu}{\hbar^2}k(\beta+\beta')$, $\Gamma_{absorp} = \frac{4\mu}{\hbar^2}\alpha(\beta'-\beta)$, $\Gamma = \Gamma_{scatt} + \Gamma_{absorp}$. The formulae (42)-(44) generalize the known results obtained in the model description of nuclear reactions (see, for instance [27]).

5. The properties of the non-unitary S-matrix for the unknown non-central and parity-violating interactions surrounded by the centrifugal barrier and a potential, the tails of which are decreasing as described in Sections 2-4

We shall study this problem, following [14]. Let us suppose that the interaction between two colliding particles is such that the S-matrix is diagonal, as regards the total momentum j, does not depend on the total-momentum projection onto an arbitrary axis, and contains both diagonal and non-diagonal elements regarding the orbital momentum l with the mixed neighboring values $l, l' = j \pm \lambda$ of equal ($\lambda = 1$) or opposite ($\lambda = \frac{1}{2}$) parities. Particularly, there is a mixture of values $l, l' = l \pm 1$ (in the case of a tensor interaction admixture) or there is no mixture at all ($l = l' = j, \lambda = 0$), and there is a mixture $l, l' = j + \frac{1}{2}$ in the case of a parity-violating interaction like $\upsilon(r)\hat{\vec{\sigma}}\hat{\vec{p}} + \hat{\vec{\sigma}}\hat{\vec{p}}\upsilon(r)$, where r is the relative distance between two particles, $\hat{\vec{\sigma}}$ is the Pauli pseudo-vector matrix, $\hat{\vec{p}}$ is the momentum operator for the relative motion of a nucleon and a nucleus with spin 0. Of course, in the case of central interactions always $l = l' = j$ and $\lambda = 0$.

Thus, we consider the unknown non-central or parity-violating interaction inside the sphere $r < a$ surrounded by the centrifugal barrier and a central potential, which is decreasing more rapidly then any exponential function $V(r)$. Supposing that there is not only the scattering but also the absorption or the creation of particles, it is natural as usual to put, generalizing (10a, 10b), the following conditions for the elements $S^j_{ll'}$ of the S-matrix:

$$0 < \sum_{l'} \left| S^j_{ll'}(\gamma, k) \right|^2 \leqslant 1, \tag{45a}$$

$$1 \geqslant \sum_{l'} \left| S^j_{ll'}(\gamma, k) \right|^2 < \infty, \tag{45b}$$

and, generalizing (1a)-(3a), the extended "unitarity" condition

$$\sum_{l} S^j_{l_1 l}(\gamma, k) S^{j\bullet}_{l l_2}(\gamma^\bullet, k^\bullet) = \delta_{l_1 l_2} \tag{46}$$

and symmetry condition

$$S^{j\bullet}_{ll'}(\gamma^\bullet, k^\bullet) = (-1)^{l+l'} S^j_{ll'}(\gamma, -k) \tag{47}$$

(as regards the axis $\operatorname{Im} k$), and also the condition of $S^j_{ll'}$ symmetry regarding the lower indices:

$$S^j_{ll'}(\gamma, k) = S^j_{l''l}(\gamma, k). \tag{48}$$

One can easily check that the conditions (42)-(45) are automatically fulfilled in the case of central complex potential (5).

A system state for $r \geqslant a$ can be described by the wave functions

$$R^{j\bullet}_{ll'}(\gamma, k, r) = \frac{i}{2kr}\big[\delta_{ll'} f_{l'-}(k, r) \exp(il'\pi/2) - S^j_{ll'}(\gamma, k) f_{l'+}(k, r) \exp(-il'\pi/2)\big] \tag{49}$$

in the continuous spectrum and

$$R^{j(n)}_{l}(\gamma, k_{nl}, r) = (2\pi)^{-1/2} B_l(\gamma, k_{nl}) f_{l+}(k_{nl}, r) r^{-1} \tag{50}$$

in the discrete part of the spectrum.

Generalizing the completeness relation (11) for the unknown non-central or parity-violating interaction inside the sphere $r < a$ surrounded by the centrifugal barrier and a central potential, which is decreasing more rapidly then any exponential function $V(r)$, we can write

$$\frac{2}{\pi} \sum_{l} \int_0^\infty k^2 dk R^{j(+)}_{l'l}(\gamma, k, r) R^{j(+)\bullet}_{l''l}(\gamma, k, r') + \sum_{n} R^{j(n)}_{l'}(\gamma, k_{nj}, r) R^{j(n)}_{l''}(\gamma^\bullet, k_{nj}, r') = \frac{\delta(r - r')}{r^2} \delta_{l'l''}. \tag{51}$$

Relation (51) is a generalization of the completeness condition for eigen functions of a class of non-hermitian Hamiltonians [22, 23] for which all eigen values are simple (not multiple) and are situated outside the axis $\Re k$.

As usual in order that one be sure of the possibility of analytic continuation of $S^j_{ll'}(\gamma, k)$ in the complex plane k, one needs to put the limitation (12) in the external region ($r \geqslant a$).

Using the properties (49) for real k and conditions (46)-(48), one can rewrite (51) in the form

$$\frac{1}{rr'}\int\limits_C dk f_{l'-}(k,r)f_{l''+}(k,r')\delta_{l'l''} - \frac{\exp[-i(l'+l'')\pi/2]}{rr'}\int\limits_C dk S^j_{l'l''}(\gamma,k)f_{l'+}(k,r)f_{l''+}(k,r') \\ +\frac{1}{rr'}\sum_n B_{l'}(\gamma,k_{nj})B_{l''}(\gamma,k_{nj})f_{l'+}(k_{nj},r)f_{l''+}(k_{nj},r') = \frac{2\pi\delta(r-r')}{r^2}\delta_{l'l''}, \tag{52}$$

where the integration path C goes along the axis $\Re k$ from $-\infty$ to ∞, passing near the point $k=0$ (here $f_{l\pm}(k,r)$ have poles of l-th order) along a semi-circle of infinitely small radius in D^+.

We shall limit ourselves by the case when $f_{l\pm}(k,r)$ behaves as $\exp(\pm ikr)$ in all the complex plane at $|k|\to\infty$. Then, shifting the integration contour into D^+, enclosing all the singularities by closed singularities (as near to them as we like) and using equalities

$$\int\limits_{\Gamma+} dk f_{l-}(k,r)f_{l+}(k,r') = \int\limits_{\Gamma+} dk e^{ik(r'-r)} = \int\limits_{-\infty}^{\infty} dk e^{ik(r'-r)} = 2\pi\delta(r-r'), \tag{53}$$

$$\int\limits_{\Gamma+} dk S^j_{l'l''}(\gamma,k)f_{l'+}(k,r)f_{l''+}(k,r') = \int\limits_{\Gamma+} dk S^j_{l'l''}(\gamma,k)e^{ik(r+r')}, \tag{54}$$

we obtain

$$\delta_{ll'}\sum_m \oint\limits_{k_m} dk f_{l-}(k,r)f_{l'+}(k,r') + \delta_{ll'}\sum_p \oint\limits_{k_p} dk f_{l-}(k,r)f_{l'+}(k,r) \\ -\exp[-i(l+l')\pi/2]\Bigg[\sum_\nu \oint\limits_{k_\nu} dk S^j_{ll'}(\gamma,k)f_{l+}(k,r)f_{l'+}(k,r') \\ +\sum_q \oint\limits_{\gamma_q} dk S^j_{ll'}(\gamma,k)f_{l+}(k,r)f_{l'+}(k,r') + \int\limits_{\Gamma+} dk S^j_{ll'}(\gamma,k)f_{l+}(k,r)f_{l'+}(k,r')\Bigg] \\ +\sum_n B_l(\gamma,k_{nj})B_{l'}(\gamma,k_{nj})f_{l+}(k_{nj},r)f_{l''+}(k_{nj},r') = 0, \tag{55}$$

where $\int\limits_{\Gamma+}$ is the integral over the infinitely large semi-circle above the real axis, $\oint\limits_{k_n}$ is the integral over an infinitesimal circle around an isolated singular point, $\oint\limits_{\gamma_p}$ is the integral over a contour which envelops a non-isolated singularity (for instance, over the edges of the cut conducted for a branch point). Since all these contours are independent, equality (55) is equivalent to the following system of equalities:

$$\exp[-i(l+l')\pi/2]\oint\limits_{k_{nj}} dk S^j_{ll''}(\gamma,k)f_{l+}(k,r)f_{l'+}(k,r') \\ = B_l B_{l'} f_{l+}(k_{nj},r)f_{l'+}(k_{nj},r'), \tag{56}$$

$$\exp[-i(l+l')\pi/2]\oint\limits_{k_\nu} dk S^j_{ll''}(\gamma,k)f_{l+}(k,r)f_{l'+}(k,r') \\ = \oint\limits_{k_\nu} dk f_{l-}(k,r)f_{l''+}(k,r')\delta_{ll'}, \tag{57}$$

$$\exp[-i(l+l')\pi/2]\oint\limits_{\gamma_p} dk S^j_{ll'}(\gamma,k)f_{l+}(k,r)f_{l'+}(k,r') \\ = \oint\limits_{\gamma_p} dk f_{l+}(k,r)f_{l'-}(k,r')\delta_{l'l}, \tag{58}$$

$$\int\limits_{\Gamma} dk S^j_{ll'}(\gamma,k)e^{ik(r+r')} = 0. \tag{59}$$

Quite similar to the previous cases of the central unknown interactions inside the sphere $r\leqslant a$, it follows from Eq. (56) that all the elements $S^j_{ll'}(\gamma,k)$ have in D^+ poles of the first order (for $\gamma=\Re\gamma$ they are situated on the half-axis $\Im k>0$ and correspond to the bound states) with the residues

$$\exp[-i(l+l')\pi/2](2\pi i)^{-1}B_l B_{l'}.$$

It follows directly from Eq. (57) that the diagonal elements $S^j_{ll}(\gamma,k)$ must have in D^+ additional isolated singularities which coincide with those isolated singularities $f_{l-}(k,r)$ in D^+, near which Eq. (22) is also valid, with the function $D_m(k)$ which also does not depend on r and has an isolate singular point k_m. Similarly, it follows from Eq. (58) that the diagonal elements $S^j_{ll}(\gamma,k)$ must have in D^+ branch points and non-isolated singularities which coincide with the appropriate singularities of $f_{l-}(k,r)$ in D^+.

As for the unknown central interactions inside the small sphere of radius a, we consider several cases of potential tails in the external region with $r \geqslant a$.

When there is only a centrifugal barrier there, then Eq. (8a) and (9a) are valid and according to Eq. (47)-(50) all the functions $\tilde{S}^j_{ll'}(\gamma,k) = S^j_{ll''}(\gamma,k)\exp(2ika)$ are regular and limited everywhere in D^+ except the isolated points. No poles can appear on the half-axis $\Re r$ because of the conditions (47)-(48) and also because of the finite values of $R^{j(+)}_{l'l}(\gamma,k,r)$ at the point $k=0$.

It is easy to conclude from the finite value of $R^{j(+)}_{l'l}(\gamma,k,r)$ for $k\to 0$ by recalling the known behavior of $f_{l\pm}(k,r)$ and of $h^{(1,2)}_l(kr) = j_l(kr) \pm in_l(kr)$ at the point $k\to 0$ that

$$S^j_{ll'}(\gamma,k) \underset{k\to 0}{\to} \delta_{ll'}[1+\mathcal{O}(k^{l_>+1})] + [1-\delta_{ll'}]\mathcal{O}(k^{l_>+1}), \quad (60)$$

where $l_>$ is the larger of the two numbers l and l'.

One can determine the analytic continuation of the functions $S^j_{ll'}(\gamma,k)$ in D^- as usual on the basis of the symmetry condition (47) and the general theorem on the analytic continuation.

Solving system (46) with the use of (47) and (48) relatively to $S^j_{ll'}(\gamma,-k)$, we obtain

$$\begin{aligned} S^j_{ll}(\gamma,-k) &= S^j_{l'l'}(\gamma,k)/d_j(\gamma,k), \\ S^j_{ll'}(\gamma,-k) &= -S^j_{ll'}(\gamma,k)/d_j(\gamma,k) \end{aligned} \quad (61)$$

(with $l \neq l'$ and $d_j(\gamma,k) = S^j_{ll}(\gamma,k)S^j_{l'l'}(\gamma,k) - [S^j_{ll'}(\gamma,k)]^2$), from which we can see that in D^- all the elements $S^j_{ll'}(\gamma,k)$ have the same poles k_{nj} (on the half-axis Im $k<0$), k_s (in the 4-th quadrant), $k_{s'}$ (in the 3-rd quadrant), which correspond to the zeros of the function d_j in D^+, and also the zeros - k_{nj}, which correspond to the poles k_{nj} in D^+. Besides that, every diagonal element $S^j_{ll'}(\gamma,k)$ can have additional poles on the half-axis $\Re k < 0$ (k_μ), in the 4-th quadrant (k_σ) and in the 3-rd quadrant ($k_{\sigma'}$), which correspond to the zeros - k_μ, - k_σ and - $k_{\sigma'}$ of two functions $S^j_{ll}(\gamma,k)$ and $S^j_{ll'}(\gamma,k)$ in D^+. Moreover, one can conclude from the formulae (48) that the zeros k_p (on the axis $\Im k$), k_r (on the axis $\Re k$), k_t (in the 1-st and 4-th quadrant) and $k_{t'}$ (in the 2-nd and 3-rd quadrant) of the diagonal element $S^j_{ll}(\gamma,k)$ correspond to the zeros - k_p, - k_r, - k_t and - $k_{t'}$ of the second diagonal element $S^j_{l'l'}(\gamma,k)$, $l'\neq l$, and also that the zeros of the non-diagonal element $S^j_{ll'}(\gamma,k)$, $l'\neq l$, can appear only in pairs $\pm k_\pi$ (on the half-axis $\Im k$), $\pm k_\rho$ (on the half-axis $\Re k$), $\pm k_t$ (in the rest of the complex plane). Evidently, the last assertion is true for those zeros which are not general zeros of all the elements $S^j_{ll'}(\gamma,k)$.

In the considered case, $S^j_{ll'}(\gamma,k)$ cannot have any singular points in D^- besides poles, since there will be a singular point - k_x of $S^j_{l'l}(\gamma,k)$ in D^+ for every singular point k_x of $S^j_{l'l'}(\gamma,k)$ in D^- because of (61), but this is in contradiction with our previous result on the analyticity of $S^j_{l'l'}(\gamma,k)$ in D^+. Thus all the elements $S^j_{l'l'}(\gamma,k)$ are meromorphic functions and consequently they can be represented in the form of a ratio of two integer analytic functions:

$$\tilde{S}^j_{ll'}(\gamma,k) = A_{ll'}(\gamma,k)\exp[g_{ll'}(k)]\prod_n \frac{1+k/k_{nl}}{1-k/k_{nl}} \cdot \prod_{m,s,s',p,r,t,t'} \frac{(1-k/k_p)(1-k/k_r)(1-k/k_t)(1-k/k_{t'})}{(1-k/k_m)(1-k/k_s)(1-k/k_{s'})}, \quad (62)$$

where $A_{ll'} = \delta_{ll'} + (1-\delta_{ll'})Ck^{l_>+1}$, $C = i\Im C$ is a constant, the topology of the poles $k_{nj}, k_m, k_s, k_{s'}$ and of the zeros - $k_{nj}, k_p, k_r, k_t, k_{t'}$ was specified before, $g_{ll'}(k) = u_{ll'}(k) + i\theta_{ll'}(k)$.

The real function $u_{ll'}(k)$ must be non-positive in D^+ because of the analyticity of $\tilde{S}^j_{ll'}(\gamma,k)$ (and, consequently, the convergence of the infinite products of (62) in D^+) and must be non-negative in D^- owing to (61). Then the "Cauchy-Riemann conditions"

$$0 \geqslant \partial u_{ll'}/\partial\Im k = -\partial\theta_{ll'}/\partial k \qquad (\Im k = 0)$$

must be satisfied on the real axis. From these conditions one can conclude that the function $\theta_{ll'}(k)$ is monotonically increasing and reaches a real value not more than once. Then $g_{ll'}(k) = u_{ll'}(k) + i\theta_{ll'}(k)$ reaches any imaginary value not more than once, and hence must be a linear function: $g_{ll'}(k) = 2i\beta_{ll'}(k) + \gamma_{ll'}(k)$. Evidently $\beta_{ll'}(k) \geqslant 0$ and, since $S^j_{l'l'}(\gamma,0) = \delta_{ll'}$, $\gamma_{ll'} = 0$. Thus, considering that $\beta_{ll'} = (\beta_{ll} + \beta_{l'l'})/2$ owing to (61), we obtain the following final expression

$$S^j_{ll'}(\gamma,k) = A_{ll'}(\gamma,k)\exp[-i(\alpha_l+\alpha_{l'})]\prod_n \frac{1+k/k_{nl}}{1-k/k_{nl}} \cdot \prod_{m,s,s',p,r,t,t'} \frac{(1-k/k_p)(1-k/k_r)(1-k/k_t)(1-k/k_{t'})}{(1-k/k_m)(1-k/k_s)(1-k/k_{s'})}, \quad (63)$$

where $\alpha_l = a - \beta_l \leqslant a$. Considering (63) and on the basis of (46), (47), we can write

$$S^j_{ll'}(\gamma^\bullet, k) = A_{ll'}(\gamma^\bullet, k)\exp[-i(\alpha_l + \alpha_{l'})]\prod_n \frac{1 + k/k^\bullet_{nl}}{1 - k/k^\bullet_{nl}} \cdot \prod_{m,s,s',p,r,t,t'} \frac{(1 - k/k^\bullet_p)(1 - k/k^\bullet_r)(1 - k/k^\bullet_t)(1 - k/k^\bullet_{t'})}{(1 - k/k^\bullet_m)(1 - k/k^\bullet_s)(1 - k/k^\bullet_{s'})}. \quad (63a)$$

In the case of $\gamma = \Re\gamma$, the zeros appear in the pairs $\pm k_r$ and $k_{s'} = -k^*_s, k_{t'} = -k^*_t$ because of the symmetry condition (47) and then

$$S^j_{ll'}(\mathrm{Re}\ \gamma, k) = A_{ll'}\exp[-i(\alpha_l + \alpha_{l'})]\prod_n \frac{1 + k/k_{nl}}{1 - k/k_{nl}} \cdot \prod_{m,s,s',p,r,t,t'} \frac{(1 - k/k_p)(1 - k/k_r)^2(1 - k/k_t)(1 - k/k^\bullet_t)}{(1 - k/k_m)(1 - k/k_s)(1 - k/k^\bullet_s)}. \quad (64)$$

It may appear a possible physical phenomenon of the sharp enhancement of $S^j_{ll'}(\gamma, k)(l' \neq l)$ in comparison with $S^j_{ll}(\gamma, k)$ near an isolated resonance, noted in [14, 15] and described in Appendix E.
When $l = l', \lambda = 0$ (particularly, $\gamma = \gamma_c$),

$$k_p = -k_m,\ k_t = -k_s,\ k_{t'} = -k_{s'},$$

the zeros k_r are absent and then

$$S_l(\gamma_c, k) \equiv S^l_{ll}(\gamma_c, k) = \exp[-2i\alpha_l k]\prod_n \frac{1 + k/k_{nl}}{1 - k/k_{nl}} \cdot \prod_{m,s,s'} \frac{(1 - k/k_p)(1 - k/k_r)(1 - k/k_t)(1 - k/k_{t'})}{(1 - k/k_m)(1 - k/k_s)(1 - k/k_{s'})}, \quad (65)$$

that corresponds to results [10]. In the particular case in which $l = l' = j$ and $\gamma = \gamma_c$ we have also $k_{s'} = -k^*_s$ and hence

$$S_l(\Re\gamma_c, k) = \exp[-i2\alpha_l k]\prod_n \frac{1 + k/k_{nl}}{1 - k/k_{nl}} \cdot \prod_{m,s} \frac{(1 + k/k_m)(1 + k/k_s)(1 - k/k^\bullet_s)}{(1 - k/k_m)(1 - k/k_s)(1 + k/k^\bullet_s)}, \quad (66)$$

that corresponds to the results [19].

If for $r > a$ there is a centrifugal barrier and a potential decreasing more rapidly than any exponential function, results (63) and (63a) are valid because in that case $f_{l\pm}(k, r)$ are also analytic in all the plane k except the points $k = 0$ and $k = \infty$ and for $|k| \to \infty$ have the limit $\exp[\pm ikr]$ in all directions.
If for $r > a$ there is also an exponential potential of the type $V = V_0\exp(-br), V_0, b > 0$, then the functions $f_{l\pm}(k, r)$ have the simple poles in points $k = \mp i\frac{b}{2}m$ ($m = 1, 2, \ldots$) and at the limit $|k| \to \infty$ they tend to $\exp(\pm ikr)$. Similar results can be obtained for the Eckart, Hulthén and Woods-Saxon potentials [19]. On the basis of (47)-(50) we also obtain in this case results (63), (63a), where in $\prod_n$ the factor

$$\prod_\nu \frac{(1 + k/\kappa_\nu)}{(1 - k/\kappa_\nu)}$$

($\kappa_\nu = ib_\nu/2 \quad (\nu = 1, 2, \ldots)$ being the "redundant" poles which do not correspond to bound states) must be included for the diagonal elements $S^j_{ll}(\gamma, k)$.

And if at the external region (with $r \geqslant a$) there are the centrifugal barrier and a potential of the type $V = V_0 P_n(r)\exp(-br)$, where $P_n(r)$ is an nth-order polynomial and $b > 0$, then the function $f_{l\pm}(-k, r)$ has poles of an order not higher than $n + 1$ at the points $\mp ib/2, \mp ib, \mp 3ib/2, \ldots$, and it is analytic at all other points of the complex plane. Thus, coming from (18)-(21), one will also in this case obtain the results (23a, 23b), where in $\prod_m$ one must include the factors, corresponding to "redundant" poles $\frac{i}{2}bm' \quad (m' = 1, 2, \ldots)$ of the first order at the presence of an exponential potential tail $V = V_0\exp(-br)$, and the factors of the type $\prod_{m''}\left(\frac{k_{m''} - k}{k_{m''} + k}\right)^n$ corresponding to multiple "redundant" poles $\frac{i}{2}bm'' \quad (m'' = 1, 2, \ldots)$ at the presence of the potential tail of the type $V_0\sum_n P_n(r)\exp(-br)$. In this case the factor with the "redundant" poles in the diagonal elements

$S^j_{ll}(\gamma, k)$, the number and the degree of which depend on the type of the potential, also appears [10, 19].
If at the external region, where $r \geqslant a$, there are the centrifugal barrier and a central Yukawa potential of the type $V = V_0[(br)^{-1}\exp(-br)]$, $V_0, b^{-1} \sim a$, then the functions $f_{l-}(k, r)$ must have the factor $\left[1 + \frac{i\rho}{2k}\ln\left(1 + \frac{2ik}{b}\right)\right]^{-1}$ and the diagonal elements $S^j_{ll}(\gamma, k)$ must have the factor $F(k)$ with the logarithmic branch point at $k_\gamma = ib/2$ [16].

6. The properties of the non-unitary multi-channel S-matrix for the unknown interactions surrounded by the centrifugal barrier and a potential, which is decreasing more rapidly then any exponential function

Following [11, 15], we consider an idealized case when the structure of colliding particles is described by an infinite discrete set of non-degenerate energy levels ε_n and wave functions $\varphi_n(\xi)$ of bound states with zero (or "frozen") spin. Let the conditions of the ortho-normalization

$$\int d\xi \varphi_m(\xi)\varphi_n(\xi) = \delta_{mn} \qquad (m, n = 0, 1, 2, \dots) \tag{67}$$

and completeness

$$\sum_{n=0}^{\infty} \varphi_n(\xi)\varphi_n(\xi') = \delta(\xi - \xi') \tag{68}$$

(ξ is a totality of the internal particle coordinates) be fulfilled.
We assume that the central interaction between colliding particles inside the sphere of radius a is unknown as before, but at $r > a$ contains the centrifugal barrier $\hbar^2 l(l+1)/r^2$ and a central potential $V(r)$, which is decreasing more rapidly then any exponential function, and there is not only a scattering but also a partial particle absorption or generation. We describe the state of the total system in the external region ($r \geqslant a$) by the function

$$R^{(+)}_{nl}(E, r, \xi) = \sum_{m=0}^{\infty} \frac{i}{2r}\sqrt{\frac{\mu}{\hbar^2 k_m}}\left[f_{l-}(k_n, r)\exp(il\pi/2)\delta_{mn} - S^{(mn)}_l(\gamma, E) f_{l+}(k_m, r)\exp(-il\pi/2)\right]\varphi_m(\xi) \tag{69}$$

in the region of the continuum part of the energy spectrum

$$E = \varepsilon_n + \frac{\hbar^2 k_n^2}{2\mu}, \tag{70}$$

i.e. for $k_n^2 \geqslant 0$ $\quad (n = 0, 1, 2, \dots; \ \varepsilon_0 < \varepsilon_1 < \varepsilon_2 < _{\dots})$, and functions

$$R_{\nu l}(E_{\nu l}, r, \xi) = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} B^{(n)}_{\nu l}(\gamma, k^{(n)}_{\nu l}) f_{l+}(k^{(n)}_{\nu l}, r) r^{-1}\varphi_n(\xi) \tag{71}$$

in the region of the discrete part of the energy spectrum

$$E_{\nu l} = \varepsilon_n + \frac{\hbar^2\left[k^{(n)}_{\nu l}\right]^2}{2\mu}, \tag{72}$$

i.e. for $\left[k^{(n)}_{\nu l}\right]^2 < 0$, $n = 0, 1, 2, \dots$ (γ is a complex parameter which characterizes the scattering ($\Re\gamma$) and the absorption or generation ($\Im\gamma$) of particles during their interaction), if we consider that $\sum_m$ and $\sum_n$ do uniformly converge.

If $\gamma = \Re\gamma$ and $E \geqslant \varepsilon_0$, the elements $S^{(mn)}_l(\gamma, E) \equiv S^{(mn)}_l(\gamma, k_0, k_1, \dots)$ must satisfy the *conditions of the unitarity*

$$\sum_{n=0}^{N-1} S^{(mn)}_l(\gamma, E) S^{(nm')\bullet}_l(\gamma, E) = \delta_{mm'} \tag{73}$$

with $\varepsilon_{N-1} \leqslant E \leqslant \varepsilon_N, 0 \leqslant m, m' \leqslant N-1, N = 1, 2, \dots$ ($N-1$ being the number of the last open channel) which follows from the law of the conservation of the particle number (see, for instance, [17, 18, 27, 28]), *of the symmetry regarding the indexes of the open channels*

$$S^{(mn)T}_l(\gamma, E) = S^{(mn)}_l(\gamma, E) = S^{(nm)}_l(\gamma, E) \tag{74}$$

and *of the symmetry regarding the wave numbers*

$$\tilde{S}^{(mn)\bullet}_l(\gamma, k_0, k_1, \dots) = \tilde{S}^{(mn)}_l(\gamma, -k_0^\bullet, -k_1^\bullet, \dots), \tag{75}$$

where

$$\tilde{S}^{(mn)}_l = \sqrt{\frac{k_m}{k_n}} S^{(mn)}_l \tag{76}$$

for $E \geqslant \varepsilon_{N-1} \quad (m, m' \leqslant N-1)$, or, considering that

$$e^{i\pi} k_n = \begin{cases} e^{i\pi} k_n = -k_n, & \text{if the } n\text{-th channel is open,} \\ k_n, & \text{if the } n\text{-th channel is closed,} \end{cases}$$

$$S_l^{(mn)\bullet}(\gamma, k_0, k_1, \ldots) = S_l^{(mn)}(\gamma, -k_0, -k_1, \ldots, -k_{N-1}, k_N, k_{N+1}, \ldots). \tag{75a}$$

Relations (74)-(75a) follow from the principle of the reversibility of time (T-invariance) in the external region ($r \geqslant a$). Since the values $k_n \quad (n = 0, 1, 2, \ldots)$, on which the elements $S_l^{(mm')}$ depend, relative to (75)-(75a), represent irrational (radical) functions of one independent variable (for instance, k_m), we shall study the elements $S_l^{(mm')}$ on the Riemann surface, which can be obtained, making the cuts on the infinite number of k_m-planes from points $+\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_n - \varepsilon_m)}$ till points $-\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_n - \varepsilon_m)}$, respectively, symmetrically relative to the imaginary axis (because of the symmetry condition (75)), and connecting these planes along the cuts. Let us draw these cuts in such a manner as is shown in Fig. 3. Considering that it follows from the usual physical boundary conditions imposed on the asymptotic of functions (69) that

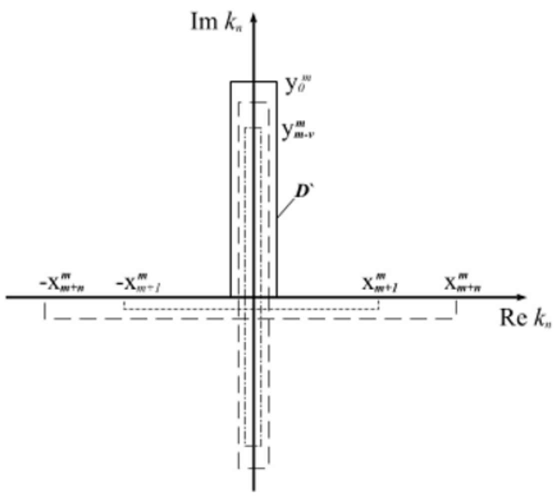

Figure 3. The physical k_m-plane with cuts between the channel thresholds $\pm x_{m+n}^m = \pm\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_{m+n} - \varepsilon_m)}, n = 0, 1, 2, \ldots; \quad \pm y_{m-v}^m = \pm\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_m - \varepsilon_{m-v})}, m \geqslant v, m, v = 0, 1, 2, \ldots$ and the contour D'.

$$\begin{aligned} k_n > 0 \quad &\text{when } E > \varepsilon_n, \\ \Im k_n > 0 \quad &\text{when } E < \varepsilon_n, \end{aligned} \tag{77}$$

we choose as the physical sheet of the Riemann surface those k_m-planes on which

$$\text{sign}\Re k_n = \text{sign}\Re k_m, \quad \text{sign}\Im k_n = \text{sign}\Im k_m \quad (n = 0, 1, 2, \ldots). \tag{78}$$

Generalizing relations (73)-(76) for the case $\gamma \neq \Re\gamma$ and complex values of $k_0, k_1, \ldots$, we can write

$$\sum_{n=0}^{N-1} \tilde{S}_l^{(mn)}(\gamma, k_0, k_1, \ldots) k_n \tilde{S}_l^{(m'n)}(\gamma, -k_0, -k_1, ... - k_{N-1}, k_N, k_{N+1}, \ldots) = \delta_{mm'} k_m, \quad 0 \leqslant m, m' \leqslant N-1, N = 1, 2, \ldots; \tag{79}$$

$$S_l^{(mn)T}(\gamma, k_0, k_1, \ldots) = S_l^{(mn)}(\gamma, k_0, k_1, \ldots) = S_l^{(nm)}(\gamma, k_0, k_1, \ldots) \tag{80}$$

and

$$\tilde{S}_l^{(mn)\bullet}(\gamma^\bullet, k_0^\bullet, k_1^\bullet, \ldots) = \tilde{S}_l^{(mn)}(\gamma, -k_0, -k_1, \ldots, -k_{N-1}, k_N, k_{N+1}, \ldots), \quad 0 \leqslant m, n \leqslant N-1, N = 1, 2, \ldots. \tag{81}$$

The completeness condition of the system wave functions in the region $r \geqslant a$ can be written in the form

$$\frac{2}{\pi}\sum_{n=0}^{\infty}\int_{\varepsilon_n}^{\infty} dE R_{nl}^{(+)}(\gamma, E; r, \xi) R_{nl}^{(+)\bullet}(\gamma, E; r', \xi') + \sum_\nu R_{\nu l}(\gamma, E_{\nu l}; r, \xi) R_{\nu l}(\chi, E_{\nu l}; r', \xi') = \frac{\delta(r-r')}{r^2}\delta(\xi - \xi'). \tag{82}$$

Substituting Eq. (69) and (71) for $R_{nl}^{(+)}$ and $R_{\nu l}$, respectively, in (82), multiplying (82) by $\varphi_{m'}(\xi)\varphi_m(\xi'')$, assuming $m' \leqslant m$ for certainness, integrating over $d\xi$ and $d\xi'$ with the utilization of (67) and (68), changing variables with utilization of (70) and (75) and utilizing the properties (13) for $f_{l\pm}(k_m, r)$ on the real axis k_m, we obtain after simple transformations, fulfilled along the upper edge of the cut on the real axis, that

$$\begin{aligned}
&\int_{C'} dk_m f_{l-}(k_m, r) f_{l+}(k_m, r')\delta_{m''m} - (-1)^l \int_{C'} dk_m \frac{k_m}{k_{m'}} \tilde{S}_l^{(m'm)}(\gamma, k_0, k_1, \ldots) f_{l+}(k_{m'}, r) f_{l+}(k_m, r') \\
&+ (1 - \delta_{m0}) \sum_{N=1}^{m} \int_{\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_{N-1} - \varepsilon_m)}}^{\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_N - \varepsilon_m)}} dk_m \sum_{n=0}^{N-1} \tilde{S}_l^{(m'n)}(\gamma, k_0, k_1, \ldots) \tilde{S}_l^{(mn)}(\gamma, -k_0, \ldots, -k_{N-1}, k_N, k_{N+1}, \ldots) \frac{k_n}{k_{m'}} \\
&\cdot f_{l+}(k_{m'}, r) f_{l+}^\bullet(k_m, r') - (-1)^l \int_{\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_{m'} - \varepsilon_m)}}^{0} dk_m \frac{k_m}{k_{m'}} \tilde{S}_l^{(m'm)}(\gamma, -k_0^\bullet, -k_1^\bullet, \ldots) f_{l+}^\bullet(k_{m'}, r) f_{l-}(k_m, r') \\
&+ \sum_\nu B_{\nu l}^{(m')}(\gamma, k_{\nu l}^{(m')}) B_{\nu l}^{(m)}(\gamma, k_{\nu l}^{(m)}) f_{l+}(k_{\nu l}^{(m')}, r) f_{l+}(k_{\nu l}^{(m)}, r') = \delta_{mm'} 2\pi\delta(r - r'),
\end{aligned} \tag{83}$$

where the contour C' is the total real axis k_m except the point $k_m = 0$. Shifting in (83) the integration contour C' into D^+, enveloping all singularities and, in particular, the cut along the imaginary axis with the contour D' depicted in Fig. 3, utilizing Eq. (79)-(80) and also taking into account the independence of the contours Γ^+ and D',

$$\int_{\Gamma^+} dk_m f_{l+}(k_m, r) f_{l-}(k_m, r') = \int_{\Gamma^+} dk_m \exp[ik_m(r - r')] = \int_{-\infty}^{\infty} dk_m \exp[ik_m(r - r'') = 2\pi\delta(r - r')],$$

$$\int_{D'} dk_m f_{l+}(k_m, r) f_{l-}(k_m, r') \equiv 0$$

and

$$\int_{\Gamma^+} dk_m \tilde{S}_l^{(m'm)}(\gamma, k_0, k_1, \ldots) f_{l+}(k_{m'}, r) f_{l+}(k_m, r') = \int_{-\infty}^{\infty} dk_m \tilde{S}_l^{(m'm)}(\gamma, k_0, k_1, \ldots) \exp[i(k_{m'} r + k_m r')],$$

after some principally simple but somewhat fine transformations with regard to the independence of contours around isolate and non-isolate singular points of $S_l^{(m'm)}$ and $f_{l-}(k_m, r)$, and the analyticity of $S_l^{(m'm)}(\gamma, k_0, k_1, \ldots)$, $S_l^{(m'm)}(\gamma, -k_0^\bullet, -k_1^\bullet, \ldots)$, $f_{l+}(k_{m'}, r)$ and $f_{l+}(k_m, r'')$ on the edges of the cut D' (see also [15] and Appendix G), we obtain the following system of the equalities:

$$(-1)^l \oint_{k_{\nu l}^{(m)}} dk_m \frac{k_m}{k_{m'}} \tilde{S}_l^{(m'm)}(\gamma, k_0, k_1, \ldots) f_{l+}(k_{m''}, r) f_{l+}(k_m, r') = B_{\nu l}^{(m')}(\gamma, k_{\nu l}^{(m')}) B_{\nu l}^{(m)}(\gamma, k_{\nu l}^{(m)}) f_{l+}(k_{\nu l}^{(m')}, r) f_{l+}(k_{\nu l}^{(m)}, r'), \tag{84}$$

$$(-1)^l \oint\limits_{k_\mu^{(m)}} dk_m \frac{k_m}{k_{m'}} \tilde{S}_l^{(m'm)}(\gamma, k_0, k_1, \ldots) f_{l+}(k_{m''}, r) f_{l+}(k_m, r') = \delta_{m'm} \oint\limits_{k_\mu^{(m)}} f_{l-}(k_m' r) f_{l+}(k_m, r'), \quad (85)$$

$$(-1)^l \oint\limits_{\gamma_p^{(m)}} dk_m \frac{k_m}{k_{m'}} \tilde{S}_l^{(m'm)}(\gamma, k_0, k_1, \ldots) f_{l+}(k_{m''}, r) f_{l+}(k_m, r') = \delta_{m'm} \oint\limits_{\gamma_p^{(m)}} f_{l-}(k_m' r) f_{l+}(k_m, r'), \quad (86)$$

$$\int\limits_{\Gamma^+} dk_m \tilde{S}_l^{(m'm)}(\gamma, k_0, k_1, \ldots) \exp[i(k_{m'} r + k_m r')] = 0, \quad (87)$$

$$(1-\delta_{m'o})\Bigg[(-1)^l \tilde{S}_l^{(m'm)}(\gamma, k_0, k_1, \ldots) - (-1)^l \tilde{S}_l^{(m'm)}(\gamma, -k_0, \ldots, -k_{N-1}, k_N, k_{N+1}, \ldots) + (-1+\delta_{m0}) \cdot \sum_{n=0}^{N-1} \tilde{S}_l^{(m'n)}(\gamma, k_0, k_1, \ldots) \tilde{S}_l^{(mn)}(\gamma, -k_0, \ldots, -k_{N-1}, k_N, k_{N+1}, \ldots) \frac{k_n}{k_m}\Bigg] = 0, \quad (88)$$

$$(1-\delta_{m'm})\left[(-1)^l \tilde{S}_l^{(m'm)}(\gamma, k_0, k_1, \ldots) + (-1+\delta_{m0}) \cdot \sum_{n=0}^{N-1} \tilde{S}_l^{(m'n)}(\gamma, k_0, k_1, \ldots) \tilde{S}_l^{(mn)}(\gamma, -k_0, \ldots, -k_{N-1}, k_N, k_{N+1}, \ldots) \frac{k_n}{k_m}\right] = 0,$$
$$m \neq 0, N = m'+1, \ldots, m, \quad (89)$$

$$\int\limits_{D'} dk_m f_{l+}(k_m, r) f_{l-}(k_m, r') \equiv 0, \quad (90)$$

where $\oint\limits_{k_\mu}, \oint\limits_{\gamma_\mu}, \int\limits_{D'}$ and $\int\limits_{\Gamma}$ signify the integration along the contours around the isolate singular points, non-isolate singular point, along the cut on the imaginary axis from $\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_0 - \varepsilon_m)}$ till 0 and along Γ^+, respectively.

It follows from (84)-(89) that all the elements $S_l^{(m'm)}$ are analytic functions of k_m in D^+ of the physical plane of k_m, except the poles $k_{\nu l}^{(m)}$, isolated singular points $k_\mu^{(m)}$, non-isolated singularities $\gamma_p^{(m)}$, the essentially singular point in ∞ and the range of the cut inside the contour D'. It then follows from (84) that all the elements $S_l^{(m'm)}$ have the same poles of the first order with residues $(-1)^{l+1}\frac{k_{m'}}{k_m}\frac{i}{2\pi}B_{\nu l}^{(m')}B_{\nu l}^{(m)}$ (for $\gamma = \Re\gamma$ they are located on the imaginary axis above the contour D' and correspond to the bound states). And finally it follows from (85) and (86) that all the diagonal elements $S_l^{(mm)}$ have in D^+ the same isolated and non-isolated singularities which coincide with the respective singularities of $f_{l-}(k_m, r)$. But the non-diagonal elements $S_l^{(m'm)}$ ($m' \neq m$) *do not have such singularities*. It is clear from (87) that the functions $\exp[i(\alpha_m k_m + \alpha_{m'} k_{m'})]S_l^{(m'm)}$ with $\alpha_m + \alpha_{m'} \leqslant 2a$ have no essential singular point in $k_m = \infty$, and from which one can assume that $S_l^{(m'm)}$ have essentially singular points of the type $\exp[-i(\alpha_m k_m + \alpha_{m'} k_{m'})]$ with $\alpha_m + \alpha_{m'} \leqslant 2a$. Relations (88) and (89) are obtained first in [11, 15]. Their analytic continuations form the range of physical values of the wave numbers for *close channels* onto the all Riemann surface impose, as (79)-(81), additional strong limitations on the analytic structure of $S_l^{(m'm)}$.

Eq. (90) is a trivial identity because $f_{l-}(k_m, r)$ has no singularity in D^+ on the interval from 0 till $\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_m - \varepsilon_0)}$ on the imaginary axis (otherwise $f_{l-}(k_0, r)$ must have the same singularity in the interval from $-\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_m - \varepsilon_0)}$ till $\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_m - \varepsilon_0)}$ which is impossible for the chosen potential tails).

Being quite similar to the behavior of $S(\gamma, k)$ for $k \to 0$ in the one-channel case (see Appendix A), one can easy obtain

$$S_l^{(mm)}(\gamma, k_0, k_1, \ldots) \underset{k_m \to 0}{\to} 1 + O(k_m^q), \quad q \geqslant l+1, \quad (91)$$

$$S_l^{(m'm)}(\gamma, k_0, k_1, \ldots) \underset{k_m \to 0}{\to} O(k_m^{q'}), \quad m' \neq m, \quad q' \geqslant l + 1/2, \quad (92)$$

coming from the finiteness of the functions (69) on the physical boundary at $k_m \to 0$.

These results generalize relations, obtained in the model study (see, for instance, [25]), and also confirm the known specificity of scattering cross sections near thresholds of inelastic scattering.

The analytic continuations of $S_l^{(m'm)}$ into all D^- of unphysical k_m-planes can be obtained with the help of Rel. (79), (88) and (89). Besides non-isolated singularities and cuts obtained by substituting k_m by $-k_m$ in ranges, limited by γ_p, there the elements $S_l^{(m'm)}$ can also contain the poles. For instance, in the case of $N = 1$ the zeros k_s of the n-th order of $S_l^{(00)}$ in D^+ of the physical plane k_0 must correspond to the poles $-k_s$ of the n-th order of the same function according to Rel. (79) in D^- of such unphysical plane of k_0 which is characterized by the condition

$$\begin{aligned} \mathrm{sign}\Re k_n &\neq \mathrm{sign}\Re k_0, \\ \mathrm{sign}\Im k_n &\neq \mathrm{sign}\Im k_0, \\ &(n = 1, 2, \ldots). \end{aligned} \tag{93}$$

Similarly, according to Rel. (89), the same zeros k_s (also in the case of $N = 1$) must correspond to the poles $-k_s$ of the n-th order of $S_l^{(0m)}$ in D^- of the unphysical plane of k_0, defined by the condition (93), if points k_s are not the zeros of $S_l^{(0m)}$ $(m = 1, 2, \ldots)$. Then, according to Rel. (88) the same zeros k_s (again also in the case of $N = 1$) must correspond to the poles $-k_s$ of the n-th order of all other elements $S_l^{(m'm)}$ $(m \geqslant m' = 1, 2, \ldots)$. It is not principally difficult to also examine the case when points k_s are zeros of one of the elements $S_l^{(0m)}$ with $m \neq 0$. And similar study can be fulfilled for any finite number of open channels.

The results obtained here are completely valid in a more general case when one uses the elements $S_{J\Pi}^{(mn)}$, instead $S_l^{(mn)}$, with J and Π being quantum numbers of total momentum and parity of the system and n being a set of quantum numbers $\{\varepsilon_n, I_n, s_n, l_n, \alpha_n\}$, corresponding to energy levels (ε_n), spins (I_n and s_n) of the bombarding and the target particles(nuclei), orbital momentum of their relative motion (l_n) and other discrete quantum numbers (α_n). And in this case the wave functions of the system are being expanded not over a set of $\varphi_m(\xi)$ but over a set of $\varphi_m^{(JM\Pi)}(\xi)$ which represent the results of the vector sum of the wave functions of the internal motion of a target nucleus and of joint eigen functions of the operators $\hat{l}_n^2$ and $\hat{l}_{n,z}$; M is the quantum number of the projection of the system total momentum.

We note that Eq. (79), (88) and (89) also provide the possibility to fulfill analytic continuation for $S_l^{(m'm)}$ or $S_{J\Pi}^{(mn)}$ from D^- of the physical plane till D^+ of the unphysical planes of the Riemann surface and the inverse. However, it is impossible to establish the analytic behavior of the elements of the S-matrix on these parts of the Riemann surface on the basis of only those relations.

In the general case one cannot obtain the explicit analytic expression for $S_l^{(mm)}$ or $S_{J\Pi}^{(mn)}$ on the Riemann surface. Making some simplifications (neglecting the threshold particularities, of bound and virtual states, presence of the same resonance poles in all the elements of the S–matrix etc), one can usually use one of three parametrizations, described in the following section.

7. Three typical general representations of the multi-channel S-matrix for the unknown interactions inside the sphere $r > a$

In the study of resonance nuclear reactions (usually low-energy and partially medium-energy, but sometimes even high-energy nuclear reactions) usually three parametrizations of the S-matrix in the energy representation are used [15]:

$$\begin{aligned} \hat{S}^{(\alpha)} = \hat{S}_0^{(\alpha)} - i \sum_\lambda \frac{\hat{g}_\lambda^{(\alpha)} \times \hat{g}_\lambda^{(\alpha)}}{E - E_\lambda^{(\alpha)} + i\Gamma_\lambda^{(\alpha)}/2}, \\ \hat{S}_0^{(\alpha)} = \hat{S}_0^{(\alpha)T}, \\ \Gamma_\lambda^{(\alpha)} = \sum_{k-1}^{n} \Gamma_{\lambda,k} \equiv N_\lambda^{-1} \sum_{k-1}^{n} |g_{\lambda,k}|^2, \\ N_\lambda \geqslant 1 \end{aligned} \tag{94}$$

(n is the number of channels),

$$\begin{aligned} \hat{S}^{(\alpha)} = (1 - i\hat{K}^{(\alpha)})^{-1}(1 + i\hat{K}^{(\alpha)}), \\ \hat{K}^{(\alpha)} = \hat{K}_0^{(\alpha)} + \sum_{\mu=1}^{\Lambda^{(\alpha)}} \frac{\hat{\gamma}_\mu^{(\alpha)} \times \hat{\gamma}_\mu^{(\alpha)}}{E - \mathrm{E}_\mu^{(\alpha)}}, \\ \hat{K}_0^{(\alpha)} = \hat{K}_0^{(\alpha)T}, \end{aligned} \tag{95}$$

$$\begin{aligned} \hat{S}^{(\alpha)} = \hat{U}^{(\alpha)} \prod_\nu \left(1 - \frac{i\Gamma_\nu^{(\alpha)} \hat{P}_\nu^{(\alpha)}}{E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\varepsilon)}/2}\right) \hat{U}^{(\alpha)T}, \\ \hat{U}^{(\alpha)} \hat{U}^{(\alpha)*} = 1, \\ \hat{P}_\nu^{(\alpha)} = \hat{P}_\nu^{(\alpha)*} = \hat{P}_\nu^{(\alpha)2}, \\ \mathrm{Trace}\ \hat{P}_\nu^{(\alpha)} = 1. \end{aligned} \tag{96}$$

Index α in (94)-(96) signifies the set of quantum numbers of conserved quantities (usually α={J,Π}, where J and Π are the quantum numbers of the total momentum (spin) and parity of the system).

In all these three parametrizations, resonances are described by the general poles of all elements of the S-matrix. According to the causality these poles must be located in the lower half-plane of the complex plane E (in order to describe the decays of the resonance states).

The first parametrization (94) is the T-invariant Mittag-Leffler expansion of the meromorphic functions, possessing only simple poles. Usually $\hat{S}_0^{(\alpha)}$ and $\hat{g}_\lambda^{(\alpha)}$ assumed to be smooth functions of the energy or are approximated by constants, as $E_\lambda^{(\alpha)}$ and $\Gamma_\lambda^{(\alpha)}$. Although resonance parameters $\hat{g}_\lambda^{(\alpha)}$ (which are complex), $E_\lambda^{(\alpha)}$ and $\Gamma_\lambda^{(\alpha)}$ (both are real) have a simple physical sense only for isolated resonances and the application of (94) meets a lot of difficulties at the range of overlapped resonances, the representation (94) is used rather often (see, for instance [31–35].

The second parametrization (95), being unitary and also T-invariant, is found in the R-matrix theory of nuclear reactions [36], in the shell model approach to nuclear reactions [37] and in some later investigations [34, 35, 37–39]. Parameters $E_\lambda^{(\alpha)}$ and $\hat{\gamma}_\mu^{(\alpha)}$ also have a simple physical sense only in the approximation of isolated resonances when $E_\mu^{(\alpha)} \cong E_\lambda^{(\alpha)}$, $(\gamma_{\mu,i}^{(\alpha)})^2 \cong \Gamma_{\mu,i}^{(\alpha)}/2$, where $\Gamma_{\mu,i}^{(\alpha)}$ and $\gamma_{\mu,i}^{(\alpha)}$ are the i-th partial width and the amplitude of the i-th partial width of the μ-th resonance, respectively.

The third parametrization (96) was obtained in [40], coming from the general principles of unitarity, meromorphy and T-invariance of the S-matrix. With this, in [40] it was noted that there is a practical difficulty of the explicit consideration of T-invariance in the general case that projectors $\hat{P}_\nu^{(\alpha)}$ are non-symmetric and non-commuting with each other. This parametrization is mostly convenient for overlapping and strongly overlapping resonances (see, for instance, [41–43]) and was utilized for revealing the time resonances (explosions) of compounds clots and nuclei in high-energy nuclear reactions at the range of strongly overlapping energy resonances [44]. It was shown in [45] that when the projectors $\hat{P}_\nu^{(\alpha)}$ do not depend on the values of any other resonance parameters ($E_\lambda^{(\alpha)}$ and $\Gamma_\nu^{(\varepsilon)}$), then $\hat{S}^{(\alpha)} = \hat{S}^{(\alpha)T}$. Really, in that case one can rewrite the resonance part $\hat{S}_{res}^{(\alpha)} \equiv \prod_{\nu=1}^{\Lambda^{(\alpha)}} (1 - \frac{i\Gamma_\nu^{(\alpha)} P_\nu^{(\alpha)}}{E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\alpha)}/2})$ from the expression (96) for $\hat{S}^{(\alpha)}$ in the form of a sum

$$\hat{S}_{res}^{(\alpha)} = 1 - i \sum_\nu \frac{\Gamma_\nu^{(\alpha)} P_\nu^{(\alpha)}}{E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\alpha)}/2} - \sum_{\nu'>\nu} \frac{\Gamma_\nu^{(\alpha)}\Gamma_{\nu'}^{(\alpha)} \hat{P}_\nu^{(\alpha)}\hat{P}_{\nu'}^{(\alpha)}}{(E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\alpha)}/2)(E - E_{\nu'}^{(\alpha)} + i\Gamma_{\nu'}^{(\alpha)}/2)} + \ldots \tag{97}$$

which can be transformed to the Mittag-Leffler expansion:

$$\hat{S}_{res}^{(\alpha)} = 1 - i \sum_\nu \frac{iG_\nu^{(\alpha)}}{E - E_\nu^{(\alpha)} + i\Gamma_\nu^{(\alpha)}/2},$$
$$G_\nu^{(\alpha)} = \Gamma_\nu^{(\alpha)} P_\nu^{(\alpha)} - i \sum_{\nu'>\nu} \frac{\Gamma_\nu^{(\alpha)}\Gamma_{\nu'}^{(\alpha)} \hat{P}_\nu^{(\alpha)}\hat{P}_{\nu'}^{(\alpha)}}{E_\nu^{(\alpha)} - E_{\nu'}^{(\alpha)} + i(\Gamma_\nu^{(\alpha)} - \Gamma_{\nu'}^{(\alpha)})/2} - i \sum_{\nu''<\nu} \frac{\Gamma_\nu^{(\alpha)}\Gamma_{\nu''}^{(\alpha)} \hat{P}_\nu^{(\alpha)}\hat{P}_{\nu''}^{(\alpha)}}{E_\nu^{(\alpha)} - E_{\nu''}^{(\alpha)} + i(\Gamma_\nu^{(\alpha)} - \Gamma_{\nu''}^{(\alpha)})/2} + \ldots \tag{97a}$$

Taking into account (97a) and the T-invariance of the expression (96) for $\hat{S}^{(\alpha)}$, one can write

$$\hat{S}_{res}^{(\alpha)} = \hat{S}_{res}^{(\alpha)T} \tag{98}$$

and then one can further rewrite (98) in the following form (see the last ref. from [35]):

$$G_\nu^{(\alpha)} = G_\nu^{(\alpha)T} \qquad \nu = 1, 2, \ldots, \Lambda^{(\alpha)}. \tag{99}$$

Relations (99) are in general too bulky as correlations between the matrices $\hat{P}_\nu^{(\alpha)}$ with different ν. But if the $\hat{P}_\nu^{(\alpha)}$ do not depend on the values of $E_\lambda^{(\alpha)}$ and $\Gamma_\nu^{(\varepsilon)}$, then the relations

$$\hat{P}_\nu^{(\alpha)} = \hat{P}_\nu^{(\alpha)T}, \tag{100}$$

$$\hat{P}_\nu^{(\alpha)}\hat{P}_{\nu'}^{(\alpha)} = \hat{P}_{\nu'}^{(\alpha)}\hat{P}_\nu^{(\alpha)}, \qquad \nu, \nu' = 1, 2, \ldots, \Lambda^{(\alpha)} \tag{101}$$

(i.e. the matrices $\hat{P}_\nu^{(\alpha)}$ will be symmetric and commute with each other) are the direct consequences of (99). By the way, such simplification (the independence of $\hat{P}_\nu^{(\alpha)}$ from any other resonance parameters) is justified at least when $\Lambda^{(\alpha)}$ and the number N of open channels are very large. It then follows from the properties of (100) and $\hat{P}_\nu^{(\alpha)} = \hat{P}_\nu^{(\alpha)*} = \hat{P}_\nu^{(\alpha)2}$, Trace $\hat{P}_\nu^{(\alpha)} = 1$ (from (96)) that the $\hat{P}_\nu^{(\alpha)}$ are real, i.e.

$$\hat{P}_\nu^{(\alpha)} = \hat{P}_\nu^{(\alpha)\bullet}. \tag{102}$$

8. Connection of analytic properties of the S-matrix with duration of the partial-wave scattering and the usual (temporal) causality condition

Let us clarify how results obtained on analytic properties of the S-matrix agree with the causality. Following [15, 46–48], we define the mean duration of the l-partial-wave scattering as the difference between mean time moments averaged over outgoing and ingoing wave-packer durations through the sphere surface with radius $r \geqslant a$ according to

$$\langle \tau_l(\gamma, r) \rangle = \frac{\int_{-\infty}^{\infty} dt t j_{l,out}}{\int_{-\infty}^{\infty} dt j_{l,out}} - \frac{\int_{-\infty}^{\infty} dt t j_{l,in}}{\int_{-\infty}^{\infty} dt j_{l,in}}, \tag{103}$$

where $j_{l,in}$ and $j_{l,out}$ are the probability flux densities, corresponding to wave packets

$$r\varphi_{l,in}(r,t) = \int_0^\infty dkA(k)[(i/2)\exp(il\pi/2)]f_{l-}(k,r)\exp(-iEt/\hbar) \tag{104a}$$

and

$$r\varphi_{l,out}(\gamma,r,t) = \int_0^\infty dkA(k)[(i/2)\exp(il\pi/2)]f_{l+}(k,r)S_l(\gamma,k)\exp(-iEt/\hbar), \tag{104b}$$

respectively.

Integrating over dt in (103) with help of the simple technique of Fourier-Laplace transformations similar to that made in [15, 46–48], we obtain the following final expression:

$$\langle\tau_l(\gamma,r)\rangle = \frac{\int_0^\infty dk\,|A(k)S_l(\gamma,k)f_{l+}(k,r)|^2\,\hbar(\partial\arg AS_lf_{l+}/\partial E)}{\int_0^\infty dk\,|AS_lf_{l+}|^2} - \frac{\int_0^\infty dk\,|A(k)f_{l-}(k,r)|^2\,\hbar(\partial\arg Af_{l-}/\partial E)}{\int_0^\infty dk\,|Af_{l-}|^2}. \tag{103a}$$

We note that, unlike the physical radial wave packet

$$\varphi_l^{(+)}(\gamma,r,t) = \varphi_{l,in}(r,t) - \varphi_{l,out}(\gamma,r,t), \tag{104a}$$

which is finite at the limit $k\to\infty$, functions $f_{l\pm}(k,r)$ have the pole of the l-th order, and so for the finiteness of wave packets (104a) and (104b) it is necessary that wave-packet amplitudes $A(k)$ would have zero at point $k=0$, at least of the l-th order, or would be zero in the finite interval $(0,\kappa)$, $\kappa>0$. With such limitations for $A(k)$, it is natural to try to clear up at what conditions the orthodox causality is fulfilled, for one formulate it thus must be true that: *for any square integrable function $A(k)$, with only the above-mentioned limitation, the mean duration $\langle\tau(\gamma,r)\rangle$ of the l-partial-wave scattering for sufficiently large $r\geqslant a$ cannot be negative, i.e.*

$$\langle\tau_l(\gamma,r)\rangle \geqslant 0. \tag{105}$$

Following [15], it is not difficult to check that in the case of unitary $S_l(\gamma,k)$ for the fulfillment of the condition (105) it is necessary and sufficient that eq.

$$\tau_l(\gamma,r) = \hbar\frac{\partial\arg S_l(\gamma,k)[f_{l+}(k,r)/f_{l-}(k,r)]}{\partial E} \geqslant 0 \tag{106}$$

were fulfilled. Really, in this case according to (103a)

$$\langle\tau_l(\gamma,r)\rangle = \frac{\left[\int_0^\infty dk\,|A(k)f_{l+}(k,r)|^2\,\tau_l(\gamma,r)\right]}{\left[\int_0^\infty dk\,|A(k)f_{l+}(k,r)|^2\right]}$$

and also in view of non-negative values of k and $|A(k)f_{l+}(k,r)|^2$ the validity of (105) follows directly and necessarily from (106). And inversely, if one assumes the validity of (105), but in the vicinity of a certain point k_0 the relation $\tau_l(\gamma,r)<0$ is valid for $r\geqslant a$, then, choosing $A(k)$ identically equal to 0 out of this vicinity, one shall violate the condition (105) which contradicts the initial assumption and therefore proves the sufficiency of our theorem.

Let us study first the validity of the condition (106) in the case when out of the interaction sphere there is only the centrifugal tail. Then

$$\begin{aligned}\tau_l(\gamma,r) &= \hbar\frac{\partial \arg S_l(\gamma,k)[h_l^{(1)}(k,r)/h_l^{(2)}(k,r)]}{\partial E}\\ &= \hbar\frac{\partial \arg S_l}{\partial E} + 2\hbar\frac{(\partial n_l/\partial k)j_l - (\partial j_l/\partial k)n_l}{(j_l)^2+(n_l)^2}\\ &= \hbar\frac{\partial \arg S_l}{\partial E} + \frac{2r/v}{[krj_l(kr)]^2+[krn_l(kr)]^2},\end{aligned} \tag{107}$$

since $[dn_l(x)/dk]j_l(x) - [dj_l(x)/dx]n_l(x) = x^{-2}$. Here $v = \hbar k/\mu$. Utilizing (32) for the calculation of $\frac{\partial \arg S_l}{\partial E}$, we obtain

$$\begin{aligned}\hbar\frac{\partial \arg S_l}{\partial E} &= -\frac{2\alpha}{v} + \frac{1}{v}\sum_\lambda \frac{2\chi_\lambda}{k^2+\chi_\lambda^2}\\ &+\frac{1}{v}\sum_s \frac{4\text{Im }k_s(k^2+|k_s|^2)}{(|k_s|^2-k^2)^2+(2k\text{Im }k_s)^2},\end{aligned} \tag{108}$$

where $\chi_\lambda = -ik_\lambda$. Since $\alpha \leqslant a$, the sum $\sum_s$ is always positive, the quantity $[krj_l(kr)]^2+[krn_l(kr)]^2$ is finite when $k>0$ and $r>a$ and tends to 1 when $r\to\infty$, and

$$\sum_s \frac{2\chi_\lambda}{k^2+\chi_\lambda^2} \geqslant \frac{2}{\chi_1}$$

when χ_1 corresponds to the first bound state (remembering that there is at least one pole, corresponding to zero on the positive imaginary semi-axis between every two adjacent zeros k_λ, located in the order of increasing $|k_\lambda|$ on the negative imaginary semi-axis), then the condition (106) is fulfilled for sufficiently large values of r when

$$\frac{r}{[krj_l(kr)]^2+[krn_l(kr)]^2} \geqslant a + \frac{1}{\chi_1}.$$

$$\tau_0(\gamma,k,r) \geqslant 0 \tag{106a}$$

(in the case that out of the interaction sphere there is only the centrifugal tail) for l= 0 and $r \geqslant a + 1/\chi_1$ this is concordant with the Goebel-Carplus-Ruderman inequality (see, for instance [49]). For $l \neq 0, k\neq 0$ and $r \geqslant a$ the quantity $Q_l = \{[krj_l(kr)]^2+[krn_l(kr)]^2\}^{-1}$ is positive, finite, tends to 0 as $(kr)^l$ when $kr\to 0$, and monotonically grows, approaching to 1, with increasing kr. In this last case

$$\tau_l(\gamma,k,r) \geqslant 0 \quad \text{for } r \geqslant R_l(k), \tag{106b}$$

where $R_l(k)$ is the largest real solution of equation $rQ_l(kr) = a + 1/\chi_{1l}$.

Let us consider what contribution for the time delay (108) would give for every separate factor of representing type (66) for $S_l(\gamma,k)$. The factor $\exp[-i2\alpha_l k]$, which is typical for the hard repulsive barrier of radius α_l, causes the negative time delay $-\alpha_l/v$. The factor $\frac{1+k/k_{nl}}{1-k/k_{nl}}$ with $\chi_{nl} = k_{nl}/i > 0$, correspondent to a bound state, causes the negative time delay $-2\chi_{nl}/v[k^2+\chi_{nl}^2]$. A similar negative time delay will be caused by the factor with a "redundant" pole. The factor $\frac{1+k/k_{ml}}{1-k/k_{ml}}$ with $\chi_{ml} = k_{ml}/i < 0$, correspondent to a virtual (anti-bound) state, causes a positive time delay. For small k $(k\to 0)$ both formulas (for bound and anti-bound states) are particular cases of the following expression for time delay - $A/v[1+k^2A^2]$, where A is the scattering length. The factor $\frac{(1+k/k_s)(1-k/k_s^\bullet)}{(1-k/k_s)(1+k/k_s^\bullet)}$ with $\text{Im } k_s > 0$, correspondent to a resonance state, causes the positive timer delay $\frac{1}{v}\cdot\frac{4\text{Im }k_s(k^2+|k_s|^2)}{(|k_s|^2-k^2)^2+(2k\text{Im }k_s)^2}$. For every corresponding factor the signs of the correspondent scattering l−th partial time delays (see Appendix F) will be the same, differing from those time delays studied here of the l−partial-wave scattering by the reduction with a factor of two in absolute value.

In the more general case when at the external region $r > a$, besides the centrifugal barrier, there is a potential, decreasing more rapidly than any exponential function,

$$\begin{aligned}\tau_l &= \hbar\frac{\partial \arg S_l(\gamma,k)[f_{l+}(k,r)/f_{l-}(k,r)]}{\partial E}\\ &= \hbar\frac{\partial \arg S_l}{\partial E} + \frac{2r}{v} + \frac{2}{v}\Im\frac{\partial\varphi_{l+}/\partial k}{\varphi_{l)}},\end{aligned}$$

where $\varphi_{l+}(k,r) = \exp(-ikr)f_{l+}(k,r)$.

Taking into account that the expression $2\Re i\frac{\partial\varphi_{l+}/\partial k}{\varphi_{l+}}$ tends to 0 when $r\to\infty$, and utilizing the result (108), one can also easily show that also in this case the inequality (105) is valid for sufficiently large values of r.

Finally, in the case when $f_{l-}(k,r)$ has in D^+ singularities of the type (22) for the potentials with decreasing exponential law, it is convenient to introduce the function $\tilde{S}_l(\gamma,k,r) = S_l(\gamma,k)\frac{\varphi_{l+}(k,r)}{\varphi_{l-}(k,r)}$ instead of $S_l(\gamma,k)$. Then, rewriting Eq. (18) and (19) in the forms:

$$\begin{aligned}(-1)^l\oint_{k_{nl}} \tilde{S}_l(\gamma,k,r')\frac{\varphi_{l-}(k,r')}{\varphi_{l+}(k,r')}f_{l+}(k,r)f_{l-}(k,r')dk\\ = (B_{nl})^2 f_{l+}(k,r)f_{l+}(k,r'),\end{aligned} \tag{18a}$$

$$\begin{aligned}(-1)^l\oint_{k_m} \tilde{S}_l(\gamma,k,r')\varphi_{l-}(k,r')\exp(ikr')f_{l+}(k,r)dk\\ = \oint_{k_m} f_{l+}(k,r)f_{l-}(k,r'),\end{aligned} \tag{19c}$$

one can easily conclude that the function $\tilde{S}_l(\gamma,k,r)$ has poles of the first order on the upper imaginary semi-axis which correspond to the bound states with the residues

$$(-1)^{l+1} i \frac{(B_{nl})^2}{2\pi} \frac{\varphi_{l+}(k_{nl},r')}{\varphi_{l-}(k_{nl},r')}$$

and, unlike $S_l(\gamma,k,r)$, it has no "redundant" poles. If one chooses sufficiently large finite values of r', for which at the fixed l the relation $\frac{\varphi_{l+}(k_{nl},r')}{\varphi_{l-}(k_{nl},r')}$ will have the same sign independent from k_{nl} (it can always be obtained, because $\varphi_{l\pm}(k_{nl},r) \to 1$ when $r \to \infty$, if k_{nl} does not coincide with a "redundant" pole; but if such coincidence occurs, the correspondent residue will be 0, since the correspondent pole vanishes!). Then the direct calculation of the quantity

$$\tau_l(\gamma,r) = \hbar \frac{\partial \arg \tilde{S}_l(\gamma,k,r)}{\partial E} + \frac{2r}{v}$$

relative to the scheme (107)-(108) will show the validity of (106) for sufficiently large r in this case also.

The same procedure (8a)-(9a) etc can also be repeated for the case of the presence in the external region $r > a$ of a potential tail of the Yukawa type because of the coincidence of the logarithmic divergence at points $k_\gamma = ib/2$ of the factor $F(k)$ in the expression (40a) for $S_l(\gamma,k,r)$ and of the term $\left[1 + \frac{i\rho}{2k} \ln\left(1 + \frac{2ik}{b}\right)\right]^{-1}$ in the expression (24a) for the function $f_{l-}(k,r)$ and hence its vanishing in $\tilde{S}_l(\gamma,k,r)$.

The case with the non-unitary $S_l(\gamma,k,r)$ appears to be somewhat more complicated. Let us rewrite (103a) in the following form:

$$\langle \tau_l(\gamma,r)\rangle = \left\langle \hbar \frac{\partial \arg S_l(\gamma,k) f_{l+}(k,r)}{\partial E} \right\rangle_1 + \left\langle \hbar \frac{\partial \arg f_{l-}(k,r)}{\partial E} \right\rangle_2 + \left\langle \hbar \frac{\partial \arg A(k)}{\partial E} \right\rangle_1 - \left\langle \hbar \frac{\partial \arg A(k)}{\partial E} \right\rangle_2, \tag{103b}$$

where $\langle\cdots\rangle_1$ and $\langle\cdots\rangle_2$ signify the average in the momentum space with the weights $|AS_l f_{l+}|^2$ and $|Af_{l-}|^2$, relatively, then choose without the limitation of the generality such an $A(k)$ in order that the quantity $\hbar\partial \arg A/\partial E$ would be limited (such a choice of $A(k)$ does physically signify that the mean time moment of the incoming-wave entrance into the sphere of radius r around the scatterer, which is equal to $-\langle([v\,|f_{l-}|^2]^{-1}\rangle_1 + \langle \hbar\partial \arg A/\partial E\rangle_1$, would be finite). Then, since the two last terms in (103b) are finite, and the quantities $\langle \hbar\partial \arg f_{l\pm}/\partial E\rangle_{1,2}$ are positive and proportional to r for sufficiently large r, one can affirm that $\langle \tau_l(\gamma,r)\rangle \geqslant 0$ at least at the range $r \gg a$.

Thus, the completeness condition of type (11) together with the conditions of symmetry and generalized unitarity of $S_l(\gamma,k)$ guarantee the fulfillment of the causality (105) for sufficiently large values of r but, in general, do not ensure the fulfillment of the micro-causality for $r \geqslant a$ (mainly because of the influence of the centrifugal barrier and partially because of the distortion of the wave-packet form during scattering).

In the case of non-central or parity-violating interactions the relation

$$\hbar \frac{\partial \arg S^j_{l'l}}{\partial E} = \frac{2}{v}\left[-\alpha - \sum_n \frac{2\mathrm{Im}\, k_{nj}(k^2 + |k_{nj}|^2)}{[|k_{nj}|^2 - k^2]^2 + [2k\mathrm{Im}\, k_{nj}]^2} - \sum_m \frac{\chi_m}{k^2+\chi_m^2} + \sum_p \frac{\chi_p}{k^2+\chi_p^2} + \sum_{t,t'} \frac{\mathrm{Im}\, k_{t,t'}}{(\mathrm{Re}\, k_{t,t'} - k)^2 + (\mathrm{Im}\, k_{t,t'})^2} - \sum_{s,s'} \frac{\mathrm{Im}\, k_{s,s'}}{(\mathrm{Re}\, k_{s,s'} - k)^2 + (\mathrm{Im}\, k_{s,s'})^2}\right] \tag{109}$$

must be valid instead of (108) for $S^j_{l'l}$. Here $\mathrm{Im}\, k_{nj} > 0, \chi_m = -ik_m < 0, \mathrm{Im}\, k_{s,s'} < 0$, and χ_p, $\mathrm{Im}\, k_{t,t'}$ can be not only positive but also negative and, moreover, the number of points with χ_p, $\mathrm{Im}\, k_t$, $\mathrm{Im}\, k_{t'}$ can be infinite. Therefore, a causality condition like (106a) and (106b) demands certain restrictions for the topology of zeros and poles of $S^j_{l'l}$, namely

$$-\sum_m \frac{\chi_m}{k^2+\chi_m^2} + \sum_p \frac{\chi_p}{k^2+\chi_p^2} + \sum_{t,t'} \frac{\mathrm{Im}\, k_{t,t'}}{(\mathrm{Re}\, k_{t,t'} - k)^2 + (\mathrm{Im}\, k_{t,t'})^2} - \sum_{s,s'} \frac{\mathrm{Im}\, k_{s,s'}}{(\mathrm{Re}\, k_{s,s'} - k)^2 + (\mathrm{Im}\, k_{s,s'})^2} \geqslant 0. \tag{110}$$

Now we pass to the durations of many-channel scattering. According to the Simonius approach (96), we use for the scattering l−th partial time delay the expression (F, 5) from Appendix F:

$$\left\langle \Delta\tau^{(J\Pi)}_{ji}(E)\right\rangle = \frac{\left\langle \left|T^{(J\Pi)}_{ji}(\gamma,k)\right|^2 \partial \arg T^{(J\Pi)}_{ji}(\gamma,k)/\partial E \right\rangle}{\left\langle \left|T^{(J\Pi)}_{ji}(\gamma,k)\right|^2 \right\rangle},$$

where in a simplified form $|A(k)|^2 \to C\delta(E-\bar{E})$, $T_{ji}^{(J\Pi)}(\gamma,k) = [\delta_{ji} - S_{ji}^{(J\Pi)}(\gamma,k)]B$ (B and C are unessential constants). Near an isolated resonance, distorted by a non-resonance background, from (F, 5) we obtain, according to Appendix F, the result (F, 6). From (F, 6) it is obvious that if $0 < \Re(\tilde{S}_{ji}^{(J\Pi)} - \delta_{ji}^{-1} 2\beta_{ji}^{(J\Pi)}) < 1$ and $\Delta\tilde{\tau}_{ji}^{(J\Pi)} = 0$, then $0 < \Re\gamma^{(J\Pi)} < \Gamma_r^{(J\Pi)}$ and hence the quantity $\Delta\tau_{ji}^{(J\Pi)}(E)$ appears to be *negative* in the energy interval $\sim \Re\gamma^{(J\Pi)}$ near energy $E_r^{(J\Pi)} + \Im\gamma^{(J\Pi)}/2$. When $0 < \Re((\tilde{S}_{ji}^{(J\Pi)} - \delta_{ji})^{-1} 2\beta_{ji}^{(J\Pi)}) \ll 1$, the minimal delay time can obtain the value $-2\hbar/\Re\gamma^{(J\Pi)} < 0$. Thus, when $\Re\gamma^{(J\Pi)} \to 0^+$, the interference of the resonance and the background scattering can bring to *arbitrarily large value of the time advance* instead of the time delay! Such a paradoxical situation is connected with the presence of the zero $E_r^{(J\Pi)} - i\gamma^{(J\Pi)}/2$, besides the pole $E_r^{(J\Pi)} - i\Gamma_r^{(J\Pi)}/2$, of the $T-$matrix in the lower unphysical half-plane of the Riemann surface. This *phenomenon of delay-advance* (i.e. of the transition of time delay to time advance near isolated resonance of many-channel collisions, distorted by the non-resonance background in the center-of-mass system) was first revealed in [50–52] and was confirmed in [53, 54]. Then, later (see, for instance, [53, 54]), such a paradoxical phenomenon was resolved in passing to the laboratory system of reference, where, unlike the center-of-mass system, the compound nucleus moves during the resonance decay and virtual time advance, caused by the distortion of isolated resonance by the non-resonance background in the center-of-mass system, is eliminated by correct phase analysis with the introduction of an additional phase parameter, describing the space-time shift caused by the real motion of the decaying compound nucleus in the laboratory system.

9. Final remarks, conclusions and perspectives

In the presented review there are the results of almost the complete study of the non-relativistic S-matrix analytic structure for unspecified (or *unknown*) central, non-central (tensor) and parity-violating T-invariant interactions, linear or non-linear, with unknown physical dynamics and kinetics, with possible absorption and/or generation of bombarding particles inside a sphere of small radius $r \leqslant a$, surrounded in the external range ($a < r < \infty$) by a centrifugal barrier with the possible presence of decreasing potential tails (decreasing more rapidly than any exponential function, or according to the exponential law, or the Yukawa law etc) for one-channel and discrete-many-channel scattering. This study was based on some general mathematical assumptions like the possibility of the S-matrix analytic continuation into the regions of complex values of particle wave numbers or kinetic energies and the completeness conditions for external wave functions, and on physical principles like the causality and some kinds of symmetry for the S-matrix.

It is rather curious how the results of research, based on the well-known cognitive principle "with the least number of assumptions to obtain the most number of results of rather general physical and mathematical character", can also help to reveal some concrete physical phenomena and effects: (a) the enhancement phenomena caused by parity violations, indicated in Appendix E; (b) the phenomena of time resonances (explosions), formed from the strongly overlapping energy resonances of high-energy many-channel nuclear reactions, mentioned in Section 5 (see [44]); (c) the paradox delay-advance phenomena in the center-of mass system at the range of nuclear isolated resonances, distorted by the non-resonance background, and such a paradoxical phenomenon was resolved in passing to the laboratory system of reference, where it was eliminated by correct phase analysis with the introduction of an additional phase parameter, describing the space-time shift caused by the real motion of the decaying compound nucleus in the laboratory system (see Section 5 and [50, 53, 54]).

In the existing publications on the analytic structure of the S-matrix and scattering (collision) amplitude the topic of *dispersion relations* (certain integral relations for the scattering amplitude) often attracts a lot of attention. A great number of papers and the majority of manuals on quantum mechanics, even when considering less than very high energy ranges, are dedicated to this topic (see, for instance, [8, 22, 30, 49, 55–59] with an extensive bibliography therein). This topic is studied in detail, both for known potential interactions and microscopically unknown interactions. There are also several authors papers on dispersion relations connected with a field of extending the dispersion relations to the nuclear optical model and compound-nucleus processes [60–62]. In the presented review I have limited myself to only the less known field of the analytic structure of the S-matrix for unknown (inside a microscopic sphere with $r \leqslant a$) interactions and motion equations, because it had not been reflected till now in practically any review or monograph on quantum mechanics written in English.

An interesting perspective on future investigations follows from consideration of all the presented results - *a research program of concrete tasks and problems, and then the continuation, extension and application of the rigorous study of the analytic properties of the S-matrix on the basis of general physical principles and general mathematic assumptions together with a search for the observable phys-*

ical manifestations of microscopic quantum collisions for unknown interactions, dynamics and kinetics:

(1) Between the remaining concrete tasks it is possible to propose (a) the study of enhancement phenomena caused by violations of T-invariance, quite similar to enhancement phenomena caused by parity violations; (b) the study of the S-matrix analytic structure for unknown interactions, surrounded by a centrifugal barrier and a screened Coulomb barrier — the last one is namely the Yukawa-potential type, differing from the Yukawa potential by the positive sign (repulsion instead of attraction) and by the scale.

(2) As a possible continuation of the presented approach there remains open a way for the study of other types of many-channel non-relativistic collisions (for instance, collisions with rearrangement of colliding systems, with the multiple generation of particles, chain reactions etc), the classes of T-violating interactions, including the interactions with microscopic quantum dissipation (quantum friction), various relativistic collisions, collisions in the presence of external fields, scattering with accompanied processes like bremsstrahlung etc.

(3) A somewhat unexpected perspective also appeared - how the rigorous mathematical method or approach can help to reveal concrete physical phenomena and effects (enhancement phenomena caused by parity violations or by T-invariance violations, virtual delay-advance phenomena, time resonances,...).

(4) And finally there are problems that remain regarding the generalization of the results obtained for the analytic properties of the S-matrix in the cases of relativistic quantum mechanics, quantum field theory and quantum gravity.

Appendix A The derivation of the product expansion for $S_l(\gamma, k)$ with the generalized "unitarity condition" (1a) for unknown interactions surrounded by a centrifugal barrier and potentials decreasing more rapidly that any exponential function (within approach outlined in [14, 16])

First, we consider some intermediate products. Taking the behavior of (20) into account, we can easily see that the factor $\exp(2i\alpha k)$, with $\alpha \leqslant a$, is one of the multipliers of $S(\gamma, k)$.

The product

$$\prod_n \frac{k_{nl}+k}{k_{nl}-k} \tag{A, 1}$$

contains all the poles of $S(\gamma, k)$ in D^+, has no other singularities or zeros, and satisfies conditions (1a)-(2a). It is bounded for $\Re k > 0$ due to the final number of k_{nl} (we also remind that all the eigen values k_{nl} are simple (non-multiple) and are situated outside the real axis k) and, if $\gamma = \Re\gamma$, it has the absolute value equal to 1 for $\Re k > 0$.

The product

$$\prod_n \frac{k_{nl}+k}{k_{nl}-k} \prod_\lambda \frac{k_\lambda - k}{k_\lambda + k} \prod_s \frac{k_s - k}{k_s + k} \prod_{s'} \frac{k_{s'} - k}{k_{s'} + k} \tag{A, 2}$$

contains all the poles and the zeros of $S_l(\gamma, k)$ in D^+, has no other singularities in D^+, satisfies conditions (1a)-(2a), and is regular in D^+ in the case of convergence. It is bounded for $\Re k > 0$ due to conditions (10a), (10b) and, if $\gamma = \Re\gamma$, it has the absolute value equal to 1 for $\Re k > 0$.

The function

$$J_{lN}(k) = \tilde{S}(\gamma, k)\left[\prod_n \frac{k_{nl}+k}{k_{nl}-k} \prod_\lambda^{N_1} \frac{k_\lambda - k}{k_\lambda + k} \prod_s^{N_2} \frac{k_s - k}{k_s + k} \prod_{s'}^{N_3} \frac{k_{s'} - k}{k_{s'} + k}\right]^{-1}, \tag{A, 3}$$

with $\tilde{S}(\gamma, k) = e^{2i\alpha k} S(\gamma, k)$, for finite numbers $N = N_1 + N_2 + N_3$ is regular and bounded on the real axis k. If the limit

$$J_l(k) = \lim_{N\to\infty} J_{lN}(k)$$

exists, then it has the same properties, and then

$$S_l(\gamma, k) = e^{-2i\alpha k} \prod_n \frac{k_{nl}+k}{k_{nl}-k} \prod_\lambda \frac{k_\lambda - k}{k_\lambda + k} \prod_s \frac{k_s - k}{k_s + k} \prod_{s'} \frac{k_{s'} - k}{k_{s'} + k}. \tag{A, 4}$$

To be certain of the validity (correctness) of (A, 4), it is necessary to show that three infinite products in (A, 4) converge. The condition for their absolute convergence is the convergence of the sum

$$\sum_{\lambda}\left|\frac{k_\lambda - k}{k_\lambda + k} - 1\right| + \sum_{s}\left|\frac{k_s - k}{k_s + k} - 1\right| + \sum_{s'}\left|\frac{k_{s'} - k}{k_{s'} + k} - 1\right| = 2\,|k|\left\{\sum \frac{1}{|k_\lambda + k|} + \sum_{s} \frac{1}{|k_s + k|} + \sum_{ss'} \frac{1}{|k_{s'} + k|}\right\} \quad \text{(A, 5)}$$

In turn, convergence of (A, 5) is determined by the convergence of the sum

$$\sum_{\lambda} \frac{1}{|k_\lambda|} + \sum_{s} \frac{1}{|k_s|} + \sum_{s'} \frac{1}{|k_{s'}|} \quad \text{(A, 6)}$$

because $|k_\lambda| \to \infty$ as $\lambda \to \infty$, $|k_s| \to \infty$ as $s \to \infty$ and $|k_{s'}| \to \infty$ as $s' \to \infty$. It is easy to see that sum (A, 6) converges if the analyticity of the function

$$\tilde{J}_l(k) = \frac{\tilde{S}_l(\gamma, k)}{\prod_n \frac{k_{nl}+k}{k_{nl}-k}} \quad \text{(A, 7)}$$

in D^+ is taken into account together with the absence of its zeros above the real axis k and if the following theorem is used.

Theorem [23]. *Let a function f(z) be bounded and analytic for $\Re z \geqslant 0$, and let its zeros in the right half-plane z be $r_1 e^{\theta_1}, r_2 e^{\theta_2}, \ldots$. Then the series $\sum_{n=1}^{\infty} r_n^{-1} \cos\theta_n$ converges.*

Because $\cos\theta_n = |\cos\theta_n| \geqslant \varepsilon$, where $\varepsilon \neq 0$, for $\tilde{J}_l(\rho)$ with $\rho = ik$, we have

$$\varepsilon \sum_{n=1}^{\infty} r_n^{-1} < \sum_{n=1}^{\infty} r_n^{-1} \cos\theta_n < \infty,$$

which proves that sum (A, 6) converges. Hence, the infinite products in (A, 4) converge uniformly and give a meromorphic function with poles $-k_\lambda, -k_s$ and $-k_{s'}$.

Now let consider the behavior of $S(\gamma, k)$ when $k \to 0$. From the symmetry conditions (2a)-(3a) and the finiteness of the wave function $R_l^{(+)}(\gamma, k, r)$ at point $k = 0$ (outside the interaction sphere) it almost evidently follows that $S(\gamma, 0)$=1. Indeed, for rapidly decreasing potential tails ($r > a$) of type (5) simple analysis of Eq. (8) shows that the behavior of the functions $f_{l\pm}(k, r)/kr$ at $k \to 0$ is defined by the behavior of $h_l^{(1,2)}(kr) = j_l(kr) \pm i n_l(kr)$, i.e. by $n_l(kr) \underset{k\to 0}{\to} -\frac{(2l-1)!!}{(kr)^{l+1}}$ (to within a constant). So, the behavior of the wave function $R_l^{(+)}(\gamma, k, r) = \frac{i}{2kr}[f_{l-}(k, r)\exp(il\pi/2) - S_l(\gamma, k) f_{l+}(k, r)\exp(-il\pi/2)]$ at $k \to 0$ is defined by the behavior of the expression $\frac{i}{2}\{-i j_l(kr)[1 + S_l(\gamma, k)] - n_l(kr)[1 - S_l(\gamma, k)]\}$ from which, taking into account the divergence of $n_l(kr)$ at $k \to 0$, it does immediately follow that $S(\gamma, 0)$=1 (and this automatically satisfies the symmetry conditions (2a)-(3a) at the point k= 0) and also for small k the evaluation $S(\gamma, k) = 1 + o(k^{l+1})$ is, at least, valid.

To the function $J_l(k)$, we note that it is not only analytical in D^+ but being an entire function without zeros it can be written in the form $\exp(u + iv)$ where $u + iv$ is an entire function (see, for example, the relevant theorem in [9]). The real function $u(k)$ must be negative in D^+ due to equality (21) and be positive in D^- due to the conditions (4) and (5). Therefore, according to the "Caushy-Riemann" equations the condition

$$0 = \partial u / \partial \Im k = -\partial v / \partial k, \qquad \Im k = 0, \quad \text{(A, 8)}$$

must be satisfied over the real axis k. From (A, 8) it follows that the function $v(k)$ increases monotonically and does not take any real value more than once. Then the function $u + iv$ does not take any imaginary value more than once and consequently must be a linear function of k:

$$u + iv = 2i\alpha_1 k + \alpha_2. \quad \text{(A, 9)}$$

Obviously, $\alpha_1 = 0$ and, due to the equality $S_l(0) = 1$, $\alpha_2 = 0$.

Thus, we finally obtain:

$$S_l(\gamma, k) = e^{-2i\alpha k} \prod_n \frac{k_{nl} + k}{k_{nl} - k} \prod_\lambda \frac{k_\lambda - k}{k_\lambda + k} \prod_s \frac{k_s - k}{k_s + k} \prod_{s'} \frac{k_{s'} - k}{k_{s'} + k}, \quad \text{(A, 10)}$$

with $\alpha = a - \alpha_1 \leqslant a$, which is just (23a). In a similar way we can obtain (23b).

Appendix B Necessary and sufficient conditions for the existence of the "redundant" poles $(m/2)ib$ with $b > 0$, $m = 1, 2, \ldots$, in $f_0(k,r)$, and hence also in $S_0(k)$[1]

Using the Jost equation

$$f_0(k,r) = \exp(ikr) + k^{-1}\int_r^\infty \sin k(r'-r)V(r')f_0(k,r')dr' \quad \text{(B, 1)}$$

and solving it formally by the method of successive approximations, we obtain the sum

$$f_0(k,r) = \sum_{\nu=0}^{\infty} f_{0\nu}(k,r), \quad \text{(B, 2)}$$

where

$$f_{00}(k,r) = \exp(ikr), \quad \text{(B, 3)}$$

$$f_{0\nu}(k,r) = [\exp(ikr)](2ikr)^\nu \int_r^\infty \{\exp[2ik(r_1-r)]-1\}V(r_1)\cdots\int_{r_\nu-1}^\infty \{\exp[-2ik(r_\nu-r_{\nu-1})]\}V(r_\nu)dr_\nu\ldots dr_1, \quad \nu = 1,2,\ldots \quad \text{(B, 4)}$$

for the cases when

$$|V| < M/r^{2+\delta}, \quad M < \infty \text{ and } \delta > 0 \quad [24]. \quad \text{(B, 5)}$$

Further, we use the following theorems:

Theorem A [23]. *Let F(k,r) be a function of complex variables k and r which is definite and continuous for all values of k in some domain D and for all values of r over the contour C. Then the function*

$$\Phi(k) = \int_C F(k,r)dr$$

is an analytical function of k in the domain D. When the contour C is infinite, the uniform convergency of the integral is also necessary.

Theorem B [23]. *Let all the functions of the series $u_1(z), u_2(z), \ldots$ be analytic functions of z in the domain D and the sum $\sum_{n=1}^{\infty} u_n(z)$ be uniformly convergent in every domain D', inside D. Then the function $u(z) = \sum_{n=1}^{\infty} u_n(z)$ is an analytic function of z inside D.*

It follows from (B, 4) and theorem A that $f_{0\nu}(k,r)\,(\nu = 0,1,2,\ldots)$ are analytical functions of k in the upper half-plane. If Born series (B, 2) is uniformly convergent, then by theorem B, the function $f_0(k,r)$ is analytic in the upper half-plane. Similarly, $f_0(-k,r)$ is analytic in the lower half-plane. Moreover, if $f_{0\nu}(k,r)$, $\nu \geqslant 1$, is analytic in the lower half-plane, then all successive terms are also analytic.
And now it is shown that the following theorems holds.

Theorem 3. *If $f_{0\nu}(k,r)$ $(\nu > 1)$ has singular points, then $f_{01}(k,r)$ has to have them also.*

[1] The translation from the Ukrainian publication [9], dedicated to the memory of Yu.V. Tsekhmistrenko – see also [16]

Indeed, let $f_{01}(k,r)$ be analytic everywhere. Then all successive terms are also analytic everywhere. This contradiction proves theorem 3.

We use the analytic structure of $f_{01}(k,r)$ in the upper half-plane. Obviously, the problem can be reduced to studying the analytical structure of the integral

$$I = \int_r^\infty [\exp(-2ikr')]V(r')dr'$$

because all other terms give functions which are analytic in the whole plane.

As shown in [23, 24], in the case of a potential

$$V(r) = P_n(r)\exp(-br), \tag{B, 6}$$

where $P_n(r)$ is an nth-order polynomial and $b > 0$, the function $f_0(-k,r)$ has poles of an order not higher than $n+1$ at the points $ib/2, ib, 3ib/2, \dots$, and it is analytic at all other points of the complex k-plane.

We now show that if $f_{01}(-k,r)$ has a pole of order not higher than $n+1$ at the point $ib/2$, then the potential must have a term of type (B, 6). It follows from such assertion that the integral

$$I_1 = \int_r^\infty [\exp(-2ikr')]V_1(r')dr' \tag{B, 7}$$

with $V_1 = V - V_2$ (where term V_2 does not give poles), can be represented on the real axis k in the form

$$I_1 \equiv \sum_{\mu=0}^{n} \frac{\varphi_\mu(-2ik,r)}{(2ik+b)^\mu}, \tag{B, 7a}$$

where $\varphi_\mu(-2ik,r)$ is analytic at all the poles and is nonzero at the point $ib/2$. Further, re-writing I_1 in the form

$$I_1 = \int_r^\infty [\exp(-(2ik+b)r')][V_1(r')\exp(br')]dr'$$

and successively integrating by parts, we can transform the right-hand side of identity (B, 7a), into the series:

$$\begin{aligned}
-\exp(-2ikr)V_1 - \frac{1}{2ik+b}\exp(-2ikr)\left(bV_1 + \frac{dV_1}{dr}\right) - \ldots - \\
-\frac{1}{(2ik+b)^n}\exp(-2ikr)\left(b^nV_1 + nb^{n-1}\frac{dV_1}{dr} + \ldots + \frac{d^nV_1}{dr^n}\right) - \ldots \equiv \\
\equiv \sum_{\mu=0}^{n}\frac{\varphi_\mu(-2ik,r)}{(2ik+b)^\mu}
\end{aligned} \tag{B, 8}$$

Comparing the coefficients of equal powers of $2ik+b$, we obtain a successive system of the corresponding equalities. Because there is no term containing $\frac{1}{(2ik+b)^m}$, $m > n$, in the right-hand side of (B, 8), we obtain

$$b^{n+1}V_1 + (n+1)b^n\frac{dV_1}{dr} + \frac{(n+1)n}{2} - b^{n-1}\frac{d^2V_1}{dr^2} + \ldots + \frac{d^{n+1}V_1}{dr^{n+1}} = 0,$$

whence follows

$$V = P_n(r)\exp(-br) + V_2, \tag{B, 9}$$

where V_2 is an arbitrary function which cannot be brought to the form $P_s(r)\exp(-b_sr)$.

The following general theorem can be proved analogously.

Theorem 4. *For $f_0(-k,r)$ to have poles of order not higher than (n_1+1) at the points $ib_1/2, ib_1, 3ib_1/2, \ldots$, not higher than (n_2+1) at the points $ib_2/2, ib_2, 3ib_2/2, \ldots$, not higher than (n_m+1) at the points $ib_m/2, ib_m, 3ib_m/2, \ldots$, it is necessary and sufficient that the corresponding potential would have a term* $\sum\limits_{m,n_m} P_{n_m}(r)\exp(-b_m r)$.

Obviously, to have *essentially singular points*, it is necessary and sufficient that the corresponding potential contain a term $X(r)\exp(-br)$, where $X(r)$ is a uniformly convergent infinite series of the type $\sum\limits_{n=0}^{\infty} \alpha_n r^n$ not equal to $\exp(\text{const } r^{\alpha})$, $0 < \alpha < \infty$.
Investigating the behavior of I on the axis $k = ib/2$ in the case where

$$V(r) = v(r)\exp(-br),$$

where $v(r)$ is an arbitrary function not having a factor exp (const r^{α}), $\alpha \geqslant 1$, one can easily conclude that the branch points can appear on that axis. One of the simplest cases is the potential $[\exp(-br)\sin(cr)]/r^q$. For various $q>0$, this potential can give branch points of different types at $k = ib/2 \pm c/2$.
If $I(r) = v(r)\exp(-br^{\alpha})$ and $\alpha > 1$ is an integer, then I and hence $f_0(k,r)$ are analytic functions in the whole plane.
Thus, the presence of the factor $\exp(-br)$ leads to the function $f_0(-k,r)$ becoming non-analytic in the upper half-plane.

Appendix C The generalization of the results of Appendix A for the presence of the centrifugal barrier with $l > 0$ outside the unknown interaction

Let it be that at the external region (with $r \geqslant a$) the centrifugal barrier $\frac{\hbar^2 l(l+1)}{2\mu r^2}$, $l > 0$, and a potential of the type $V = V_0 P_n(r)\exp(-br)$, where $P_n(r)$ is an nth-order polynomial and $b > 0$. Adopting from [16, 19], we can write the integral equation

$$f_{l-}(k,r) = f_{0-}(k,r) + l(l+1)\int\limits_r^{\infty} G(k;r,r')(r')^{-2} f_{l-}(k,r')dr', \tag{C, 1}$$

where $r > a$ and the Green's function has the form

$$G(k;r,r') = (2ik)^{-1}[f_{0-}(k,r)f_{0+}(k,r') - f_{0-}(k,r')f_{0+}(k,r)] = [f_{0+}(k,0)]^{-1}[\Phi(k,r)f_{0+}(k,r') - \Phi(k,r')f_{0+}(k,r)]. \tag{C, 2}$$

Because the function $\Phi(k,r) = \frac{1}{2ik}[f_{0-}(k,0)f_{0+}(k,r) - f_{0+}(k,0)f_{0-}(k,r)]$ is regular everywhere, the Green's function in Eq. (B, 2) has no singularity for any finite k and Eq. (B, 1) allows the computation of $f_{l-}(k,r)$ from $f_{0-}(k,r)$ in a univalent way. The solution $f_{l-}(k,r)$ of Eq. (B, 1) for any l contains the same isolate singularities at the same points as the function $f_{0-}(k,r)$. For instance, (1) if for $r > a$ there is a centrifugal barrier and a potential decreasing more rapidly than any exponential function, in that case $f_{l\pm}(k,r)$ are analytical in all of the plane k except the points k=0 and $k=\infty$, as well as $f_{0\pm}(k,r)$; (2) if the external potential tail contains a term $V_0\sum\limits_{m,n_m} P_{n_m}(r)\exp(-b_m r)$, the functions $f_{l\pm}(k,r)$ for any $l \geqslant 0$ are analytical in all of the plane k except the points k=0 and $k=\infty$ and also have the poles of order not higher than $(n_1$+1) at the points $ib_1/2, ib_1, 3ib_1/2, \ldots$, not higher than $(n_2$+1) at the points $ib_2/2, ib_2, 3ib_2/2, \ldots$, not higher than $(n_m$+1) at the points $ib_m/2, ib_m, 3ib_m/2, \ldots$; (3) if the external potential tail contains a term $V_0[(br)^{-1}\exp(-br)]$, $V_0 > 0, b^{-1} \sim a$, the functions $f_{l\pm}(k,r)$ for any $l \geqslant 0$ have the logarithmic singularity at the point $k_\gamma = ib/2$.

Appendix D The derivation of the formula (32)

We rewrite expression (24a) for $f_{0-}(k,r)$ inside the circle γ_c around the point $k_\gamma = ib/2$ in the form

$$\begin{aligned} f_{0-}(k,r) &\to \lim_{k\to k_\gamma}\left\{1+\rho\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}\int_b^{\infty} db'\frac{\exp(-b'r)}{b'(b'\mp 2ik)}\right\}\exp(\pm ikr) \\ &= \lim_{\eta\to 0}\left\{1+\rho\left[1+\frac{i\rho}{2k}\ln\left(1+\frac{2ik}{b}\right)\right]^{-1}\int_0^{\infty} d\tilde{b}\frac{\exp[-(\tilde{b}+b)r]}{(\tilde{b}+\eta)(\tilde{b}+b)}\right\}\exp(-ikr), \end{aligned} \tag{D, 1}$$

where we introduce the variables $\tilde{b} = b' - b, \eta = 2ik + b$. We then make the following simple transformations of the right-hand part of (D, 1):

$$\begin{aligned} f_{0-}(k,r) \to \lim_{\eta\to 0} \left\{ 1 + \rho \left[1 + \frac{i\rho}{2k} \ln\left(1 + \frac{2ik}{b}\right)\right]^{-1} \int_0^\infty d\tilde{b} \frac{\exp(-\tilde{b}r)\exp(2ikr)}{(\tilde{b}+\eta)(\tilde{b}-2ik+\eta)} \right\} \exp(-ikr) \\ = \exp(-ikr) + \rho \left[1 + \frac{i\rho}{2k} \ln\left(1 + \frac{2ik}{b}\right)\right]^{-1} \int_b^\infty d\tilde{b} \frac{\exp(-\tilde{b}r)}{\tilde{b}(\tilde{b}-2ik)} \exp(ikr) \\ + \rho \left[1 + \frac{i\rho}{2k} \ln\left(1 + \frac{2ik}{b}\right)\right]^{-1} \lim_{\eta\to 0} \int_0^b d\tilde{b} \frac{\exp(-\tilde{b}r)}{(\tilde{b}+\eta)(\tilde{b}-2ik+\eta)} \exp(ikr) \\ = \exp(-ikr) + \frac{A_-}{A_+}[f_{0+}(k,r) - \exp(ikr)] + \rho \left[1 + \frac{i\rho}{2k} \ln\left(1 + \frac{2ik}{b}\right)\right]^{-1} \lim_{\eta\to 0} \int_0^b d\tilde{b} \frac{\exp(-\tilde{b}r)}{(\tilde{b}+\eta)(\tilde{b}-2ik+\eta)} \exp(ikr), \end{aligned} \qquad \text{(D, 2)}$$

where $A_\mu = [1 \pm \frac{i\rho}{2k} \ln(1 \pm \frac{2ik}{b})]^{-1}$ and we use definition (24a) for $f_{0+}(k,r)$.
We now analyze the last integral $J = \lim_{\eta\to 0} \int_0^b d\tilde{b} \frac{\exp(-\tilde{b}r)}{(\tilde{b}+\eta)(\tilde{b}-2ik+\eta)}$ in the last right-hand part of (D, 2), using formulas (3.352.1) and (8.214.1) from [26]:

$$\begin{aligned} J = \lim_{\eta\to 0} \int_0^b d\tilde{b} \frac{\exp(-\tilde{b}r)}{(\tilde{b}+\eta)(\tilde{b}-2ik+\eta)} = \lim_{\eta\to 0} \frac{1}{b-\eta} \int_0^b d\tilde{b} \exp(-\tilde{b}r) \left[\frac{1}{\tilde{b}+\eta} - \frac{1}{\tilde{b}+b} \right] \\ = \lim_{\eta\to 0} \left\{ \frac{1}{b-\eta} \exp(\eta r)[\mathrm{Ei}(-br-\eta r) - \mathrm{Ei}(-\eta r)] - \frac{1}{b-\eta} \exp(br)[\mathrm{Ei}(-2br) - \mathrm{Ei}(-br)] \right\} \\ = \lim_{\eta\to 0} \frac{1}{b} [-\ln(\eta r) + X(\eta, r)], \end{aligned} \qquad \text{(D, 3)}$$

where $X(\eta,r) = \sum_{k=1}^{\infty} \frac{(-br+\eta r)^k}{k \cdot k!} - \sum_{k=1}^{\infty} \frac{(-\eta r)^k}{k \cdot k!} - \exp(br) \left[\sum_{k=1}^{\infty} \frac{(-2br)^k}{k \cdot k!} - \sum_{k=1}^{\infty} \frac{(-br)^k}{k \cdot k!} \right]$ is analytic function of $\eta = b + 2ik$ at the point $\eta = 0$ and in a small circle $|\eta| < b$ and the function Ei is defined in [26]. Further we can obviously rewrite (D, 3) as

$$J = \frac{1}{b} \left\{ \left[1 + \frac{i\rho}{2k} \ln\left(1 + \frac{2ik}{b}\right)\right] \frac{2ik}{\rho} + Z(k,r) \right\}, \qquad \text{(D, 3a)}$$

where $Z(k,r) = X(\eta,r) - \ln(br) - \frac{2ik}{\rho}$ is an analytical function of k at the point $k = k_\gamma = ib/2$ and in a small circle $|k| < b/2$.
Using (D, 3a), we continue the transformations of (D, 2):

$$\begin{aligned} f_{0-}(k,r) \lim_{k\to k_\gamma} \exp(-ikr) + \frac{A_-}{A_+}[f_{0+}(k,r) - \exp(ikr)] \\ + \rho \left[1 + \frac{i\rho}{2k} \ln\left(1 + \frac{2ik}{b}\right)\right]^{-1} \lim_{\eta\to 0} \exp(ikr) \frac{1}{b} \left\{ \left[1 + \frac{i\rho}{2k} \ln\left(1 + \frac{2ik}{b}\right)\right] \frac{2ik}{\rho} + Z(k,r) \right\} \\ = W(k,r) + \left[1 + \frac{i\rho}{2k} \ln\left(1 + \frac{2ik}{b}\right)\right]^{-1} U(k,r), \end{aligned} \qquad \text{(D, 2a)}$$

where $W(k,r) = \exp(-ikr) + \frac{2ik}{b}\exp(ikr)$ and $U(k,r) = \left[1 - \frac{i\rho}{2k}\ln(1 - \frac{2ik}{b})\right]^{-1} f_{0+}(k,r) + \exp(ikr)\{\frac{\rho}{b} Z(k,r) - [1 - \frac{i\rho}{2k}\ln(1 - \frac{2ik}{b})]\}$ are analytic functions of k at the point $k = k_\gamma = ib/2$ and in a small circle $|k| < b/2$.

Appendix E A possibility of sharp enhancement of $S^j_{l'l}$ $(l' \neq l)$ in comparison with $S^j_{ll}(\gamma, k)$ near an isolated resonance

Let us assume that a factor like

$$S^j_{ll'}(\gamma,k) \approx \frac{\delta_{ll'}(E - E_t^{(ll)}) + i\Gamma_t^{(l'l)}/2}{E - E_s + i\Gamma_s/2}, \qquad l', l = 0, 1, \qquad \text{(E, 1)}$$

where $E_t^{(ll)} = \frac{\hbar^2 |k_t^{(ll)}|^2}{2\mu}$, $E_s = \frac{\hbar^2 |k_s|^2}{2\mu}$, $\Gamma_t^{(ll)} = -2ik\text{Im}\ k_t^{(ll)}$, $\Gamma_t^{(00)} = -\Gamma_t^{(11)}$, $\Gamma_s = 2ik\text{Im}\ k_s$, plays an essential role in (64) in some energy region. If $E_t^{(ll)} \approx E_s$, $\Gamma_{1s}^2 + \Gamma_{2s}^2 = \Gamma_s^2$, where $\Gamma_{1s}^2 = (\Gamma_t^{(ll)})^2$, $\Gamma_{2s}^2 = (\Gamma_t^{(l'l)})^2$ for $l' \neq l$, (46) is fulfilled and the scattering cross section is

$$\sigma_{el} \approx \frac{\pi}{k^2} \frac{(\Gamma_s - \Gamma_{1s})^2/4 + \Gamma_{2s}^2/4}{(E - E_s)^2 + \Gamma_s^2/4}. \tag{E, 2}$$

When $\Gamma_{1s} \ll \Gamma_{2s} \approx \Gamma_s$ a sharp enhancement of $S_{l'l}^j (l' \neq l)$ may happen in comparison with $S_{ll}^j(\gamma, k)$ at the resonance. In the extreme case in which $\Gamma_{1s} = 0$, a resonance of $S_{l'l}^j (l' \neq l)$ corresponds to an anti-resonance of $S_{ll}^j(\gamma, k)$. Therefore, the influence of non-central or parity-violating interactions in these resonance regions may be essential even if their strength is very small (but, of course, non-zero).

Appendix F The scattering time delay and the scattering l-th partial time delay

Usually one uses the scattering time delay and the scattering l-th partial time delay (see, for instance, [15, 46–48]), which in the typical approximation of neglecting by the influence of distortions caused by the wave-packet form during the scattering, are defined by the expressions

$$\langle \Delta\tau(E, \Omega_{fi}) \rangle = \frac{\langle |f(E, \Omega_{fi})|^2 \, \partial \arg f(E, \Omega_{fi})/\partial E \rangle}{\langle |f(E, \Omega_{fi})|^2 \rangle} \tag{F, 1}$$

and

$$\langle \Delta\tau_l(E) \rangle = \frac{\langle |T_l(E)|^2 \, \partial \arg T_l(E)/\partial E \rangle}{\langle |T_l(E)|^2 \rangle}, \tag{F, 2}$$

where the scattering amplitude $f(E, \Omega_{fi})$ and functions $T_l(E)$ are connected with the functions $S_l(\gamma, k)$ by relations

$$f(E, \Omega_{fi}) = \frac{4\pi\mu}{-\hbar^2} T(E, \Omega_{fi}) \equiv \sum_l (2ik)^{-1} [S_l(\gamma, k) - 1] \sqrt{4\pi(2l+1)} Y_{l0}(\Omega_{fi}) \tag{F, 3}$$

(Ω_{fi} being the angle of scattering $\vec{k}_i \to \vec{k}_f$) and

$$T_l(\gamma, k) = [1 - S_l] \frac{\hbar^2}{8\pi^{1/2} i\mu k}, \tag{F, 4}$$

respectively.
In the case of the multi-channel scattering (collisions or reactions), generalizing (F, 2) according to the parametrization (96), we define for the partial time delay in the collision $i \to j$:

$$\langle \Delta\tau_{ji}^{(J\Pi)}(E) \rangle = \frac{\left\langle \left| T_{ji}^{(J\Pi)}(\gamma, k) \right|^2 \partial \arg T_{ji}^{(J\Pi)}(\gamma, k)/\partial E \right\rangle}{\left\langle \left| T_{ji}^{(J\Pi)}(\gamma, k) \right|^2 \right\rangle}. \tag{F, 5}$$

Using in a simplified form $|A(k)|^2 \to C\delta(E - \bar{E})$, $T_{ji}^{(J\Pi)}(\gamma, k) = [\delta_{ji} - S_{ji}^{(J\Pi)}(\gamma, k)]B$ (B and C are unessential constants), we obtain near an isolated resonance, distorted by a non-resonance background:

$$\begin{aligned} \Delta\tau_{ji}^{(J\Pi)}(E) = \partial \arg[\delta_{ji} - S_{ji}^{(J\Pi)}(\gamma, k)]/\partial E = \Delta\tilde{\tau}_{ji}^{(J\Pi)} + \hbar\Gamma_r^{(J\Pi)}/2[(E - E_r^{(J\Pi)})^2 + \Gamma_r^{(J\Pi)2}/4] \\ -\hbar \text{Re}\ \gamma^{(J\Pi)}/2[(E - E_r^{(J\Pi)} - \text{Im}\ \gamma^{(J\Pi)}/2)^2 + (\text{Re}\ \gamma^{(J\Pi)}/2)^2], \end{aligned} \tag{F, 6}$$

where $\hat{S}^{(J\Pi)} = \hat{U}^{(J\Pi)} \left[\hat{I} - \frac{i\Gamma_r^{(J\Pi)} \hat{P}_r^{(J\Pi)}}{E - E_r^{(J\Pi)} + i\Gamma_r^{(J\Pi)}/2} \right] \hat{U}^{(J\Pi)T}$, $\Delta\tilde{\tau}_{ji}^{(J\Pi)} = \hbar\partial \arg[\delta_{ji} - \tilde{S}_{ji}^{(J\Pi)}]/\partial E$, $\gamma^{(J\Pi)} = \Gamma_r^{(J\Pi)}[1 + (\delta_{ji} - \tilde{S}_{ji}^{(J\Pi)})^{-1} 2\beta_{ji}^{(J\Pi)}]$, $\hat{\beta}^{(J\Pi)} = \hat{U}^{(J\Pi)} \hat{P}_r^{(J\Pi)} \hat{U}^{(J\Pi)T}$.

Appendix G The derivation of formulae (88) and (89)

If we shift the integration contour C' in (83) into D^+, enveloping all singularities and, in particular, the cut along the imaginary axis with the contour D' depicted in Fig. 3, and utilizing Eq. (79)-(81) with regard to the independence of contours around all the singularities of $S_l^{(m'm)}$ and $f_{l-}(k_m, r)$, and also the analyticity of $S_l^{(m'm)}(\gamma, k_0, k_1, \ldots)$, $S_l^{(m'm)}(\gamma, -k_0^\bullet, -k_1^\bullet, \ldots)$, $f_{l+}(k_{m'}, r)$ and $f_{l+}(k_m, r'')$ on the edges of the cut D', we obtain (80)-(87) and also the following two equalities:

$$\sum_{N=1}^{m'} \int\limits_{\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_{N-1}-\varepsilon_m)}}^{\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_N-\varepsilon_m)}} dk_m \frac{k_m}{k_{m'}} \Bigg[(-1)^l \tilde{S}^l_{m'm}(\gamma, k_0, k_1, \ldots) - (-1)^l \tilde{S}^l_{m'm}(\gamma, -k_0^\bullet, -k_1^\bullet, \ldots)$$
$$+(-1+\delta_{mo}) \cdot \sum_{n-0}^{N-1} \tilde{S}^l_{m'n}(\gamma, k_0, k_1, \ldots) \tilde{S}^l_{mn}(\gamma, -k_0^\bullet, -k_1^\bullet, \ldots) \frac{k_n}{k_m} \Bigg] f_{l+}(k_m, r) f_{l+}(k_m, r') = 0, \quad (G, 1)$$

$$\sum_{N=1}^{m} \int\limits_{\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_{N-1}-\varepsilon_m)}}^{\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_N-\varepsilon_m)}} dk_m \frac{k_m}{k_{m'}} \Bigg[(-1)^l \tilde{S}^l_{m'm}(\gamma, k_0, k_1, \ldots)$$
$$+(-1+\delta_{mo}) \cdot \sum_{n-0}^{N-1} \tilde{S}^l_{m'n}(\gamma, k_0, k_1, \ldots) \tilde{S}^l_{mn}(\gamma, -k_0^\bullet, -k_1^\bullet, \ldots) \frac{k_n}{k_m} \Bigg] f_{l+}(k_{m'}, r) f_{l+}(k_m, r') = 0. \quad (G, 2)$$

Passing to analysis of Eq. (G, 1) and (G, 2), we note that if we would initiate not from the contour C' but from contour C' previously shifted in a parallel way from C' upwards by the variable interval $y\,(0 < y < \sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_m - \varepsilon_{m'})})$ or $x\,(\sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_m - \varepsilon_m)} < x < \sqrt{\frac{2\mu}{\hbar^2}(\varepsilon_m - \varepsilon_0)})$. Then, instead of Eq. (89) and (90), we should obtain the equalities differing from them by the substitution of the upper integration limits on y or x respectively. Differentiating such equalities over y or x, considering the analyticity of the integrands along the integration contours, we obtain equalities (89) and (90).

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