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Dynamical properties of an asymmetric bistable system with quantum fluctuations in the strong-friction limit

Research Article

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Abstract: The dynamical properties of an overdamped Brownian particle moving in an asymmetric bistable system with quantum fluctuations are investigated. Within the strong-friction limit (the quantum Smoluchowski regime), the analytic expression for the relaxation time of the system is derived by means of the projection-operator method, in which the effects of the memory kernels are taken into account. Based on the relaxation time, we consider both the overdamped quantum case and its classical counterpart.In these contexts, the effects of the quantum fluctuations and the asymmetry of the potential are discussed. It is found that: (i) The quantum effects in an asymmetric bistable system on time scales of the relaxation process are more prominent for lower temperatures and smaller asymmetries of the potential. (ii) The quantum effects speed up the rate of fluctuation decay of the state-space variable for lower temperatures. (iii) The asymmetry of the potential first slows down the rate of fluctuation decay of the state-space variable and then increases it.

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Keywords: quantum fluctuations • asymmetric bistable system • relaxation time

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1. Introduction

In classical statistical physics, the description of a system coupled to a thermal bath of temperature T is formulated in terms of Langevin-type equations and corresponds to the Fokker-Planck or the master equations [1]. Applying the Fokker-Planck equation or the master equations, Bouzat and Wio [2, 3] studied the effect of a potential asymmetry on the stochastic resonance in a bistable system driven by Gaussian white noise. Nikitin and co-workers [4, 5] presented a detailed theoretical analysis of switching time distributions of an asymmetric bistable system subject to a periodic and stochastic force. The power spectrum of the same model has been studied in [6]. The stochastic resonance in an asymmetric bistable system subject to correlated multiplicative and additive noise has also been investigated [7, 8].

All of the above works on asymmetric bistable system focus

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on the classical theory. However, systems in the strong-friction limit have attracted a great deal of interest [9–11]. Within the strong-friction limit, the corresponding quantum Langevin equation and the quantum master equations (e.g., of Lindblad form) have been derived [10, 11]. It has recently been shown that for a quantum system coupled to a heat-bath environment the exact dynamics of a dissipative quantum system within the strong-friction limit can be cast into a time-evolution equation for the position distribution. In other words, the quantum Smoluchowski equation (QSE) has been derived, and the important role of quantum fluctuations in this limit has been revealed [12–15]. Based on the QSE derived in [12], the quantum decay rates for barrier potentials driven by external stochastic and periodic forces in the strong-damping regime have been studied [16], and the transient properties of a Brownian particle moving in a bistable system with quantum corrections have also been investigated [17]. This so-derived QSE contradicts the second law of thermodynamics, so a modified quantum Smoluchowski equation (MQSE) has been put forward that consistently describes thermal quantum states by a modified diffusion [18–21]. Subsequently, the stochastic resonance and transient properties in an asymmetric bistable system with quantum fluctuations have been investigated [22], and the diffusive-transport properties of a quantum Brownian particle strongly interacting with a thermostat and moving in a tilted, spatially periodic potential have also been investigated [23]. Some of these investigations were concerned with the stochastic resonance; others were concerned with the transient and transport properties.

The correlation functions and the associated relaxation time are important physical quantities that characterize the statistical behaviour of a stochastic process, and hence they are usually used to describe the fluctuation behaviour in non-equilibrium systems [24]. Early investigations of the characteristic behavior of the relaxation time were limited to the case of the classical theory [25–29]. But in order to characterize the dynamical properties of the asymmetric bistable system with quantum fluctuations, the relaxation time of systems within the strong-friction limit (the quantum Smoluchowski regime) needs to be investigated.

Sections 2 and 3 derive the analytic expressions for the steady-state probability density and the relaxation time of an asymmetric bistable system, respectively. The effects of quantum fluctuations and the asymmetry of the potential on the steady-state probability density and relaxation time are discussed. Finally, some concluding remarks are given in Sec. 4.

2. Steady-state probability density with quantum fluctuations

Based on the works of Refs. [18–23], the coordinate-diagonal elements of the density operator $\rho(t)$ of a particle of mass M moving in the potential $V(x)$ (i.e., the probability-density function $P(x,t) = \langle x|\rho(t)|x\rangle$ in position space x) obey an MQSE. Within the strong-friction limit, the dynamics of such a particle is described by the MQSE, which takes into account the leading quantum corrections. It reads explicitly [30]

$$\gamma M \frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}V'(x)P(x,t) + \frac{\partial^2}{\partial x^2}\left[\frac{1}{\beta(1-\lambda\beta V''(x))}\right]P(x,t), \tag{1}$$

where the prime denotes the derivative with respect to the coordinate x. The parameter λ describes quantum fluctuations in position space and reads

$$\lambda = \left(\frac{\hbar}{\pi M\gamma}\right)\ln\left(\frac{\hbar\beta\gamma}{2\pi}\right), \qquad \beta = \frac{1}{k_B T}, \tag{2}$$

in which $\hbar$ is the Planck constant, k_B is the Boltzmann constant, and T is the temperature.
Note that Eq. (1) is valid whenever $k_B T \ll \hbar\gamma$. From Eq. (1), we can obtain the steady-state probability density [30]:

$$P_{st}(x) = N\beta(1-\lambda\beta V''(x))\exp[-\beta U(x)]. \tag{3}$$

In this paper, we consider an asymmetric bistable kinetic system; the potential of this system is

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + cx, \tag{4}$$

where c represents the asymmetry of the potential. Substituting Eq. (4) into (3), we have

$$U(x) = c(1+\lambda\beta)x - \frac{1+\lambda\beta}{2}x^2 - c\lambda\beta x^3 + \frac{1+4\lambda\beta}{4}x^4 - \frac{1}{2}\lambda\beta x^6. \tag{5}$$

The quantity N in Eq. (3) is a normalization constant. Then the expectation values of the nth power of the state-space variable x are given by

$$\langle x^n\rangle_{st} = \frac{\int_{-\infty}^{+\infty} x^n P_{st}(x)dx}{\int_{-\infty}^{+\infty} P_{st}(x)dx}. \tag{6}$$

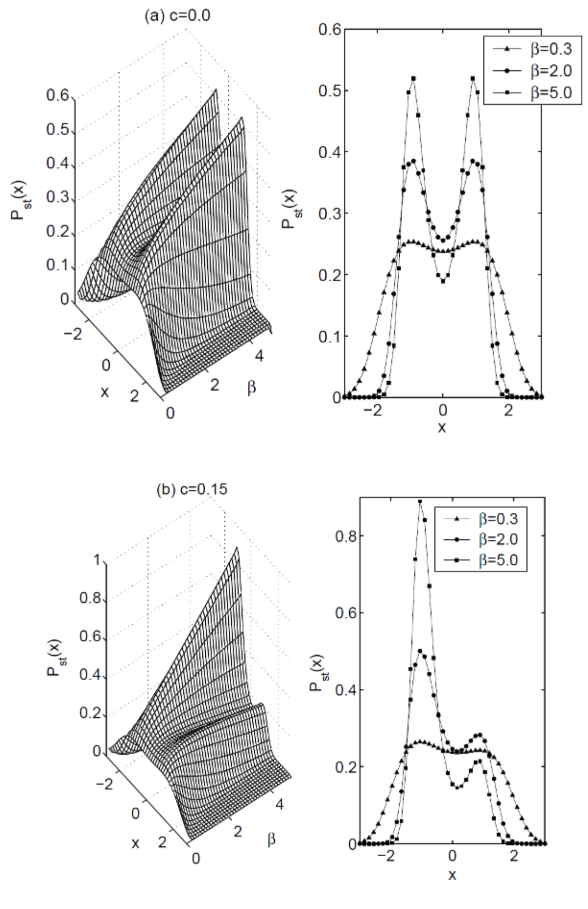

Figure 1. The steady-state probability density $P_{st}(x)$ vs. x and β with $\lambda = 10^{-3}\ln(10^4\beta)$. (a) $c = 0.0$, (b) $c = 0.15$.

According to the expression $P_{st}(x)$ [Eq. (3)], the effects of both β and c on $P_{st}(x)$ can be studied. $P_{st}(x)$ as a function of β and as a function of c are plotted in Figs. 1 and 2, respectively. When the system is symmetric (see $c = 0$ in Fig. 1a), there are two symmetric peaks in $P_{st}(x)$, and the height of the two peaks at $x < 0$ and $x > 0$ increases with increasing β. That is, for the case of $c = 0$, the larger β, the larger the fluctuation of the state-space variable x. However, when the system is asymmetric (see $c = 0.15$ in Fig. 1b), the height of the peak at $x < 0$ increases and the height of the peak at $x > 0$ first increases and then decreases as β increases. That is, for the case of $c \neq 0$, up to a certain point we have the behavior that a larger β leads to larger fluctuations of the state-space variable x. When β is increased beyond this point, the behavior is reversed: a larger β results in smaller fluctuations of the state-space variable x. Therefore, the effects of β on the steady-state variable $\langle x\rangle_{st}$ for the case of $c \neq 0$ are

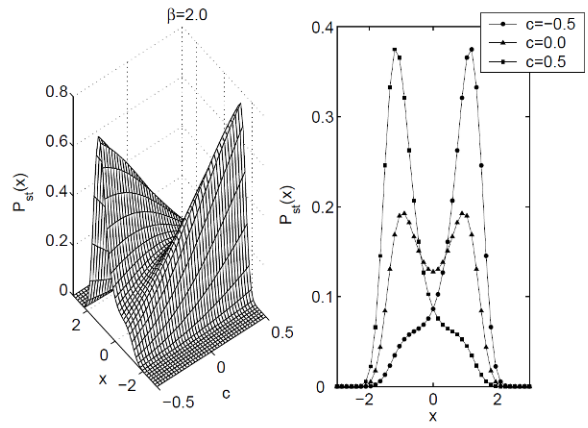

Figure 2. The steady-state probability density $P_{st}(x)$ vs. x and c with $\lambda = 10^{-3}\ln(10^4\beta)$ and $\beta = 2.0$.

more complex than those for the case of $c = 0$. It is clear from Fig. 2 that $P_{st}(x)$ changes from having two peaks to having one peak when $|c|$ is increased. That is, the larger $|c|$, the smaller the fluctuations of the state-space variable x.

3. Relaxation time with quantum fluctuations

The stationary correlation function of the state-space variable x is defined by [31, 32]

$$C(s) = \langle \delta x(t+s)\delta x(t)\rangle_{st} = \lim_{t\to\infty}\langle \delta x(t+s)\delta x(t)\rangle. \quad (7)$$

It describes the fluctuation decay of the state-space variable $\delta x(t) = x(t) - \langle x(t)\rangle$ in the stationary state. The stationary normalized correlation function of the state-space variable x is given by

$$C(s) = \frac{\langle \delta x(t+s)\delta x(t)\rangle_{st}}{\langle(\delta x)^2\rangle_{st}}, \quad (8)$$

where $\delta x(t+s)$ in Eq.(8) can be written as

$$\delta x(t+s) = \exp(L_{FP}^{+}s)\delta x(t), \quad (9)$$

with

$$L_{FP}^{+} = -V'(x)\frac{\partial}{\partial x} + \frac{1}{\beta(1-\lambda\beta V''(x))}\frac{\partial^2}{\partial^2 x}. \quad (10)$$

Thus, one can easily rewrite Eq. (8), and obtain the associated Laplace transform

$$\widetilde{C}(\omega) = \int_0^\infty ds \exp(-\omega s) C(s) = \frac{1}{\langle(\delta x)^2\rangle_{st}} \left\langle \delta x \frac{1}{\omega - L_{FP}^+} \delta x \right\rangle_{st}. \quad (11)$$

The fluctuation decay of the state-space variable can also be represented by the relaxation time T_c , which is defined by

$$T_c = \int_0^\infty C(s) ds. \quad (12)$$

Obviously, from Eqs. (11) and (12), the relaxation time can also be written as

$$T_c = \widetilde{C}(0). \quad (13)$$

Applying the projection-operator method [25, 33], the following continued-fraction expression for $\widetilde{C}(\omega)$ in Eq. (11) can be obtained

$$\widetilde{C}(\omega) = \frac{1}{\omega + \gamma_0 + K_0(\omega)}. \quad (14)$$

The memory kernel $K_0(\omega)$ and the memory of the memory kernel $K_1(\omega)$ read

$$K_0(\omega) = \frac{\Gamma_1}{\omega + \gamma_1 + K_1(\omega)}, \quad (15)$$

$$K_1(\omega) = \frac{\Gamma_2}{\omega + \gamma_2 + K_2(\omega)}, \quad (16)$$

$$K_2(\omega) = \frac{\Gamma_3}{\omega + \gamma_3 + \cdots}, \quad (17)$$

where

$$\Gamma_i = \frac{\langle(\delta x_i)^2\rangle_{st}}{\langle(\delta x_{i-1})^2\rangle_{st}}, \quad \gamma_i = \frac{\langle \delta x_i L_{FP}^+ \delta x_i \rangle_{st}}{\langle(\delta x_i)^2\rangle_{st}}, \quad i = 0, 1, 2, \cdots. \quad (18)$$

If the memory effects are completely neglected (setting $\Gamma_1 = 0$), we have the zeroth-order approximation of the relaxation time

$$T_c^0 = \gamma_0^{-1} = \frac{\langle(\delta x)^2\rangle_{st}}{\left\langle \frac{1}{\beta(1-\lambda\beta V''(x))} \right\rangle_{st}}. \quad (19)$$

It is clear that the zeroth-order approximation of the relaxation time $T_c^0 = \gamma_0^{-1}$ is in good agreement with the result calculated by virtue of a Stratonovich-like ansatz [34].

If the memory of the memory kernel is neglected (setting $\Gamma_2 = 0$), we have the first-order approximation expression for the relaxation time

$$T_c = T_c^0 \left[1 + \frac{\Gamma_1}{\gamma_0 \gamma_1} \right]^{-1} = \frac{T_c^0}{1 - \kappa}, \quad (20)$$

with

$$\Gamma_1 = -\frac{\left\langle \frac{V''(x)}{\beta(1-\lambda\beta V''(x))} \right\rangle_{st}}{\langle(\delta x)^2\rangle_{st}} + \gamma_0^2,$$

$$\gamma_1 = -\frac{\left\langle \frac{(V''(x))^2}{\beta(1-\lambda\beta V''(x))} \right\rangle_{st}}{\Gamma_1 \langle(\delta x)^2\rangle_{st}} + \frac{\gamma_0^3}{\Gamma_1} - 2\gamma_0, \quad (21)$$

where $\kappa = -\Gamma_1/\gamma_0\gamma_1$, and κ can be used to measure the effects of the memory on the relaxation time [35, 36]. It is clearly seen from Eq. (20) that the relaxation time $T_c = T_c^0$ when κ is zero, which completely neglects the effects of the memory kernels. However, Fujisaka and Grossmann have pointed out that earlier experience with the Duffing oscillator or with laser fluctuations has shown that the effects of the memory kernels are significant [35]. Figure 3 depicts the curves of κ as a functions of β for various values of c for both the overdamped quantum case (solid line) and its classical counterpart (dashed line).

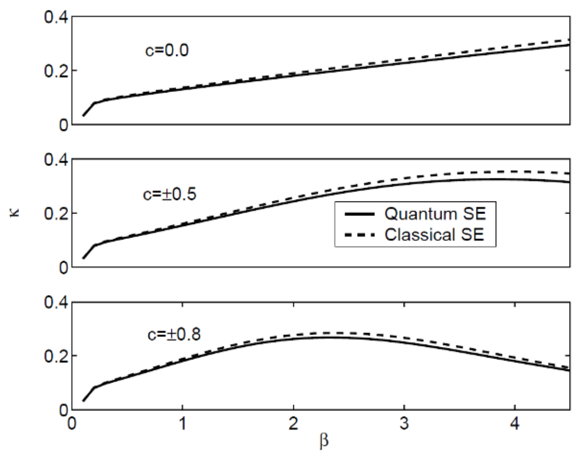

Figure 3. The memory parameter κ vs. β for both the overdamped quantum case (solid line) and its classical counterpart (dashed line) with $\lambda = 10^{-3}\ln(10^4\beta)$. $|c|$ takes the values 0.0, 0.5, and 0.8.

When the system is symmetric (see $c = 0$), the curve of κ increases with increasing β. However, when the system is asymmetric (see $|c| = 0.5$ and 0.8), κ first increases and then decreases as β increases, exhibiting a maximum. That is to say, for the case of $c = 0$, β amplifies the effects of the memory on the relaxation time. However, for the case of $|c| \neq 0$, β amplifies the effects of the memory on the relaxation time up to a certain point, and beyond this point, it then suppresses them. Simultaneously, we compare also the resulting quantum (solid line) and classical cases (dashed line) for the influences of c and β. For high temperatures (i.e., small β), the quantum κ agrees with the classical one. With decreasing temperature (i.e., increasing β), the quantum κ is lower than the classical κ (see Fig. 3). Therefore, the quantum effects and the asymmetry of the potential play an important role in the asymmetric bistable system, especially when the effects of the memory kernels are taken into account.

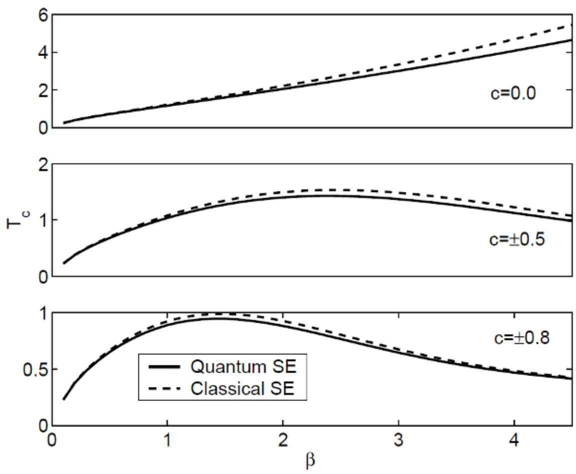

Figure 4. The relaxation time T_c vs. β for both the overdamped quantum case (solid line) and its classical counterpart (dashed line) with $\lambda = 10^{-3}\ln(10^4\beta)$. $|c|$ takes the values 0.0, 0.5, and 0.8.

Equation (20) is the expression of the relaxation time of an asymmetric bistable system, in which the effects of the memory kernels are taken into account. Large T_c values mean that there is a slow-down of the fluctuation decay of the state-space variable in the stationary state, while small T_c values correspond to a speed-up of the fluctuation-decay process. The relaxation time T_c as a functions of β and as a function of c for both the quantum case (solid line) and the classical case (dashed line) are plotted in Figs. 4 and 5, respectively. In Fig. 4, when the system is symmetric (i.e., $c = 0$), T_c increases with increasing β, i.e., β slows down the rate of fluctuation decay of the state-space variable for the case of $c = 0$. However, when the system is asymmetric (see $|c| = 0.5$ and 0.8), T_c first increases and then decreases as β increases, exhibiting a maximum, i.e., β first slows down the rate of fluctuation decay of the state-space variable and then enhances it. In Fig. 5, T_c exhibits a maximum with

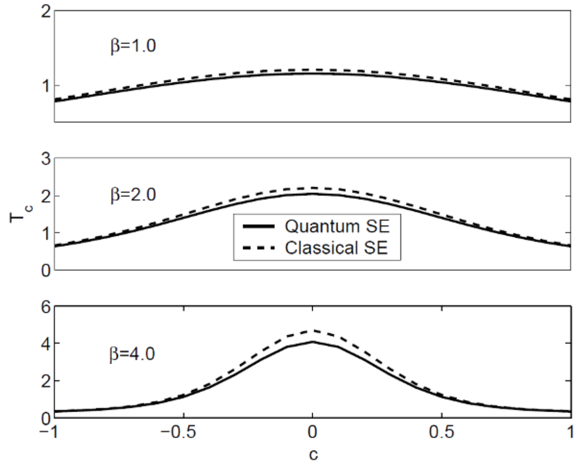

Figure 5. The relaxation time T_c vs. c for both the overdamped quantum case (solid line) and its classical counterpart (dashed line) with $\lambda = 10^{-3}\ln(10^4\beta)$. β takes the values 1.0, 2.0, and 4.0.

increasing c, i.e., c first slows down the rate of fluctuation decay of the state-space variable and then enhances it. We can also compare the quantum case (solid line) with the classical case (dashed line) for the influences of c and β on T_c. For high temperatures (i.e., small β), T_c in the quantum theory is in good agreement with the classical theory. However, for lower temperatures (i.e., large β), T_c in the quantum case is lower than in the classical case, i.e., the quantum effects speed up the rate of fluctuation decay of the state-space variable. The difference between the quantum case and the classical case increases with increasing β, and decreases with increasing $|c|$. That is, the quantum effects stand out for lower temperatures, and weaken for larger $|c|$. Therefore, the quantum effects in an asymmetric bistable kinetic system on time scales of the relaxation process are prominent for lower temperatures and smaller asymmetries of the potential.

4. Concluding remarks

In this paper, we have studied the dynamical properties of an overdamped Brownian particle moving in an asymmetric bistable system with quantum fluctuations. Within the strong-friction limit, the quantum Brownian motion is described by the modified quantum Smoluchowski equation, which takes into account leading quantum corrections. First of all, we have discussed the effects of both β and c on the steady-state variable $\langle x \rangle_{st}$. The effects of β on the steady-state variable $\langle x \rangle_{st}$ for the case of $c \neq 0$ are more complex than those for the case of $c = 0$. Second, the analytic expression for the relaxation time of the system with quantum fluctuations has been derived by means of the projection-operator method, in which the effects of the memory kernels are taken into account. The quantum effects and the asymmetry of the potential play an important role in an asymmetric bistable system, especially when the effects of the memory kernels are taken into account. Third, the effects of both β and c on the relaxation time have been analyzed for both the quantum case and the classical case. The behavior of the relaxation time for the case of $c = 0$ (i.e., the system is symmetric) and for the cases of $|c| \neq 0$ (i.e., the system is asymmetric) are very different. For the case of $c = 0$, β slows down the rate of fluctuation decay of the state-space variable and for the case of $|c| \neq 0$, β first slows down the rate of fluctuation decay of the state-space variable and then enhances it. The asymmetry c first slows down the rate of fluctuation decay of the state-space variable and then enhances it. Moreover, we have compared the resulting quantum and classical cases. For higher temperatures, the relaxation time in the quantum theory is in good agreement with the classical theory. For lower temperatures, however, the relaxation time in the quantum case is lower than in the classical case, i.e., quantum effects speed up the rate of fluctuation decay of the state-space variable for lower temperatures. The difference between the quantum case and the classical case increases with increasing β, while it decreases with increasing $|c|$. Therefore, quantum effects in an asymmetric bistable kinetic system are prominent for lower temperatures and smaller asymmetries of the potential.

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