Cent. Eur. J. Phys. • 9(6) • 2011 • 1503-1508
DOI: 10.2478/s11534-011-0071-y

VERSITA

Central European Journal of **Physics**

Arbitrary *l*-state solutions of the Schrödinger equation for the Eckart potential with an improved approximation of the centrifugal term

Research Article

Jerzy Stanek*

Faculty of Chemistry, Adam Mickiewicz University, Grunwaldzka 6, 60-780 Poznań, Poland

Received 14 March 2011; accepted 28 July 2011

Abstract: Applying an improved approximation scheme to the centrifugal term, the approximate analytical solutions of the Schrödinger equation for the Eckart potential are presented. Bound state energy eigenvalues and the corresponding eigenfunctions are obtained in closed forms for the arbitrary radial and angular momentum quantum numbers, and different values of the screening parameter. The results are compared with those obtained by the other approximate and numerical methods. It is shown that the present method is systematic, more efficient and accurate.

PACS (2008): 03.65.Ge, 34.20.Cf

Keywords: bound states • Eckart potential • centrifugal term

1. Introduction

The search for exact solution of the Schrödinger equation (SE) for arbitrary angular momentum states has been an important research area since the birth of quantum mechanics. However, for most potentials, exact analytical solutions of SE are known only for zero angular momentum states. For nonzero ones the corresponding SE can only be solved approximately using a suitable approximation scheme. For the exponential-type potential the most general approximation is based on the use of a second-order exponential expansion of the centrifugal term about a suitably chosen equilibrium coordinate. This method, originally proposed by Pekeris [1], has been used to obtain approximate analytical expressions for the energy levels and wavefunctions for the rotating Morse oscillator. For the other exponential-type potentials such as Eckart, Hulthén, Pöschl-Teller, Rosen-Morse, Tietz or Manning-Rosen, many authors have used the approximation scheme suggested by Greene and Aldrich [2] taking the centrifugal term as $1/r^2 \approx c_0 e^{-r/a}/\left(1-e^{-r/a}\right)^2$, where c_0 is an adjustable parameter. Although using this approximation scheme the corresponding SE can be solved in a closed form for many exponential-type potential models it is valid only for high values of the parameter a i.e., for small values of r/a . A better approximation scheme was proposed by Badawi et al. [3]. They introduced the centrifugal term in a form formally homogeneous to the original potential

*E-mail: stanek@amu.edu.pl

Springer

to keep the factorizability of the corresponding SE. By equating the rotational term to the expression

$$1/r^2 \approx c_0 + c_1 e^{-r/a}/\left(1-e^{-r/a}\right) + c_2 e^{-2r/a}/\left(1-e^{-r/a}\right)^2, \quad (1)$$

where c_i parameters can be determined as a function of the specific potential parameters, they showed that direct factorization becomes possible for potentials like Morse-Pekeris, Rosen-Morse, Manning-Rosen or Tietz. A similar expansion has been used by Lu [4] or Ikhdair and Sever [5] for the empirical potential suggested by Schiöberg or for the Hulthén potential by Jia and collaborators [6]. It has also been applied by Ikhdair and Sever [7] to the Manning-Rosen potential. Although expansion (1) has proved its power and efficiency when compared with that proposed by Greene and Aldrich it does not lead to a solution in closed form for all exponential-type potentials. For the Eckart-like potential (EP) considered here, an analytical approximation to the l-wave solution of SE has been obtained by Dong et al. [8] using the conventional approximation scheme developed by Greene and Aldrich. Following the same approximation scheme, an approximate solution of SE and its parity-time-symmetric version of the EP was investigated by Zhang et al. [9] using the Nikiforov-Uvarov method. The scattering state solution and the phase shifts were derived in a paper by Chen et al. [10], whereas the relativistic bound state solutions for EP have been proposed by Liu et al. [11]. Following these papers it is easy to notice that the results are in good agreement with the accurate numerical ones only for short-range potentials. It means that the Greene and Aldrich approximation scheme for the centrifugal term is valid only when the potential parameter a is large. Recently, Taşkin and Koçak [12] investigated the same model using an approximation scheme similar to (1). They obtained quite accurate results, particularly for small values of the screening parameter a, but with c_i coefficients (denoted there as λ and ξ) taken as adjustable parameters. In this paper we propose a modified approximation scheme similar to the one proposed by Lu [4] and where the c_i coefficients are to be determined as a function of the potential parameters depending on the vibration-rotation state considered. We first calculate r_l , i.e. the value of r, which yields the minimum of the effective potential by solving numerically a transcendental equation $dV_{eff}(r)/dr = 0$. Next, by expanding a new exponential variable in a Taylor's series about l-dependent equilibrium, truncating this series after the second term and comparing it with Eq. (1) we obtain the l-dependent c_i parameters. Finally, by using expansion (1) we solve the corresponding SE in terms of the generalized hypergeometric functions. In order to verify the accuracy of our approximation scheme, we calculate the energy eigenvalues and eigenfunctions of bound states for the Eckart-like [13] potential defined by

$$V(r) = -\alpha\frac{e^{-r/a}}{1-e^{-r/a}} + \beta\frac{e^{-r/a}}{\left(1-e^{-r/a}\right)^2}, \qquad \alpha,\beta>0, \quad (2)$$

where the parameters α and β describe the depth of the potential well, while the parameter a is related to the range of the potential, and compare them with the results obtained by numerical and other methods. The rest of the paper is organized as follows. In the next Section, applying the improved approximate scheme to the centrifugal term we present the bound state solutions and the normalized radial wave functions of the Eckart-like potential for the arbitrary angular momentum number. In Section 3, our results are compared with the best results published by Taşkin et al.[12] and the accurate numerical ones [16]. Concluding remarks are given in Section 4.

2. Bound state solutions of the Eckart potential for arbitrary angular momentum number

The SE for a central molecular potential $V(r)$ can be written as

$$\left[-\frac{\hbar^2}{2\mu}\nabla^2 + V(r)\right]\Psi_{n_r lm}(r,\vartheta,\varphi) = E_{n_r l}\Psi_{n_r lm}(r,\vartheta,\varphi), \quad (3)$$

where r is the internuclear separation, μ is the reduced mass, n_r denotes the radial, and l the angular momentum quantum number. Writing the wave function in the form of $\Psi_{n_r lm}(r,\vartheta,\varphi) = r^{-1}u_{n_r l}(r)\,Y_{lm}(\vartheta,\varphi)$ and substituting Eq. (2) into Eq. (3), where $Y_{lm}(\vartheta,\varphi)$ is a spherical harmonic function, we obtain, after separating the angular part, the radial SE

$$\left[-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + V_{eff}(r)\right]u_{n_r l}(r) = E_{n_r l}u_{n_r l}(r). \quad (4)$$

Here, the effective potential $V_{eff}(r)$ is defined as a sum of the Eckart potential and the centrifugal term

$$V_{eff}(r) = V(r) + \frac{l(l+1)\hbar^2}{2\mu r^2}. \quad (5)$$

Obviously, Eq. (4) can be analytically solved only for the s-wave case ($l=0$). For the $l\neq 0$ states we must use approximate methods. Here, we apply the improved

approximation scheme for the centrifugal term introduced by Lu [4]

$$\frac{1}{x^2} = c_0 + c_1 \frac{e^{-x}}{1-e^{-x}} + c_2 \frac{e^{-2x}}{(1-e^{-x})^2}, \tag{6}$$

where $x = r/a$. Inserting (5) and (6) into equation (4) allows us to obtain

$$\left[-\frac{\hbar^2}{2\mu a^2} \frac{d^2}{dx^2} + V_{eff}(x) \right] u_{n_r l}(x) = E_{n_r l} u_{n_r l}(x), \tag{7}$$

where $V_{eff}(x) = -\alpha \frac{e^{-x}}{1-e^{-x}} + \beta \frac{e^{-x}}{(1-e^{-x})^2} + \frac{l(l+1)\hbar^2}{2\mu a^2} \left[c_0 + c_1 \frac{e^{-x}}{1-e^{-x}} + c_2 \frac{e^{-2x}}{(1-e^{-x})^2} \right]$.

Introducing a new variable $z = e^{-x}$ and setting $u_{n_r l} = g_{n_r l}$ Eq. (7) can be rearranged as

$$z^2 \frac{d^2 g_{n_r l}(z)}{dz^2} + z \frac{dg_{n_r l}(z)}{dz} + \left(\frac{[a_1 - l(l+1)c_1]z}{1-z} - \frac{b_1 z + l(l+1)c_2 z^2}{(1-z)^2} - \lambda^2 \right) g_{n_r l}(z) = 0, \tag{8}$$

where we used the dimensionless parameters given by

$$\lambda^2 = -\frac{2\mu a^2 E}{\hbar^2} - l(l+1)c_0, \quad a_1 = \frac{2\mu a^2 \alpha}{\hbar^2}, \quad b_1 = \frac{2\mu a^2 \beta}{\hbar^2}. \tag{9}$$

Because the regular singular points here are at $z = 0$ and $z = 1$, we take the following trial solution for $g_{n_r l}(z)$

$$g_{n_r l}(z) = z^{\lambda} (1-z)^{\delta} f(z), \tag{10}$$

where $\delta = \frac{1}{2}\sqrt{1 + 4b_1 + 4l(l+1)c_2} + \frac{1}{2}$. Inserting Eq. (10) into Eq. (8), we get

$$z(1-z)\frac{d^2 f(z)}{dz^2} + [2\lambda + 1 - (2\lambda + 2\delta + 1)z] \frac{df(z)}{dz} + [a_1 - b_1 - l(l+1)c_1 - \delta(2\lambda + 1)] f(z), \tag{11}$$

whose solution are nothing but the hypergeometric function [14]

$$f(z) = {}_2F_1(A, B; C; z) + z^{1-C} {}_2F_1(A - C + 1, B - C + 1; 2 - C; z), \tag{12}$$

where the parameters A, B, C are given by

$$A = \lambda + \delta - \rho, \quad B = \lambda + \delta + \rho, \quad C = 2\lambda + 1 \tag{13}$$

with $\rho = \sqrt{a_1 - b_1 - l(l+1)c_1 + \lambda^2 + \delta(\delta - 1)}$. Considering the boundary condition, i.e., $f(z)$ tending to finite value when $z \to 0$ the allowed solution is only

$$f(z) = {}_2F_1(A, B; C; z) = \sum_{k=0}^{\infty} \frac{(A)_k (B)_k}{(C)_k} \frac{z^k}{k!}, \tag{14}$$

where $(x)_k$ denotes the Pochhammer symbol given by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}. \tag{15}$$

From the properties of the hypergeometric function the series (14) approaches infinity unless $A = \lambda + \delta - \rho$ is a negative integer. Hence, the eigenfunction $g_{n_r l}(z)$ is not finite everywhere unless

$$\lambda + \delta - \rho = -n_r, \quad n_r = 0, 1, 2, \ldots \tag{16}$$

from which we have

$$\lambda = -\frac{1}{2} \frac{\left[-a_1 + b_1 + l(l+1)c_1 + \delta + 2\delta n_r + n_r^2 \right]}{n_r + \delta}. \tag{17}$$

Substitution of Eq. (9) into Eq. (17) leads to the following energy eigenvalues

$$E_{n_r l} = -\frac{\hbar^2}{2\mu a^2} \left(\frac{\left[-a_1 + b_1 + l(l+1)c_1 + \delta + 2\delta n_r + n_r^2 \right]^2}{4(n_r + \delta)^2} - l(l+1)c_0 \right). \tag{18}$$

Using Eq. (13), we can write the radial eigenfunction as

$$g_{n_r l}(z) = N z^{\lambda} (1-z)^{\delta} {}_2F_1(-n_r, \lambda + \delta + \rho; 2\lambda + 1; z), \tag{19}$$

where N is the normalization constant which can be calculated from the normalization condition

$$\int_0^{\infty} u_{n_r l}(r)^2 \, dr = a \int_0^1 z^{-1} g_{n_r l}(z)^2 \, dz = 1. \tag{20}$$

Putting the wave function (Eq. (19)) into Eq. (20) and using the series expansion in Eq. (14) we obtain

$$N^2 \sum_{j=0}^{n_r} \sum_{k=0}^{n_r} (-1)^{j+k} \gamma_{jk} \times \int_0^1 z^{2\lambda + j + k - 1} (1-z)^{2\delta} \, dz = a^{-1}, \tag{21}$$

where $\gamma_{jk} = \dfrac{\Gamma(n_r+1)^2}{\Gamma(j+1)\Gamma(n_r-j+1)\Gamma(k+1)\Gamma(n_r-k+1)}$

$$\times \frac{\Gamma(\lambda+\delta+\rho+j)\Gamma(\lambda+\delta+\rho+k)}{\Gamma(\lambda+\delta+\rho)^2}$$

$$\times \frac{\Gamma(2\lambda+1)^2}{\Gamma(2\lambda+j+1)\Gamma(2\lambda+k+1)}.$$

The definite integral in Eq. (21) is just the integral representation of the Beta function [15] $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ for $\Re(x) > 0$ and $\Re(y) > 0$. Using $x = 2\lambda + j + k$ and $y = 2\delta + 1$ as parameters for the Beta function, we can express Eq. (21) as

$$\int_0^1 z^{2\lambda+j+k-1}(1-z)^{2\delta+1-1}\,dz = \frac{\Gamma(2\lambda+j+k)\Gamma(2\delta+1)}{\Gamma(2\lambda+2\delta+j+k+1)}, \tag{22}$$

and hence the normalization constant as

$$N = \left[a \sum_{j=0}^{n_r} \sum_{k=0}^{n_r} (-1)^{j+k} \gamma_{jk} \right.$$

$$\left. \times \frac{\Gamma(2\lambda+j+k)\Gamma(2\delta+1)}{\Gamma(2\lambda+2\delta+j+k+1)} \right]^{-1/2}. \tag{23}$$

Before we get the energy eigenvalues we have to obtain the coefficients c_0, c_1 and c_2. We can use a different methods to determine them. The well-known approach, mentioned above, follows from the Pekeris model where the centrifugal term $1/r^2$ is expanded as a Taylor series of the exponential variable $z = e^{-r/a}$ around the potential minimum. It is easy to show that EP has a minimum value $V(r_0) = -\dfrac{(\alpha-\beta)^2}{4\beta}$ at $r_0 = a\ln\left(\dfrac{\alpha+\beta}{\alpha-\beta}\right)$. However, the expansion around r_0 is only valid for lower vibrational energy states [17]. Bearing in mind that the energy eigenvalues are mainly determined by the behaviour of the effective potential in the region near the l-dependent minimum point of the potential, we can obtain better results by expanding the centrifugal term $1/r^2$ around the l-dependent minimum r_l. Unfortunately, for the effective potential considered here, we can get r_l only approximately by solving the equation $dV_{eff}(r)/dr = 0$ numerically.

After determining the l-dependent minimum r_l, we start by writing

$$\frac{1}{x^2} = \frac{1}{\ln(z)^2} = \frac{1}{\ln(z_l+\Delta z)^2}, \tag{24}$$

where $\Delta z = z - z_l$. Expanding the right-hand side of Eq. (24) in a series around $\Delta z = 0$ up to second order in Δz we obtain

$$\frac{1}{x^2} = \frac{1}{\ln(z_l)^2} - \frac{2}{z_l\ln(z_l)^3}\Delta z + \left[\frac{3+\ln(z_l)}{z_l\ln(z_l)^4}\right]\Delta z^2 + O\left(\Delta z^3\right). \tag{25}$$

On the other hand, if we write the approximation (6) used for the centrifugal term as

$$\frac{1}{x^2} = c_0 + c_1\frac{z_l+\Delta z}{1-z_l-\Delta z} + c_2\frac{(z_l+\Delta z)^2}{(1-z_l-\Delta z)^2} \tag{26}$$

and expand it in a Taylor series around $\Delta z = 0$ to the second order of the Δz we get

$$\frac{1}{x^2} = c_0 + \frac{c_1 z_l}{1-z_l} + \frac{c_2 z_l^2}{(1-z_l)^2} + \left[\frac{(2c_2-c_1)z_l + c_1}{(1-z_l)^3}\right]$$

$$\Delta z + \left[\frac{(2c_2-c_1)z_l + c_1 + c_2}{(1-z_l)^4}\right]\Delta z^2 + O\left(\Delta z^3\right). \tag{27}$$

By equating terms of like powers in Eqs. (25) and (27) we obtain

$$c_0 = \frac{1}{\ln(z_l)^2} - \frac{z_l^2+2z_l-3}{\ln(z_l)^3} + \frac{3z_l^2-6z_l+3}{\ln(z_l)^4} \tag{28a}$$

$$c_1 = -\frac{2z_l^2-6z_l+4}{z_l\ln(z_l)^3} + \frac{6z_l^3-18z_l^2+18z_l-6}{z_l\ln(z_l)^4} \tag{28b}$$

$$c_2 = -\frac{z_l^4-2z_l^3+2z_l-1}{z_l^2\ln(z_l)^3}$$

$$+ \frac{3z_l^4-12z_l^3+18z_l^2-12z_l+3}{z_l^2\ln(z_l)^4} \tag{28c}$$

3. Results

To verify the accuracy and the effectiveness of our approximation scheme we have calculated the bound state energy eigenvalues $(-E_{nl})$ for a given principal quantum $n = n_r + l + 1$ and angular quantum number l, with two values of the parameter β, and various values of the parameter $\alpha = 1/a$. Our results were compared with those obtained numerically by Lucha and Schöberl [16] with the help of the MATHEMATICA package and those obtained by Taşkin and Koçak [12]. All results are shown in Table 1 and 2 together with the relative errors using $RE = \left|\frac{E(approx)-E(num)}{E(num)}\right|$. Most of the relative errors for the eigenvalues are less than 1%. Although the errors increase

Table 1. The bound state energy eigenvalues of the Eckart potential as a function of $\alpha = 1/a$ parameter for some states in atomic units ($\hbar = \mu = 1$) when $\beta = 0.0001$

States	$\alpha = 1/a$	Present (I)	RE_I (%)	Taşkin et al. [12] (II)	RE_{II} (%)	Schöberl at al. [16][a]
2p	0.025	0.1008306	0.005	0.1015944	0.752	0.1008358
	0.050	0.0978345	0.001	0.0982980	0.472	0.0978358
	0.075	0.0884174	0.001	0.0885875	0.191	0.0884183
	0.100	0.0783776	0.010	0.0784035	0.023	0.0783854
	0.150	0.0590651	0.069	0.0592870	0.306	0.0591059
	0.200	0.0415749	0.329	0.0423448	1.517	0.0417120
	0.250	0.0261474	1.377	0.0286908	8.217	0.0265124
	0.300	0.0128764	6.237	0.0174777	27.268	0.0137330
	0.350	0.0018365	51.196	0.0090235	139.798	0.0037630
3p	0.025	0.0401232	0.004	0.0403106	0.463	0.0401250
	0.050	0.0322435	0.014	0.0323958	0.458	0.0322482
	0.075	0.0235330	0.095	0.0237732	0.925	0.0235553
	0.100	0.0157843	0.470	0.0162724	2.608	0.0158588
	0.150	0.0039725	9.903	0.0054340	23.245	0.0044091
3d	0.025	0.0413638	0.001	0.0414790	0.278	0.0413642
	0.050	0.0321936	0.011	0.0321085	0.276	0.0321973
	0.075	0.0227792	0.087	0.0229644	0.725	0.0227991
	0.100	0.0142976	0.486	0.0152256	5.973	0.0143675
	0.150	0.0008190	40.002	0.0044524	226.183	0.0013650
4p	0.025	0.0184616	0.009	0.0185468	0.453	0.0184632
	0.050	0.0106995	0.153	0.0108554	1.302	0.0107159
	0.075	0.0044170	1.973	0.0047920	6.349	0.0045059
	0.100	0.0003883	46.161	0.0011400	58.070	0.0007212
4d	0.025	0.0189196	0.011	0.0189774	0.295	0.0189216
	0.050	0.0104280	0.309	0.0106863	2.161	0.0104603
	0.075	0.0035778	4.992	0.0045048	19.624	0.0037658
4f	0.025	0.0190212	0.004	0.0189461	0.399	0.0190220
	0.050	0.0098979	0.159	0.0102192	3.082	0.0099137
	0.075	0.0023963	4.458	0.0039915	59.144	0.0025081
5p	0.025	0.0086820	0.034	0.0087420	0.656	0.0086850
	0.050	0.0024778	1.797	0.0026616	5.489	0.0025231
5d	0.025	0.0088512	0.072	0.0089315	0.834	0.0088576
	0.050	0.0021233	5.121	0.0025859	15.548	0.0022379
5f	0.025	0.0088225	0.081	0.0088850	0.626	0.0088297
	0.050	0.0015838	8.488	0.0023989	38.609	0.0017307
5g	0.025	0.0086919	0.028	0.0087060	0.135	0.0086943
	0.050	0.0009085	6.870	0.0021281	118.155	0.0009755
6p	0.025	0.0037813	0.158	0.0038340	1.233	0.0037873
6d	0.025	0.0038176	0.394	0.0039244	2.393	0.0038327
6f	0.025	0.0037296	0.611	0.0038903	3.672	0.0037525
6g	0.025	0.0035709	0.584	0.0037866	5.421	0.0035919

[a] We found two misprints in Table 1 and 2 of Ref. [12] which are corrected here.

Table 2. The bound state energy eigenvalues of the Eckart potential as a function of $\alpha = 1/a$ parameter for some states in atomic units ($\hbar = \mu = 1$) when $\beta = 0.00005$

States	$\alpha = 1/a$	Present (I)	RE_I (%)	Taşkin et al. [12] (II)	RE_{II} (%)	Schöberl at al. [16]
2p	0.025	0.1064616	0.011	0.1073496	0.823	0.1064737
	0.050	0.0994116	0.005	0.0999045	0.491	0.0994162
	0.075	0.0891252	0.004	0.0893055	0.199	0.0891284
	0.100	0.0787732	0.010	0.0788015	0.026	0.0787809
	0.150	0.0592326	0.069	0.0594540	0.305	0.0592734
	0.200	0.0416619	0.328	0.0427308	2.229	0.0417989
	0.250	0.0261970	1.371	0.0287387	8.198	0.0265611
	0.300	0.0129055	6.220	0.0175051	27.203	0.0137615
	0.350	0.0018531	50.949	0.0090385	139.240	0.0037780
3p	0.025	0.0418361	0.009	0.0420572	0.519	0.0418400
	0.050	0.0326956	0.017	0.0328534	0.466	0.0327011
	0.075	0.0237237	0.095	0.0239652	0.921	0.0237464
	0.100	0.0158817	0.465	0.0163698	2.594	0.0159559
	0.150	0.0040020	9.816	0.0054640	23.130	0.0044376
3d	0.025	0.0424587	0.001	0.0425903	0.310	0.0424588
	0.050	0.0324700	0.011	0.0323839	0.276	0.0324736
	0.075	0.0228947	0.087	0.0230774	0.710	0.0229146
	0.100	0.0143561	0.483	0.0152813	5.931	0.0144257
	0.150	0.0008359	39.464	0.0044680	223.581	0.0013808
4p	0.025	0.0191762	0.013	0.0192709	0.481	0.0191787
	0.050	0.0108687	0.152	0.0110261	1.292	0.0108852
	0.075	0.0044750	1.941	0.0048493	6.260	0.0045636
	0.100	0.0004067	44.897	0.0011578	56.883	0.0007380
4d	0.025	0.0193735	0.010	0.0194354	0.310	0.0193753
	0.050	0.0105312	0.304	0.0107884	2.131	0.0105633
	0.075	0.0036127	4.931	0.0045386	19.434	0.0038001
4f	0.025	0.0193516	0.004	0.0192762	0.394	0.0193525
	0.050	0.0099719	0.158	0.0102901	3.029	0.0099876
	0.075	0.0024207	4.399	0.0040137	58.513	0.0025321
5p	0.025	0.0090287	0.037	0.0090910	0.653	0.0090320
	0.050	0.0025404	1.738	0.0027232	5.334	0.0025853
5d	0.025	0.0090707	0.068	0,0091505	0.812	0.0090768
	0.050	0.0021615	4.998	0.0026230	15.287	0.0022752
5f	0.025	0.0089820	0.080	0.0090440	0.610	0.0089892
	0.050	0.0016108	8.319	0.0024245	37.991	0.0017570
5g	0.025	0.0088169	0.028	0.0088280	0.098	0.0088194
	0.050	0.0009290	6.703	0.0021471	115.637	0.0009957
6p	0.025	0.0039588	0.151	0.0040128	1.211	0.0039648
6d	0.025	0.0039299	0.375	0.0040362	2.320	0.0039447
6f	0.025	0.0038111	0.589	0.0039716	3.597	0.0038337
6g	0.025	0.0036347	0.566	0.0038488	5.291	0.0036554

when the $\alpha = 1/a$ parameter increases, it is evident that present results are in better agreement with those obtained numerically than the previous ones. The accuracy of the present model results in up to 10-300 times better agreement than the estimations provided by [12]. Consequently, it confirms the validity and usefulness of our approximation scheme. What is more, the present method is not only accurate but quite simple and computationally efficient.

4. Concluding remarks

By using an improved approximation scheme to deal with the centrifugal term, we investigated the bound state solutions of the Schrödinger equation with the Eckart potential for arbitrary angular momentum quantum number. The bound state energy eigenvalues and the normalized radial wave functions were obtained in terms of generalized hypergeometric functions. It was shown that the present results are in better agreement with those obtained by using a numerical integration approach than the other analytical results obtained by using the conventional approximation to the centrifugal term. The method presented in this paper is accurate and a systematic. It can be successfully applied not only to the model considered here, but also to other exponential-type potentials.

Acknowledgments

I would like to thank Prof. Franz Schöberl, who provided the MATHEMATICA program that was used in this work and an anonymous referee for valuable comments and suggestions.

References

[1] C. L. Pekeris, Phys. Rev. 45, 98 (1934)
[2] R. L. Greene, C. Aldrich, Phys. Rev. A 14, 2363 (1976)
[3] M. Badawi, N. Bessis, G. Bessis, Journal of Physics B: Atomic and Molecular Physics 5, L157 (1972)
[4] J. Lu, Phys. Scripta 72, 349 (2005)
[5] S. M. Ikhdair, R. Sever, Ann. Phys-Berlin 18, 189 (2009)
[6] C. S. Jia, J. Y. Liu, P. Q. Wang, Phys. Lett. A 372, 4779 (2008)
[7] S. M. Ikhdair, R. Sever, arXiv:0807.2085v1 [quant-ph]
[8] S. H. Dong, W. C. Qiang, G. H. Sun, V. B. Bezerra, J. Phys. A-Math. Theor. 40, 10535 (2007)
[9] A. P. Zhang, W. C. Qiang, Y. W. Ling, Chinese Phys. Lett. 26, 100302 (2009)
[10] C. Y. Chen, D. S. Sun, F. L. Lu, J. Phys. A-Math. Theor. 41, 035302 (2008)
[11] X. Y. Liu, G. F. Wei, C. Y. Long, Int. J. Theor. Phys. 48, 463 (2009)
[12] F. Taşkin, G. Koçak, Chinese Phys. B 19, 090314 (2010)
[13] C. Eckart, Phys. Rev. 35, 1303 (1930)
[14] I. S. Gradshteyn, I. M. Ryzhik, Tables of Integrals, Series, and Products, 7th edition (Academic Press, New York, 2007)
[15] M. Abramowitz, I. A. Stegun, Handbook of Mathematical Functions (Dover Publications, New York, 1972)
[16] W. Lucha, F. F. Schöberl, Int. J. Mod. Phys. C 10, 607 (1999)
[17] D. A. Morales, Chem. Phys. Lett. 394, 68 (2004)