

On some aspects of the definition of scattering states in quantum field theory

Research Article

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Abstract:

The problem of extending quantum-mechanical formal scattering theory to a more general class of models that also includes quantum field theories is discussed, with the aim of clarifying certain aspects of the definition of scattering states. As the strong limit is not suitable for the definition of scattering states in quantum field theory, some other limiting procedure is needed. Two possibilities are considered, the abelian limit and adiabatic switching. Formulas for the scattering states based on both methods are discussed, and it is found that generally there are significant differences between the two approaches. As an illustration of the applications and the features of these formulas, S-matrix elements and energy corrections in two quantum field theoretical models are calculated using (generalized) old-fashioned perturbation theory. The two methods are found to give equivalent results.

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1. Introduction

Much of our knowledge about atomic and subatomic physics comes from studying the results of scattering experiments.

Therefore it is not surprising that special attention has been devoted to the theoretical investigation of scattering processes.

The central objects in the quantum theoretical description of scattering processes are the S-matrix elements, which are scalar products of scattering states, also known as *in*

and *out* states:

$$S_{wv} = \langle w, out | v, in \rangle. \quad (1)$$

$|v, in\rangle$ is a state that looks, in a suitable sense, like the free-particle state $|v\rangle$ in the remote past, and $|w, out\rangle$ is a state that looks like the free-particle state $|w\rangle$ in the far future (see, e.g., Chapter 3 of [1]). A central issue in scattering theory is to define properly these *in* and *out* states. In quantum-mechanical potential scattering, the following standard formulas (see [1–3]) define the *in* and *out* states:

$$|v, in\rangle = \lim_{T \rightarrow \infty} U(0, -T)|v\rangle, \quad (2)$$

$$|w, out\rangle = \lim_{T \rightarrow \infty} U(0, T)|w\rangle, \quad (3)$$

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where $U(t_2, t_1)$ is the time-evolution operator

$$U(t_2, t_1) = e^{iH_0 t_2} e^{-iH(t_2-t_1)} e^{-iH_0 t_1}, \quad (4)$$

H is the total Hamiltonian operator that describes the scattering process, $|v\rangle$ and $|w\rangle$ are state vectors that characterize the scattered particles in the infinite past and future, respectively, and H_0 is a free-particle Hamiltonian operator that describes the time evolution of $|v\rangle$ and $|w\rangle$. It is an interesting question whether it is possible to generalize (2) and (3) for applications to a wider class of models, and in particular to quantum field theoretical (QFT) models. It is well known that in relativistic quantum field theory there exists a standard covariant formalism, based on the Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas, for the description of scattering. This formalism, which is presented in several textbooks, is specific to field theory in the sense that it involves the fields that appear in the model to which it is applied. In the present paper we are interested in those generalizations of (2) and (3) that maintain the feature of only involving Hamiltonian operators and state vectors, but no further details (e.g., symmetry properties) of the structure of the model to which they are applied. Besides their theoretical interest, such generalizations of (2) and (3) can also be useful as starting points for old-fashioned perturbation theory, which has the distinguished property of being formulated solely on the basis of on-shell particles, and is thus more suitable for certain purposes than covariant perturbation theory. Moreover, in $1+1$ dimensional QFT there exist models for which the application of the standard formalism is unknown [4].

The problem of generalizing (2) and (3) in a not specifically field theoretical way is addressed in many quantum field theory textbooks in preparation for the presentation of the field theoretical formalism (see, e.g., Chapter 3 of [1]). However, it seems to us that the literature on this subject is incomplete, in particular regarding the normalization of the *in* and *out* states and the treatment of the $T \rightarrow \infty$ limit. Correspondingly, we feel that some details of the (generalized) old-fashioned perturbation theory should also be revisited.

The aim of this paper is to discuss these aspects using two significantly different generalizations of (2) and (3).

In Sec. 2, we consider the case in which the abelian average is used in the definition of the *in* and *out* states.

In Sec. 3, we discuss another generalization of (2) and (3), which we proposed recently in [5], and which is based on adiabatic switching.

In Sec. 4, we apply the formulas presented in Secs. 2 and 3 to two quantum field theoretical models in the framework of (generalized) old-fashioned perturbation theory. The

first model describes the scattering of a massive scalar particle on a fixed defect line in $1+1$ dimensions. The second example is the ϕ^4 model. This section is intended to provide an illustration for Secs. 2 and 3. Our conclusions are given in Sec. 5.

Although the problems discussed in Secs. 2 and 3 are somewhat mathematical in nature, we try to avoid too much mathematical sophistication; nevertheless we mention that the limits of state vectors in Hilbert spaces are always understood to be strong limits.

2. Abelian limit

The first generalization of (2) and (3) that we consider is the following:

$$|v, in\rangle = \frac{1}{\sqrt{Z_{v,in}}} \lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, -T)|v\rangle, \quad (5)$$

$$|w, out\rangle = \frac{1}{\sqrt{Z_{w,out}}} \lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, T)|w\rangle, \quad (6)$$

where

$$Z_{v,in} = \lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \frac{\langle v | \hat{U}_\epsilon(0, -T)^\dagger \hat{U}_\epsilon(0, -T) | v \rangle}{\langle v | v \rangle}, \quad (7)$$

$$Z_{w,out} = \lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \frac{\langle w | \hat{U}_\epsilon(0, T)^\dagger \hat{U}_\epsilon(0, T) | w \rangle}{\langle w | w \rangle}, \quad (8)$$

$$\hat{U}_\epsilon(0, -T) = \epsilon \int_0^T d\tau e^{-\epsilon\tau} \bar{U}(0, -\tau), \quad (9)$$

$$\hat{U}_\epsilon(0, T) = \epsilon \int_0^T d\tau e^{-\epsilon\tau} \bar{U}(0, \tau), \quad (10)$$

ϵ is a small positive real number, and

$$\bar{U}(t_2, t_1) = e^{iH^{as} t_2} e^{-iH(t_2-t_1)} e^{-iH^{as} t_1}. \quad (11)$$

Three important differences between (5), (6) and (2), (3) should be mentioned. The first one is that H^{as} is written instead of H_0 . H^{as} is a suitably chosen Hamiltonian operator that describes the time evolution of the scattered particles at times long before and after the scattering event and thus has a role similar to that of H_0 . The purpose of introducing the notation H^{as} is to emphasize the fact that due to self-interaction effects H^{as} is generally not identical to the Hamiltonian obtained from H by switching off the interaction (see, e.g., Sec. 3.1 of [1]), a feature absent in potential scattering. The superscript ^{as} is intended to refer to the word asymptotic. Sometimes H^{as} is also called

the renormalized H_0 operator. We also note that in quantum field theory H rather than H_0 is typically modified (see the second example in Sec 4).

The second difference is the replacement of the simple $T \rightarrow \pm\infty$ limits by abelian limits (i.e., using $\lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, -T)|v\rangle$ and $\lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, T)|w\rangle$ instead of $\lim_{T \rightarrow \infty} \hat{U}(0, -T)|v\rangle$ and $\lim_{T \rightarrow \infty} \hat{U}(0, T)|w\rangle$). Nevertheless, both the $T \rightarrow \infty$ and the $\epsilon \rightarrow +0$ limits are assumed to be strong limits. The abelian limit is frequently applied in the literature, e.g., in [6, 7] and in Chapter 3 of [1] (in the latter reference the term "abelian limit" is not used). It is often not mentioned in the literature, but it should be emphasized that, as explained in more detail below, the simple limits $\lim_{T \rightarrow \infty} \hat{U}(0, -T)|v\rangle$ etc. are not sufficient in general for quantum field theories, as they usually do not exist. In particular, this situation arises in the examples presented in Sec. 4.

The third main difference is the presence of the factors $\frac{1}{\sqrt{Z_{v,in}}}$ and $\frac{1}{\sqrt{Z_{w,out}}}$, which are included in order to ensure the correct normalization of $|v, in\rangle$ and $|w, out\rangle$. The need for such normalization factors is related to the use of the abelian limit, as explained below under points 1.) and 2.) in more detail. Although the need for these factors was recognized in some of the early literature on scattering in quantum field theory (see Sec. 5.7 of [7]), the reasons are not explained in much detail. In more recent books and articles, these normalization factors do not appear. The authors of [8] are concerned with the normalization of the *in* and *out* states (see Secs. 4.1 and 4.5 of [8]), but it is not the abelian limit that they apply, and the discussion is restricted mainly to the vacuum state.¹

In the remainder of this section, the discussion of the features of (5) and (6) is continued, but in order to separate clearly the various comments they are presented in the form of a numbered list.

1.) Relation between the abelian and simple limits

It is well known that the abelian limit is more effective than the simple limit; in particular if $\lim_{T \rightarrow \infty} U(0, -T)|v\rangle = |V\rangle$ exists, where $U(0, -T)$ is a unitary operator for any value of T and $|v\rangle$ is a vector of unit norm, then the abelian limit $\lim_{\epsilon \rightarrow +0} \epsilon \int_0^\infty d\tau e^{-\epsilon\tau} U(0, -\tau)|v\rangle$ also exists and is equal to $|V\rangle$. We now recall the proof of this result, which can be found in, e.g., [2]. The integral $\int_0^\infty d\tau e^{-\epsilon\tau} U(0, -\tau)|v\rangle$ is convergent for any $\epsilon > 0$, since $\int_0^\infty e^{-\epsilon\tau} \|U(0, -\tau)|v\rangle\| =$

$\int_0^\infty d\tau e^{-\epsilon\tau} < \infty$, where $\|U(0, -\tau)|v\rangle\| = \|v\| = 1$ has been used, which follows from the unitarity of $U(0, -\tau)$. We define $|V_\epsilon\rangle$ as $|V_\epsilon\rangle = \epsilon \int_0^\infty d\tau e^{-\epsilon\tau} U(0, -\tau)|v\rangle$. It has to be shown that $\lim_{\epsilon \rightarrow +0} |V_\epsilon\rangle = |V\rangle$, i.e., $\lim_{\epsilon \rightarrow +0} \|V_\epsilon - V\| = 0$. This can be done as follows: let us split the integral $|V_\epsilon\rangle - |V\rangle = \epsilon \int_0^{T_1} d\tau e^{-\epsilon\tau} [U(0, -\tau)|v\rangle - |V\rangle] + \epsilon \int_{T_1}^\infty d\tau e^{-\epsilon\tau} [U(0, -\tau)|v\rangle - |V\rangle]$. The norm of the first term goes to zero in the $\epsilon \rightarrow +0$ limit for any fixed $T_1 > 0$ because of the factor ϵ . The $\epsilon \rightarrow +0$ limit of the norm of the second term can be estimated as $\sup_{\tau \in [T_1, \infty)} \|U(0, -\tau)|v\rangle - |V\rangle\|$, which can be made arbitrarily close to zero by choosing T_1 sufficiently large because of the convergence of $U(0, -\tau)|v\rangle - |V\rangle$ to zero as $\tau \rightarrow \infty$.

2.) The reason for the normalization factors

The factors $\frac{1}{\sqrt{Z_{v,in}}}$ and $\frac{1}{\sqrt{Z_{w,out}}}$ are included in (5), (6) in order to ensure that $|v, in\rangle$ and $|w, out\rangle$ have the same norm as $|v\rangle$ and $|w\rangle$. The result above under point 1.) shows that if the simple limit $\lim_{T \rightarrow \infty} \hat{U}(0, -T)|v\rangle$ exists, then $Z_{v,in} = 1$, since in this case the abelian limit $\lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, -T)|v\rangle$ is the same as the simple limit $\lim_{T \rightarrow \infty} \hat{U}(0, -T)|v\rangle$, and the norm of the latter is the same as the norm of $|v\rangle$ because of the unitarity of $\hat{U}(0, -T)$. However, the abelian limit can exist even if the simple limit does not, and in this case the norm of the vector obtained by the abelian limit can be different from the norm of $|v\rangle$, i.e., $Z_{v,in} \neq 1$ is possible. The same also applies to the *out* states.

It is not only the normalization of the *in* and *out* states that may be affected when the abelian limit is used, but also their orthogonality properties. If $|v_1\rangle$ and $|v_2\rangle$ are orthogonal and the limits $\lim_{T \rightarrow \infty} \hat{U}(0, -T)|v_1\rangle$ and $\lim_{T \rightarrow \infty} \hat{U}(0, -T)|v_2\rangle$ exist, then $|v_1, in\rangle$ and $|v_2, in\rangle$ are also orthogonal because of the unitarity of $\hat{U}(0, -T)$. In general, however, the orthogonality of $|v_1\rangle$ and $|v_2\rangle$ does not imply the orthogonality of $|v_1, in\rangle$ and $|v_2, in\rangle$. Nevertheless, the S-matrix is often defined as $S_{ij} = \langle v_i, out|v_j, in\rangle$, where $|v_j, in\rangle$ and $|v_i, out\rangle$ are produced from a specially chosen set of orthonormal state vectors $\{|v_i\rangle\}$ of the Hilbert space. These vectors usually represent plane waves (with the exception of the vacuum state); the index i is a general multi-index and orthonormality means $\langle v_i|v_j\rangle = \delta(i, j)$, where $\delta(i, j)$ is a Dirac-delta. $|v_i\rangle$ should be regarded as a vector-valued distribution in the variable i , and S_{ij} is also a generalized function of i and j . A sufficient condition for the unitarity of S_{ij} is the orthogonality of $|v_i, in\rangle$ and $|v_j, in\rangle$ (and also $|v_i, out\rangle$ and $|v_j, out\rangle$) for $i \neq j$ together with the equivalence of the Hilbert space spanned by $\{|v_i, in\rangle\}$ and that spanned by $\{|v_i, out\rangle\}$. In principle, it has to be verified in each particular case that these conditions are satisfied. The veri-

¹ It is important to note that the normalization of the *in* and *out* states and the $T \rightarrow \infty$ limit are satisfactorily treated in those parts of the literature where the standard covariant field theoretical formalism and the LSZ reduction formulas are presented.

fication of orthogonality may be simplified by symmetries. It seems that in the literature it is usually assumed that limits like $\lim_{T \rightarrow \infty} \bar{U}(0, -T)|v\rangle$ etc. exist, in which case the normalization factors $Z_{v,in}$ and $Z_{w,out}$ are equal to 1. However, in Sec. 4 and in point 4.) below we present examples showing the relevance of these factors; i.e., values $Z_{v,in} \neq 1$ and $Z_{w,out} \neq 1$ will be obtained in these examples, which also implies the nonexistence of the simple limits taken without the abelian averaging.

If the *in* and *out* states are produced from plane-wave states in (5) and (6), then the evaluation of (7) and (8) requires some consideration, since plane wave states do not have finite norm.

3.) Intertwining property of the mappings defined by (5) and (6)

Energy conservation in scattering processes is expressed by the property that the mappings $|v\rangle \rightarrow |v, in\rangle$ and $|w\rangle \rightarrow |w, out\rangle$ defined by (5) and (6) intertwine $e^{iH^{as}\Delta t}$ and $e^{iH\Delta t}$, i.e., $|e^{iH^{as}\Delta t}v, in\rangle = e^{iH\Delta t}|v, in\rangle$ and $|e^{iH^{as}\Delta t}w, out\rangle = e^{iH\Delta t}|w, out\rangle$, where Δt is an arbitrary positive real number. In order to derive these equations, let us consider $|e^{iH^{as}\Delta t}v, in\rangle$. We have

$$\begin{aligned} & \epsilon \int_0^\infty d\tau e^{-\epsilon\tau} \bar{U}(0, -\tau) e^{iH^{as}\Delta t} |v\rangle = \\ & e^{iH\Delta t} e^{\epsilon\Delta t} \epsilon \int_0^\infty d\tau e^{-\epsilon(\tau+\Delta t)} \bar{U}(0, -(\tau+\Delta t)) |v\rangle \\ & = e^{iH\Delta t} e^{\epsilon\Delta t} \epsilon \int_{\Delta t}^\infty d\tilde{\tau} e^{-\epsilon\tilde{\tau}} \bar{U}(0, -\tilde{\tau}) |v\rangle \\ & = e^{iH\Delta t} e^{\epsilon\Delta t} \epsilon \int_0^\infty d\tilde{\tau} e^{-\epsilon\tilde{\tau}} \bar{U}(0, -\tilde{\tau}) |v\rangle \\ & - e^{iH\Delta t} e^{\epsilon\Delta t} \epsilon \int_0^{\Delta t} d\tilde{\tau} e^{-\epsilon\tilde{\tau}} \bar{U}(0, -\tilde{\tau}) |v\rangle. \end{aligned} \quad (12)$$

If $\epsilon \rightarrow 0$, then the second term in (12) obviously tends to zero and $e^{\epsilon\Delta t}$ tends to 1. Thus, we have

$$\begin{aligned} & \lim_{\epsilon \rightarrow +0} \epsilon \int_0^\infty d\tau e^{-\epsilon\tau} \bar{U}(0, -\tau) |e^{iH^{as}\Delta t}v\rangle = \\ & e^{iH\Delta t} \lim_{\epsilon \rightarrow +0} \epsilon \int_{\tau=0}^\infty d\tau e^{-\epsilon\tau} \bar{U}(0, -\tau) |v\rangle. \end{aligned} \quad (13)$$

Equation (13) shows that $Z_{v,in} = Z_{e^{iH^{as}\Delta t}v,in}$. Thus, dividing (13) by $\sqrt{Z_{v,in}}$ yields the desired result $|e^{iH^{as}\Delta t}v, in\rangle = e^{iH\Delta t}|v, in\rangle$.

Differentiating (13) with respect to Δt and then setting $\Delta t = 0$ also yields the intertwining relation

$$\begin{aligned} & \lim_{\epsilon \rightarrow +0} \epsilon \int_0^\infty d\tau e^{-\epsilon\tau} \bar{U}(0, -\tau) |H^{as}v\rangle = \\ & H \lim_{\epsilon \rightarrow +0} \epsilon \int_{\tau=0}^\infty d\tau e^{-\epsilon\tau} \bar{U}(0, -\tau) |v\rangle, \end{aligned} \quad (14)$$

i.e., $\lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, -T) |H^{as}v\rangle = H \lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, -T) |v\rangle$. This implies that if $|v\rangle$ is an eigenvector of H^{as} , then $|v, in\rangle$ is an eigenvector of H with the same eigenvalue. The case of *out* states is similar.

4.) The case of finite-dimensional Hilbert spaces

Let us now consider the case of a Hilbert space with finite dimension N . Although there is no nontrivial scattering in this situation, this case is worth discussing because of its mathematical simplicity, and because it has similarities to more complicated cases (e.g., those studied in Sec. 4). It also shows that the three features of (5) and (6) mentioned above are not specific to infinite-dimensional Hilbert spaces or to quantum field theory.

It is straightforward to verify that the abelian limit $\lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, -T) |v\rangle = |V\rangle$ exists for any $|v\rangle$ vector and any H^{as} , and if $|v\rangle$ is an eigenvector of H^{as} with eigenvalue E_v , then $|V\rangle$ is the orthogonal projection of $|v\rangle$ on the eigenspace of H belonging to the eigenvalue E_v . In particular, $|V\rangle = 0$ if E_v is not an eigenvalue of H , which shows that H^{as} should be chosen in such a way that it has the same eigenvalues as H . Similar statements can be made if $\hat{U}_\epsilon(0, T)$ is taken instead of $\hat{U}_\epsilon(0, -T)$. Moreover, the equation $\lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, -T) |v\rangle = \lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, T) |v\rangle$ holds and implies $Z_{v,in} = Z_{v,out}$.

Let $|v\rangle$ be an eigenvector of H^{as} . In this case, $Z_{v,in} = 1$ if and only if $|v\rangle$ is also an eigenvector of H and in addition H and H^{as} have the same eigenvalue on $|v\rangle$; otherwise $Z_{v,in} < 1$. A similar statement can be made for $Z_{v,out}$.

Assume now that the eigenvalues of H are nondegenerate and H^{as} has the same eigenvalues as H , and let $|v_i^0\rangle$, $i = 1 \dots N$ be a complete set of normalized eigenstates of H^{as} . With the additional assumption that $|v_i^0\rangle$ has nonzero projection on the eigenstate of H that belongs to E_{v_i} for all $i = 1 \dots N$ (i.e., $Z_{v_i,in} = Z_{v_i,out} \neq 0$), the S -matrix $S_{ij} = \langle v_i, out | v_j, in \rangle$ turns out to be the unit matrix (δ_{ij}) , as expected. We note that the abelian limit allows a much larger class of suitable H^{as} operators than the simple limit; if $|v\rangle$ is an eigenvector of H^{as} , then $\bar{U}(0, -T)|v\rangle$ is an oscillating function of T (and hence not convergent as $T \rightarrow \infty$) unless $|v\rangle$ is also an eigenvector of H with the same eigenvalue. This means that if one demands that $\lim_{T \rightarrow \infty} \bar{U}(0, -T)|v\rangle$ should exist for any vector $|v\rangle$, then $H^{as} = H$ must hold. On the other hand, if the abelian limit is used, then any H^{as} is suitable that has the same eigenvalues as H and has the property that any eigenvector of it has nonzero projection on the eigenvector of H with the same eigenvalue.

Let us now allow for H to have degenerate eigenvalues, and let H^{as} have the same eigenvalues as H with the same degeneracies. Let E_i and E_j be two eigenvalues of H , and

let $|v_i\rangle$ and $|v_j\rangle$ be two orthogonal and normalized eigenvectors of H^{as} with the respective eigenvalues E_i and E_j . If $E_i \neq E_j$, then $\langle v_i, \text{in} | v_j, \text{in} \rangle = 0$, of course. However, if $E_i = E_j$, then $\langle v_i, \text{in} | v_j, \text{in} \rangle = 0$ does not follow from $\langle v_i | v_j \rangle = 0$. Nevertheless, it is always possible to choose the (orthonormal) basis vectors within an eigenspace of H^{as} in such a way that the corresponding *in* states are also orthonormal. This can be seen in the following way: let E be an eigenvalue and V_E the corresponding eigenspace of H^{as} . A second scalar product on V_E can be defined as $\langle u | v \rangle_2 = \langle u, \text{in} | v, \text{in} \rangle \sqrt{Z_{u,\text{in}} Z_{v,\text{in}}}$. This determines a self-adjoint linear map A on V_E with the property $\langle u | Av \rangle = \langle u | v \rangle_2$. A admits a diagonal eigenbasis $|v_i\rangle$; for these basis vectors, $\langle v_i, \text{in} | v_j, \text{in} \rangle = \delta_{ij}$ and $\langle v_i, \text{out} | v_j, \text{out} \rangle = \delta_{ij}$ hold. In addition, $\langle v_i, \text{out} | v_j, \text{in} \rangle = \delta_{ij}$ also holds. Furthermore, if there is a symmetry group whose action on the Hilbert space commutes with both H and H^{as} , then A is an invariant mapping. If the representation of this symmetry group on V_E is irreducible, then A is the identity map, and thus $\langle v_i, \text{in} | v_j, \text{in} \rangle = \delta_{ij}$ and $\langle v_i, \text{out} | v_j, \text{out} \rangle = \delta_{ij}$ (as well as $\langle v_i, \text{out} | v_j, \text{in} \rangle = \delta_{ij}$) hold for any orthonormal basis of V_E .

5.) Perturbation theory

Let $\{|v_i\rangle\}$ be an orthonormal basis (i.e. $\langle v_i | v_j \rangle = \delta(i, j)$) consisting of eigenvectors of H^{as} . The eigenvalue of H^{as} on $|v_i\rangle$ is denoted by E_i . A perturbation series for $\lim_{T \rightarrow \infty} \langle v_i | \hat{U}_\epsilon(0, -T) | v_j \rangle$ can be obtained from the Dyson series; it is given by

$$\begin{aligned} \lim_{T \rightarrow \infty} \langle v_i | \hat{U}_\epsilon(0, -T) | v_j \rangle &= \langle v_i | v_j \rangle - i \frac{\langle ij \rangle}{P(ij) + \epsilon} \\ &+ \sum_{k=2}^{\infty} (-i)^k \int dm_1 dm_2 \dots dm_{k-1} \frac{\langle im_{k-1} \rangle}{P(ij) + \epsilon} \\ &\frac{\langle m_{k-1} m_{k-2} \rangle}{P(m_{k-1} j) + \epsilon} \dots \frac{\langle m_2 m_1 \rangle}{P(m_2 j) + \epsilon} \frac{\langle m_1 j \rangle}{P(m_1 j) + \epsilon}, \end{aligned} \quad (15)$$

where the notation

$$P(ij) = i(E_i - E_j), \quad \langle ij \rangle = \langle v_i | H - H^{\text{as}} | v_j \rangle \quad (16)$$

has been used. Similar formulas can be found, e.g., in Chapter 3 of [1] and in [4].

For $\lim_{T \rightarrow \infty} \langle v_i | \hat{U}_\epsilon(0, T)^{\dagger} | v_j \rangle$ we have

$$\begin{aligned} \lim_{T \rightarrow \infty} \langle v_i | \hat{U}_\epsilon(0, T)^{\dagger} | v_j \rangle &= \langle v_i | v_j \rangle - i \frac{\langle ij \rangle}{P(ji) + \epsilon} \\ &+ \sum_{k=2}^{\infty} (-i)^k \int dm_1 dm_2 \dots dm_{k-1} \frac{\langle im_1 \rangle}{P(m_1 i) + \epsilon} \\ &\frac{\langle m_1 m_2 \rangle}{P(m_2 i) + \epsilon} \dots \frac{\langle m_{k-2} m_{k-1} \rangle}{P(m_{k-1} i) + \epsilon} \frac{\langle m_{k-1} j \rangle}{P(ji) + \epsilon}. \end{aligned} \quad (17)$$

The perturbation series for S-matrix elements can be derived from the perturbation series above for

$\lim_{T \rightarrow \infty} \langle v_i | \hat{U}_\epsilon(0, -T) | v_j \rangle$ and $\lim_{T \rightarrow \infty} \langle v_i | \hat{U}_\epsilon(0, T)^{\dagger} | v_j \rangle$. In the derivation, one uses the identity $I = \int di |v_i\rangle \langle v_i|$ where necessary.

If one wants to obtain a Taylor series in powers of a coupling constant g appearing in H , then one has to take into consideration that generally H^{as} , $|v_i\rangle$, and E_i also depend on g . Therefore, the individual terms of a series obtained for a quantity of interest have to be expanded further into a series in powers of g , and terms containing the same power of g have to be collected.

6.) Lippmann–Schwinger equations

It follows from (15) and (17) that the vectors $\lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, -T) | v_j \rangle$ and $\lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, T) | v_j \rangle$ satisfy the Lippmann–Schwinger equations

$$\begin{aligned} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, -T) | v_j \rangle &= \\ | v_j \rangle + \frac{-i}{i(H^{\text{as}} - E_j) + \epsilon} (H - H^{\text{as}}) \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, -T) | v_j \rangle, \end{aligned} \quad (18)$$

$$\begin{aligned} \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, T) | v_j \rangle &= \\ | v_j \rangle + \frac{-i}{i(H^{\text{as}} - E_j) - \epsilon} (H - H^{\text{as}}) \lim_{T \rightarrow \infty} \hat{U}_\epsilon(0, T) | v_j \rangle. \end{aligned} \quad (19)$$

Equations (15) and (17) can also be derived from (18) and (19) by iteration. Note that ϵ is a positive number in these equations.

7.) Long-range potentials

Long-range potentials, like the Coulomb potential, are well known to require special treatment, i.e., (2) and (3) cannot be applied straightforwardly in such cases. This situation is not changed by switching to (5) and (6). One of the possible approaches to handling such potentials is to introduce a shielding, i.e., to approximate a long-range potential by short-range potentials, as described in detail in [2].

3. Adiabatic switching

Another possible generalization of (2) and (3), that we proposed in a slightly different form in [5], is the following:

$$|v, \text{in}\rangle = \lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \tilde{U}_\epsilon(0, -T) \Pi_\epsilon(-T) |v\rangle, \quad (20)$$

$$|w, \text{out}\rangle = \lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \tilde{U}_\epsilon(0, T) \Pi_\epsilon(T) |w\rangle, \quad (21)$$

where

$$\tilde{U}_\epsilon(t_2, t_1) = T \exp \left[-i \int_{t_1}^{t_2} \tilde{H}_{l,\epsilon}(t) dt \right], \quad (22)$$

$$\tilde{H}_{l,\epsilon}(t) = e^{-\epsilon|t|} e^{iH^{\text{as}}t} (H - H^{\text{as}}) e^{-iH^{\text{as}}t}, \quad (23)$$

and T denotes time ordering in (22). The unitary operator $\Pi_\epsilon(T)$ is diagonal with respect to a suitable, complete, orthonormal set of eigenvectors of H^{as} , and its action on a vector $|v\rangle$ belonging to this set is given by

$$\Pi_\epsilon(T)|v\rangle = \frac{\sqrt{\langle v|\tilde{U}_\epsilon(T,0)|v\rangle\langle v|\tilde{U}_\epsilon(0,T)|v\rangle}}{\langle v|\tilde{U}_\epsilon(0,T)|v\rangle}|v\rangle. \quad (24)$$

In these formulas, an adiabatic switching is applied instead of the abelian limit. Equation (23) implies the replacement of H by the time-dependent Hamiltonian operator $\tilde{H}_\epsilon(t) = H^{\text{as}} + e^{-\epsilon|t|}(H - H^{\text{as}})$, for which $\lim_{t \rightarrow \pm\infty} \tilde{H}_\epsilon(t) = H^{\text{as}}$. The time-evolution operator $\tilde{U}_\epsilon(t_1, t_2)$ is unitary for any t_1, t_2 , and ϵ . Therefore, it is not necessary to include normalization constants in (20) and (21). The phase operator $\Pi_\epsilon(T)$ is generally needed, however, to make the $\epsilon \rightarrow 0$ limit convergent; it does not affect the unitarity of the S -matrix. $\tilde{U}_\epsilon(t_1, t_2)$ has the properties $\tilde{U}_\epsilon(t_1, t_2)\tilde{U}_\epsilon(t_2, t_3) = \tilde{U}_\epsilon(t_1, t_3)$ and $\tilde{U}_\epsilon(t_1, t_2)^{-1} = \tilde{U}_\epsilon(t_2, t_1)$. Adiabatic switching also appears in the literature on scattering theory; in particular, it is known that for potential scattering (20) and (21), without the phase operator and with $H^{\text{as}} = H_0$ (where H_0 denotes the Hamiltonian operator obtained from H by switching off the interaction), yield the same *in* and *out* states as (2) and (3) [9]. A similar result was obtained in [5] in perturbation theory.

If H^{as} (and H) has a finite, discrete, and nondegenerate spectrum, then (20) is essentially the Gell-Mann-Low formula [10, 11], which produces eigenvectors of H from eigenvectors of H^{as} . This implies that if H_0 has also a nondegenerate spectrum, then $H^{\text{as}} = H_0$ can be taken; the choice of H^{as} is not restricted. In particular, it is not necessary in (20) and (21) that H and H^{as} have the same eigenvalues. The phase operator, however, is necessary for the existence of the $\epsilon \rightarrow 0$ limit.

In the case of more general Hamiltonian operators, (20) and (21) can be regarded as generalizations of the Gell-Mann-Low formula, and it is then natural to conjecture that $H^{\text{as}} = H_0$, where H_0 is defined, as above, to be the Hamiltonian operator obtained from H by switching off the interaction, will be a suitable choice for a large class of models that describe nontrivial scattering processes. Perturbative calculations in particular models support this conjecture (see [5] and Sec. 4).

The properties that H^{as} can be taken to be H_0 and $\tilde{U}_\epsilon(0, -T)$ and $\tilde{U}_\epsilon(0, T)$ are unitary (which is not the case for $\hat{U}_\epsilon(0, -T)$ and $\hat{U}_\epsilon(0, T)$) are appealing features of the formulas (20) and (21) when compared to (5) and (6). On the other hand, adiabatic switching is more difficult to handle mathematically than the abelian average.

In view of Sec. 2 and the result of [9] mentioned above, it is an interesting question whether the phase operator can be omitted if H^{as} has the same spectrum as H . By perturbative calculations applied to Hamiltonian operators with finite discrete spectra and to the examples discussed in Sec. 4, one finds that the answer is that this factor cannot be omitted in general. One finds the same result by numerical calculations with 2×2 matrices. The analytical results of [12], which apply to 2×2 matrices, can also be used to show that in general the $\epsilon \rightarrow 0$ limit of $\lim_{T \rightarrow \infty} \tilde{U}_\epsilon(0, -T)|v\rangle$ or $\lim_{T \rightarrow \infty} \tilde{U}_\epsilon(0, T)|w\rangle$ does not exist if H^{as} has the same spectrum as H .

For perturbative calculations, the following formulas can be used. Let $\{|v_i\rangle\}$ be an orthonormal basis consisting of eigenvectors of H^{as} , and let us denote by \tilde{E}_i the eigenvalues of H^{as} on these vectors. From the Dyson series one obtains the following perturbation series for $\lim_{T \rightarrow \infty} \langle v_i|\tilde{U}_\epsilon(0, -T)|v_j\rangle$:

$$\begin{aligned} \lim_{T \rightarrow \infty} \langle v_i|\tilde{U}_\epsilon(0, -T)|v_j\rangle &= \langle v_i|v_j\rangle - i \frac{\langle ij\rangle}{P(ij) + \epsilon} \\ &+ \sum_{k=2}^{\infty} (-i)^k \int dm_1 dm_2 \dots dm_{k-1} \frac{\langle im_{k-1}\rangle}{P(ij) + k\epsilon} \\ &\frac{\langle m_{k-1}m_{k-2}\rangle}{P(m_{k-1}j) + (k-1)\epsilon} \dots \frac{\langle m_2m_1\rangle}{P(m_2j) + 2\epsilon} \frac{\langle m_1j\rangle}{P(m_1j) + \epsilon}, \end{aligned} \quad (25)$$

where the notation $P(ij) = i(\tilde{E}_i - \tilde{E}_j)$ and $\langle ij\rangle = \langle v_i|H - H^{\text{as}}|v_j\rangle$ has been used.

The main difference between this formula and (15) is in the coefficients of ϵ in the denominators. This shows that in general it is essential to carefully take into consideration the precise value of these coefficients in order to obtain correct results. This point needs to be emphasized because it is usually not mentioned in the literature. Nevertheless, there are also several instances in calculations when the precise values of these coefficients are not important.

For $\lim_{T \rightarrow \infty} \langle v_i|\tilde{U}_\epsilon(T, 0)|v_j\rangle$ we have

$$\begin{aligned} \lim_{T \rightarrow \infty} \langle v_i|\tilde{U}_\epsilon(T, 0)|v_j\rangle &= \langle v_i|v_j\rangle - i \frac{\langle ij\rangle}{P(ji) + \epsilon} \\ &+ \sum_{k=2}^{\infty} (-i)^k \int dm_1 dm_2 \dots dm_{k-1} \frac{\langle im_1\rangle}{P(m_1i) + \epsilon} \\ &\frac{\langle m_1m_2\rangle}{P(m_2i) + 2\epsilon} \dots \frac{\langle m_{k-2}m_{k-1}\rangle}{P(m_{k-1}i) + (k-1)\epsilon} \frac{\langle m_{k-1}j\rangle}{P(ji) + k\epsilon}. \end{aligned} \quad (26)$$

Due to the differences between (25), (26) and (15), (17) it is not immediately clear how equations analogous to

(18) and (19) can be found for $\lim_{T \rightarrow \infty} \tilde{U}_\epsilon(0, -T)|v_j\rangle$ and $\lim_{T \rightarrow \infty} \tilde{U}_\epsilon(0, T)|v_j\rangle$.

Although outside the context of scattering theory, adiabatic switching and the Gell-Mann-Low formula have been subject of active research recently; see, e.g., [12–18] and references therein.

4. Examples

In this section, an application of (5), (6) and (20), (21) in two quantum field theoretical models is presented. Perturbation theory based on (15), (17) and (25), (26) is used to calculate S-matrix elements and energies of *in* and *out* states.

The first model describes the scattering of a massive relativistic particle (a real scalar boson) on a defect in 1 + 1 spacetime dimensions. The defect is localized at $x = 0$. The Hamiltonian operator without interaction is

$$H_0 = \frac{1}{2} \int_{-\infty}^{\infty} dx : (\partial_t \phi)^2 + (\partial_x \phi)^2 + m_0^2 \phi^2 : , \quad (27)$$

where ϕ is a boson field satisfying the commutation relation $[\phi(x, t), \partial_t \phi(x', t)] = i\delta(x - x')$. The interaction term is $H_{\text{int}} =: \phi(0, 0)^2 :$; $H = H_0 + gH_{\text{int}}$. In this model, the interaction term breaks translation and Lorentz symmetry. The second example is the ϕ^4 model in 3 + 1 dimensions. In this model, the Hamiltonian operator in the absence of interactions is that of a free scalar boson of mass m_0 :

$$H_0 = \frac{1}{2} \int d^3x : (\partial_t \phi)^2 + (\partial_x \phi)^2 + m_0^2 \phi^2 : . \quad (28)$$

The interaction term H_{int} is

$$H_{\text{int}} = \int d^3x : \phi^4 : , \quad (29)$$

and $H = H_0 + gH_{\text{int}}$. The field ϕ satisfies the commutation relation $[\phi(x, t), \partial_t \phi(x', t)] = i\delta^3(x - x')$.

In the following, the vacuum–vacuum S-matrix element, the one-particle S-matrix elements, in the second example the two-particle S-matrix elements, and the energy of the *in* and *out* states are discussed. Perturbation theory is applied to second order in g in the first example and to third order in the second example. The details of the calculations are not presented, since they are rather lengthy and can be considered straightforward. We note that the application of (20) and (21) in these examples was also discussed in [5].

First example: scattering of a scalar particle on a defect
We now turn to the discussion of the application of (5)

and (6), in which the abelian average is used, to the first example. We take H^{as} to be of the form

$$H^{\text{as}} = H_0 + \Delta E \cdot I, \quad (30)$$

where ΔE is a constant that is still to be determined, and I is the unit operator. In this case, H_0 and H^{as} have the same eigenvectors, and one uses as basis vectors the vacuum state and the multiparticle plane-wave eigenstates of H_0 . The vacuum state $|\Omega\rangle$ is normalized as $\langle \Omega | \Omega \rangle = 1$ and the one- and two-particle states $|k\rangle$ and $|k_1 k_2\rangle$ as $\langle k_1 | k_2 \rangle = \delta(k_1 - k_2)$, $\langle k_3 k_4 | k_1 k_2 \rangle = \delta(k_1 - k_3)\delta(k_2 - k_4) + \delta(k_1 - k_4)\delta(k_2 - k_3)$, etc.

The ansatz (30) for H^{as} can be regarded as a guess, but one can also infer that this should be the choice for H^{as} from the results of the application of (20) and (21) (in which the adiabatic switching is used). In that case, one takes $H^{\text{as}} = H_0$, and then one can calculate the eigenvalues that H has on the *in* and *out* states. In this way, one obtains that the energies of the *in* and *out* states are shifted by a common constant with respect to their values at $g = 0$.

The coefficients in the Taylor series for ΔE are denoted as $\Delta E = g\Delta E^{(1)} + g^2\Delta E^{(2)} + \dots$. To second order, one finds that the requirement that the $\epsilon \rightarrow +0$ limit of $\lim_{T \rightarrow \infty} \langle \Omega | \hat{U}_\epsilon(0, T)^\dagger \hat{U}_\epsilon(0, -T) | \Omega \rangle$ be convergent determines $\Delta E^{(1)}$ and $\Delta E^{(2)}$ uniquely, and one obtains $\Delta E^{(1)} = 0$ and

$$\Delta E^{(2)} = - \int dk_A dk_B \frac{1}{32\pi^2} \frac{1}{\omega_A \omega_B (\omega_A + \omega_B)}, \quad (31)$$

where the notation $\omega = \sqrt{m_0^2 + k^2}$ has been used. For $Z_{\Omega, \text{in}}$ and $Z_{\Omega, \text{out}}$, one then obtains

$$\begin{aligned} Z_{\Omega, \text{in}} = Z_{\Omega, \text{out}} = \\ 1 - g^2 \int dk_A dk_B \frac{1}{32\pi^2} \frac{1}{\omega_A \omega_B (\omega_A + \omega_B)^2} + O(g^3). \end{aligned} \quad (32)$$

The final result for the vacuum–vacuum S-matrix element to second order in g is $S_{\Omega, \Omega} = \langle \Omega, \text{out} | \Omega, \text{in} \rangle = 1 + O(g^3)$, as expected. Equation (32) is also consistent with $Z_{\Omega, \text{in}} \leq 1$, since the integrand is positive.

Regarding one-particle states (with $k \neq 0$), one finds that if $\Delta E^{(1)}$ and $\Delta E^{(2)}$ are chosen as above, then the $\epsilon \rightarrow +0$ limit of $\lim_{T \rightarrow \infty} \langle k_2 | \hat{U}_\epsilon(0, T)^\dagger \hat{U}_\epsilon(0, -T) | k_1 \rangle$ is convergent. One obtains that $\lim_{T \rightarrow \infty} \langle k_2 | \hat{U}_\epsilon(0, -T)^\dagger \hat{U}_\epsilon(0, -T) | k_1 \rangle = Z_{\Omega, \text{in}} \delta(k_1 - k_2)$, and hence $Z_{k, \text{in}} = Z_{\Omega, \text{in}}$. One also finds that $Z_{k, \text{out}} = Z_{k, \text{in}}$.

We conjecture that to all orders in g there exists a unique value of ΔE , such that the $\epsilon \rightarrow +0$ limit of

$\lim_{T \rightarrow \infty} \langle v_i | \hat{U}_\epsilon(0, T)^\dagger \hat{U}_\epsilon(0, -T) | v_j \rangle$ is convergent for any plane-wave basis vectors $|v_i\rangle$ and $|v_j\rangle$ (not containing any particle with $k = 0$), and that the Z factor is the same for all such states, to all orders in g .

We find that to second order the S -matrix element S_{k_1, k_2} is

$$S_{k_1, k_2} = T(k_2) \delta(k_1 - k_2) + R(k_2) \delta(k_1 + k_2), \quad (33)$$

where

$$\begin{aligned} T(k_2) &= 1 - g \frac{i}{2|k_2|} - g^2 \frac{1}{4k_2^2} + O(g^3) \\ R(k_2) &= -g \frac{i}{2|k_2|} - g^2 \frac{1}{4k_2^2} + O(g^3). \end{aligned} \quad (34)$$

An exact result for S_{k_1, k_2} to all orders was obtained in a different framework in [19]. It also takes the form (33), and $T(k_2)$ and $R(k_2)$ are²

$$T(k_2) = \frac{i|k_2|}{i|k_2| - g/2}, \quad R(k_2) = \frac{g/2}{i|k_2| - g/2}. \quad (35)$$

Application of Eqs. (20) and (21), with $H^{\text{as}} = H_0$, yields the same results for $S_{\Omega, \Omega}$ and S_{k_1, k_2} as above. The phase factors in (20) and (21) are necessary for obtaining convergent results as $\epsilon \rightarrow 0$. The states $|\Omega, \text{in}\rangle = |\Omega, \text{out}\rangle$, $|k, \text{in}\rangle$, and $|k, \text{out}\rangle$ are eigenvectors of H . Their energies can therefore be calculated from the eigenvalue equation. One finds that the energies of these states are all shifted by a common constant with respect to their value at $g = 0$; the value of this constant equals ΔE above (at least to second order in g).

Second example: ϕ^4 model

Continuing with the second example, it is well known that in perturbation theory the ϕ^4 model in $3 + 1$ dimensions contains ultraviolet divergences, which can for example be handled by introducing a cutoff. For simplicity we shall not introduce such a cutoff. Therefore, our results will be formal expressions, which, however, is sufficient for our present purpose. One could also consider the ϕ^4 model in $1 + 1$ dimensions; in that case, the calculations and the results are very similar to those in $3 + 1$ dimensions, but ultraviolet divergences do not occur. It is also well known that there are divergences in the model associated with translation invariance and the infiniteness of the volume of space. In perturbation theory, these divergences are

related to vacuum bubble diagrams (see, e.g., [8], Chapter 4). A rigorous treatment of these divergences could be given by introducing a suitable regularization, e.g., by considering the system in a finite volume. In this case, the size of the space volume has to be increased simultaneously with the T appearing in (5), (6) and in (20), (21). Nevertheless, it is also known that the contributions of the vacuum bubble diagrams are absent in correlation functions and S -matrix elements in the standard formalism (the demonstration of this result is usually done in a formal manner, i.e., without doing a rigorous regularization; see, e.g., Chapter 17 of [20] and Chapter 6 of [21]). In the present paper, we also restrict ourselves to a formal treatment of the divergences corresponding to vacuum bubble diagrams, mainly because a rigorous treatment would be much more complicated, and because we do not expect that it would change the results. For further details on the difficulties of defining Poincaré-symmetric quantum field theoretical models, we refer the reader to Sec. 4-1-1 of [21] and to [22]. We also note that one of our reasons for including the first example in this paper is that although it is a quantum field theory, it does not have divergences in perturbation theory that correspond to vacuum bubble diagrams, and it is also free of ultraviolet divergences.

We first consider the application of (5) and (6). If we kept $H = H_0 + gH_{\text{int}}$ as was specified at the beginning of this section, with H_0 and H_{int} given by (28) and (29), then the mass of the particles described by H would be different from m_0 . Therefore, H^{as} would depend on g , and the relation between H_0 and H^{as} would be more complicated than in the previous example (i.e., H^{as} would not be of the form $\Delta E \cdot I + H_0$). This, in turn, would complicate the application of (15) and (17), since these formulas are given in terms of eigenvalues of H^{as} and matrix elements with respect to eigenstates of H^{as} , whereas the known quantities, in terms of which the model is defined, are the eigenvalues of H_0 and the matrix elements of H_{int} with respect to the plane-wave and vacuum eigenstates of H_0 . For this reason, we modify H by adding two terms, so that the modified total Hamiltonian operator that we actually consider is

$$H' = H_0 + g \int d^3x : \phi^4 : - \Delta E \cdot I - \frac{1}{2} \delta m^2 \int d^3x : \phi^2 : . \quad (36)$$

The ΔE and δm^2 parameters introduced here should be chosen in such a way that the ground-state energy and the particle mass corresponding to H' are 0 and m_0 , respectively, so that $H^{\text{as}} = H_0$ can be taken. The asymptotic states from which the scattering states are produced by (5) and (6) are then the vacuum state and the multiparticle plane-wave eigenstates of H_0 , and the matrix elements and eigenvalues entering the formulas (15), (17) are known, thus the difficulty described above is avoided.

² This result was quoted with some wrong signs in [5].

Nevertheless, ΔE and δm^2 are initially unknown constants that depend on g and have to be determined. We note that the kind of modification of the model that we have done is a common part of renormalization.

The $-\Delta E \cdot I$ term does not have much physical significance, as it merely shifts all energies by a common constant. The $-\frac{1}{2}\delta m^2 \int d^3x : \phi^2 :$ term serves to readjust the mass of the particles described by H' to the value m_0 , which is the physical value in the present approach. The coefficients in the Taylor series of ΔE and δm^2 are denoted as $\Delta E = g\Delta E^{(1)} + g^2\Delta E^{(2)} + \dots$ and $\delta m^2 = g(\delta m^2)^{(1)} + g^2(\delta m^2)^{(2)} + \dots$. To second order, $\Delta E^{(1)}$ and $\Delta E^{(2)}$ are determined in a way similar to the first example, and one obtains $\Delta E^{(1)} = 0$ and

$$\Delta E^{(2)} = - \frac{\int d^3k_A d^3k_B d^3k_C d^3k_D \frac{\eta^2}{\omega_A \omega_B \omega_C \omega_D}}{4! \frac{[\delta^3(k_A + k_B + k_C + k_D)]^2}{(\omega_A + \omega_B + \omega_C + \omega_D)^2}}, \quad (37)$$

where $\eta = 4!/4(2\pi)^3$, and the notation $\omega = \sqrt{m_0^2 + k^2}$ has been used.

For $Z_{\Omega, in}$ and $Z_{\Omega, out}$ one obtains

$$Z_{\Omega, in} = Z_{\Omega, out} = 1 - g^2 \int \frac{d^3k_A d^3k_B d^3k_C d^3k_D \frac{\eta^2}{\omega_A \omega_B \omega_C \omega_D}}{4! \frac{[\delta^3(k_A + k_B + k_C + k_D)]^2}{(\omega_A + \omega_B + \omega_C + \omega_D)^2}} + O(g^3). \quad (38)$$

The integrand on the right-hand side is nonnegative, which is consistent with $Z_{\Omega, in} \leq 1$. The final result for the vacuum–vacuum S-matrix element is $S_{\Omega, \Omega} = \langle \Omega, out | \Omega, in \rangle = 1 + O(g^3)$, as expected.

Regarding $\lim_{T \rightarrow \infty} \langle k_2 | \hat{U}_\epsilon(0, T)^\dagger \hat{U}_\epsilon(0, -T) | k_1 \rangle$, we obtain from the requirement that the $\epsilon \rightarrow +0$ limit of this quantity should be convergent that $(\delta m^2)^{(1)} = 0$ and

$$(\delta m^2)^{(2)} = -2 \int \frac{d^3k_A d^3k_B \frac{\eta^2}{\omega_A \omega_B \omega_C (\omega_A + \omega_B + \omega_C - m_0)}}{3!} - 2 \int \frac{d^3k_A d^3k_B \frac{\eta^2}{\omega_A \omega_B \omega_C (\omega_A + \omega_B + \omega_C + m_0)}}{3!}, \quad (39)$$

where $k_C = -k_A - k_B$ in ω_C .

For $Z_{k, in}$ and $Z_{k, out}$ we obtain

$$Z_{k, in} = Z_{k, out} = Z_{\Omega, in} - g^2 \int \frac{d^3k_A d^3k_B \frac{\eta^2}{\omega_A \omega_B \omega_C \omega} \frac{1}{(\omega_A + \omega_B + \omega_C - \omega)^2}}{3!} - g^2 \int \frac{d^3k_A d^3k_B \frac{\eta^2}{\omega_A \omega_B \omega_C \omega} \frac{1}{(\omega_A + \omega_B + \omega_C + \omega)^2}}{3!} + O(g^3). \quad (40)$$

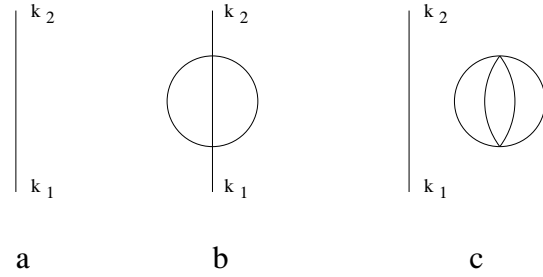


Figure 1. Graphs for the one-particle S-matrix element to second order

This result is also consistent with $Z_{k, in} \leq 1$. The final result for the one-particle S-matrix element is $\langle k_2, out | k_1, in \rangle = \delta^3(k_2 - k_1) + O(g^3)$, which is also in agreement with the expectation. Our result for δm^2 is equal to the result that can be obtained in the standard field theoretical formalism.

To second order, the contributions to $\lim_{T \rightarrow \infty} \langle k_2 | \hat{U}_\epsilon(0, T)^\dagger \hat{U}_\epsilon(0, -T) | k_1 \rangle$ arising from the interaction term $g \int d^3x : \phi^4 :$ correspond to the graphs shown in Fig. 1. (Graphs can be associated with various terms in perturbation theory in a similar way as in the standard formalism. Detailed graph rules are not described here, since they are not needed in this paper.) The contributions corresponding to 1.b and 1.c diverge as $1/\epsilon$ in the $\epsilon \rightarrow +0$ limit. These divergences are canceled by further contributions arising from the terms $-\Delta E \cdot I$ and $-\frac{1}{2}\delta m^2 \int d^3x : \phi^2 :$ in H' .

Considering $\lim_{T \rightarrow \infty} \langle k_3, k_4 | \hat{U}_\epsilon(0, T)^\dagger \hat{U}_\epsilon(0, -T) | k_1, k_2 \rangle$, we find that its $\epsilon \rightarrow 0$ limit is finite if $\Delta E^{(1)}$, $\Delta E^{(2)}$, $(\delta m^2)^{(1)}$, and $(\delta m^2)^{(2)}$ are chosen as above.

For $Z_{k_1 k_2, in}$ and $Z_{k_1 k_2, out}$ one finds

$$Z_{k_1 k_2, in} = Z_{k_1 k_2, out} \quad (41)$$

and

$$\frac{Z_{k_1 k_2, in}}{Z_{\Omega, in}} = \frac{Z_{k_1, in}}{Z_{\Omega, in}} \frac{Z_{k_2, in}}{Z_{\Omega, in}}. \quad (42)$$

It is natural to expect that these relations hold to all orders of perturbation theory, and that similar relations hold for states containing an arbitrary number of particles.

The diagrams corresponding to second-order contributions to $\lim_{T \rightarrow \infty} \langle k_3, k_4 | \hat{U}_\epsilon(0, T)^\dagger \hat{U}_\epsilon(0, -T) | k_1, k_2 \rangle$ arising from $g \int d^3x : \phi^4 :$ in H' are those shown in Fig. 2, and those that are obtained from 2.a, 2.d, 2.e, 2.f, and 2.g by interchanging k_3 and k_4 . The terms that are divergent in the $\epsilon \rightarrow +0$ limit correspond to the graphs 2.e, 2.f, and 2.g, as well as to those that are obtained from 2.e, 2.f, and 2.g by interchanging k_3 and k_4 . The rate of divergence of

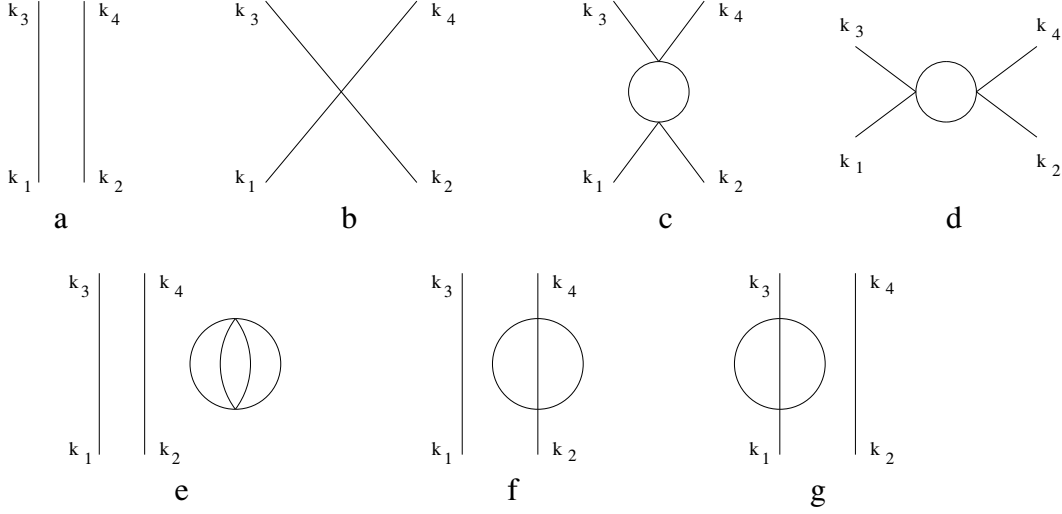


Figure 2. Graphs for the two-particle S-matrix elements to second order

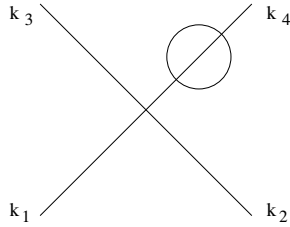


Figure 3. A third-order graph containing a radiative correction on an external line

these terms is $1/\epsilon$. Further terms arising from the counterterms $-\Delta E \cdot I$ and $-\frac{1}{2}\delta m^2 \int d^3x : \phi^2 :$ cancel these divergences.

The final result for the two-particle S-matrix contains only the terms that correspond to the graphs 2.a, 2.b, 2.c, and 2.d (and to those obtained from these graphs by interchanging k_3 and k_4). To first order, the two-particle S-matrix reads

$$\begin{aligned} S_{k_3 k_4; k_1 k_2} &= \delta^3(k_1 - k_3)\delta^3(k_2 - k_4) + \delta^3(k_1 - k_4)\delta^3(k_2 - k_3) \\ &\quad - g2\pi i \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \\ &\quad \delta^3(k_1 + k_2 - k_3 - k_4) \frac{\eta}{\sqrt{\omega_1 \omega_2 \omega_3 \omega_4}} + O(g^2). \end{aligned} \quad (43)$$

At third order, we focus on those contributions to $\lim_{T \rightarrow \infty} \langle k_3, k_4 | \hat{U}_\epsilon(0, T)^\dagger \hat{U}_\epsilon(0, -T) | k_1, k_2 \rangle$ yielded by the interaction term $g \int d^3x : \phi^4 :$ which correspond to

graphs that contain a radiative correction on an external line. One of these graphs is shown in Fig. 3, and there are three other, similar ones in which the correction is on the k_1 , k_2 , or k_3 leg. These contributions diverge as $1/\epsilon$ in the $\epsilon \rightarrow +0$ limit. This divergence is canceled by further terms arising from the counterterm $-\frac{1}{2}\delta m^2 \int d^3x : \phi^2 :$ in H' , with $(\delta m^2)^{(1)}$ and $(\delta m^2)^{(2)}$ as above. There are also third-order contributions to $S_{k_3 k_4; k_1 k_2}$ from the product of the second-order parts of $1/\sqrt{Z_{k_1 k_2}}$ and $1/\sqrt{Z_{k_3 k_4}}$ and the first-order part of $\lim_{\epsilon \rightarrow +0} \lim_{T \rightarrow \infty} \langle k_3, k_4 | \hat{U}_\epsilon(0, T)^\dagger \hat{U}_\epsilon(0, -T) | k_1, k_2 \rangle$, which is $-g2\pi i \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \delta^3(k_1 + k_2 - k_3 - k_4) \frac{\eta}{\sqrt{\omega_1 \omega_2 \omega_3 \omega_4}}$ (the graph corresponding to this factor is depicted in Fig. 2.b). The sum of these terms plus the first-order part is given by

$$\begin{aligned} &-g2\pi i \delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \\ &\delta^3(k_1 + k_2 - k_3 - k_4) \frac{\eta}{\sqrt{\omega_1 \omega_2 \omega_3 \omega_4}} \sqrt{Z_\phi^4} + O(g^4), \end{aligned} \quad (44)$$

where Z_ϕ is the constant

$$\begin{aligned} Z_\phi &= 1 - g^2 \int \frac{d^3 k_A d^3 k_B}{3!} \frac{\eta^2}{\omega_A \omega_B \omega_C m_0} \frac{1}{(\omega_A + \omega_B + \omega_C - m_0)^2} \\ &\quad + g^2 \int \frac{d^3 k_A d^3 k_B}{3!} \frac{\eta^2}{\omega_A \omega_B \omega_C m_0} \frac{1}{(\omega_A + \omega_B + \omega_C + m_0)^2} \\ &\quad + O(g^3), \end{aligned} \quad (45)$$

with $k_C = -k_A - k_B$ in ω_C . This Z_ϕ is equal to the field-strength renormalization constant in the standard formalism. Although Z_ϕ has some similarity to Z_k/Z_Ω , they are

different quantities; in particular the latter is not independent of k .

The expression (44) is the same as the one obtained using the standard formalism. Of course, there are further contributions to $S_{k_3 k_4; k_1 k_2}$ at order g^3 , which correspond to connected graphs that contain three vertex points (of order four) and do not contain corrections on external lines. Denoting these contributions, together with those at order g^2 which correspond to connected graphs that contain two vertex points of order four, by we have to third order

$$\begin{aligned} S_{k_3 k_4; k_1 k_2} &= \delta^3(k_1 - k_3)\delta^3(k_2 - k_4) + \delta^3(k_1 - k_4)\delta^3(k_2 - k_3) \\ &\quad - g2\pi i\delta(\omega_1 + \omega_2 - \omega_3 - \omega_4) \\ &\quad \delta^3(k_1 + k_2 - k_3 - k_4) \frac{\eta}{\sqrt{\omega_1 \omega_2 \omega_3 \omega_4}} \sqrt{Z_\phi^4} \\ &\quad + \dots + O(g^4). \end{aligned} \quad (46)$$

In the case of the formulas (20) and (21), in which adiabatic switching is used, we take $H^{as} = H_0$, and we also leave H unchanged. We make the choice $H^{as} = H_0$ because H_0 is the operator obtained from H by switching off the interaction, i.e., by setting g to zero. Thus, this is the choice one would try first. Another reason is that a more complicated H^{as} would entail the same difficulties as described above at the beginning of the discussion of the application of (5) and (6). In addition, by doing calculations with the choice $H^{as} = H_0$ (and with the original H), one can illustrate the feature of (20) and (21) that they do not require the ground-state energy and particle mass corresponding to H^{as} and H to be the same.

Regarding the perturbation series for the quantities $\lim_{T \rightarrow \infty} \langle \Omega | \tilde{U}_\epsilon(0, T)^\dagger \tilde{U}_\epsilon(0, -T) | \Omega \rangle$, $\lim_{T \rightarrow \infty} \langle k_2 | \tilde{U}_\epsilon(0, T)^\dagger \tilde{U}_\epsilon(0, -T) | k_1 \rangle$, and $\lim_{T \rightarrow \infty} \langle k_3, k_4 | \tilde{U}_\epsilon(0, T)^\dagger \tilde{U}_\epsilon(0, -T) | k_1, k_2 \rangle$ to second or third order in g , one finds that the coefficients of the higher than first-order terms do not have finite limits as $\epsilon \rightarrow 0$, and the divergent parts correspond to graphs that contain disconnected vacuum bubbles or radiative corrections on external lines, in analogy to the case when the abelian limit is applied. Nevertheless, the phase factors that are also included in (20) and (21) cancel these divergences. Thus, for the S-matrix elements $S_{\Omega, \Omega}$, $S_{k_1; k_2}$, and $S_{k_3 k_4; k_1 k_2}$, finite results are obtained.

The final results for $S_{\Omega, \Omega}$ and $S_{k_1; k_2}$ are the same as above, as expected; $S_{\Omega, \Omega} = 1 + O(g^3)$ and $S_{k_1; k_2} = \delta^3(k_2 - k_1) + O(g^3)$. To third order we obtain for $S_{k_3 k_4; k_1 k_2}$ the result

$$\begin{aligned} S_{k_3 k_4; k_1 k_2} &= \delta^3(k_1 - k_3)\delta^3(k_2 - k_4) + \delta^3(k_1 - k_4)\delta^3(k_2 - k_3) \\ &\quad - g2\pi i\delta(\hat{\omega}_1 + \hat{\omega}_2 - \hat{\omega}_3 - \hat{\omega}_4) \\ &\quad \delta^3(k_1 + k_2 - k_3 - k_4) \frac{\eta}{\sqrt{\hat{\omega}_1 \hat{\omega}_2 \hat{\omega}_3 \hat{\omega}_4}} \sqrt{Z_\phi^4} \\ &\quad + \dots + O(g^4), \end{aligned} \quad (47)$$

where ... denotes further terms corresponding to connected diagrams that contain two or three vertices of order four and that do not contain corrections on external lines. These terms are the same as in the case above where the abelian limit was used. In (47), $\hat{\omega}_1$ denotes $\sqrt{m^2 + k_1^2}$, etc., where $m^2 = m_0^2 + \delta m^2$ and δm^2 is here equal to (39). Z_ϕ is the same as in (45). The differences between the results (46) and (47) obtained for $S_{k_3 k_4; k_1 k_2}$ using (20), (21) and (5), (6), respectively, are due to the fact that the total Hamiltonians in the two cases are different; in particular, the particle masses corresponding to them are not the same.

The energies of $|\Omega, in\rangle$ and $|k, in\rangle$ can also be calculated from the eigenvalue equation (as these states are eigenstates of H). This yields the result that to second order the energy of $|\Omega, in\rangle$ is equal to ΔE , and the square of the mass of the scalar particle is equal to $m_0^2 + \delta m^2$.

For a more direct comparison with the case where the abelian limit is applied, one can take H' given by (36), (37), and (39) as the total Hamiltonian operator, and $H^{as} = H_0$. The results for $S_{\Omega, \Omega}$, $S_{k_1; k_2}$, and $S_{k_3 k_4; k_1 k_2}$ with these Hamiltonians are then exactly the same as above where the abelian limit was applied. The phase factors included in (20) and (21) are still necessary; without these factors one would not obtain convergent (as $\epsilon \rightarrow 0$) final results.

5. Summary and conclusion

We discussed the problem of the generalization of quantum-mechanical formal scattering theory to a wider class of models that includes quantum field theories, with the aim of clarifying certain elementary aspects that are not treated completely satisfactorily in textbooks. We intended to draw attention to the fact that in the case of quantum field theories the most straightforward strong limit that can be applied in the standard quantum-mechanical formula for producing the *in* and *out* states is not suitable, therefore some more effective limiting procedure is needed, which requires certain modifications of the formulas used in quantum mechanics.

We studied two possibilities: the application of the abelian limit and of adiabatic switching. The modifications of the quantum-mechanical formulas required by these two methods are significantly different, which shows that quantum-mechanical formal scattering theory can be generalized in at least two nontrivially different ways. Neither of the two ways appear to be distinguished.

The first method that we considered, which was the abelian limit, does not preserve the unitarity of the time-evolution operator, making it necessary to include suit-

able normalization factors in the formulas for the *in* and *out* states. The orthogonality properties of the *in* and *out* states also need verification. In addition, the application of the abelian limit requires that the particle masses and the vacuum energy corresponding to the asymptotic Hamiltonian operator that describes the particles far in time from the collision be the same as the masses and the vacuum energy corresponding to the full Hamiltonian operator.

The second method, the application of adiabatic switching, preserves the unitarity of the time-evolution operator. Therefore, it is not necessary to introduce normalization factors into the formulas for the *in* and *out* states. However, it is necessary to include suitable compensating phase factors in the formulas to make the phases of the *in* and *out* states convergent when the $\epsilon \rightarrow 0$ limit is taken. It is also not necessary that the particle masses and the vacuum energy corresponding to the asymptotic Hamiltonian operator be the same as for the full Hamiltonian operator. While a Lippmann–Schwinger equation can be written down in the case when the abelian limit is applied, this does not seem to be possible in a straightforward way in the case of adiabatic switching.

In order to provide illustrations for the features of the two approaches, we considered the case when the Hilbert space is finite dimensional, and we also studied two specific quantum field theoretical models. The case of a finite-dimensional Hilbert space is interesting because of its mathematical simplicity and because it shows that the features mentioned above are not specific to infinite-dimensional Hilbert spaces or to quantum field theory. For the two quantum field theoretical models, we calculated the vacuum–vacuum, one-particle, and two-particle *S*-matrix elements as well as the energy of the *in* and *out* states to second (and in some cases to third) order in the coupling constant, using suitably modified old-fashioned perturbation theory. The results of these calculations confirm the necessity of the modifications of the quantum-mechanical formulas, and they are also in agreement with results that can be obtained by other methods, indicating that the formulas (5), (6) and (20), (21) can indeed be applied to quantum field theories. It remains an open problem to extend our results to higher orders of perturbation theory. Although expected, it is remarkable that both methods considered in this paper, in spite of the differences between them, give equivalent final results, at least for the physical quantities above and to the orders taken into account. It would be interesting to investigate whether there exist models for which the two methods give different results. A further interesting question is whether

the *S*-matrix elements yielded by (5), (6) or (20), (21) can depend on the choice of H^{as} , with fixed total Hamiltonian operator.

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