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Path integral treatment of a noncentral electric potential

Research Article

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Abstract: We present a rigorous path integral treatment of a dynamical system in the axially symmetric potential $V(r,\theta) = V(r) + \frac{1}{r^2}V(\theta)$. It is shown that the Green's function can be calculated in spherical coordinate system for $V(\theta) = \frac{\hbar^2}{2\mu}\frac{\gamma+\beta\sin^2\theta+\alpha\sin^4\theta}{\sin^2\theta\cos^2\theta}$. As an illustration, we have chosen the example of a spherical harmonic oscillator and also the Coulomb potential for the radial dependence of this noncentral potential. The ring-shaped oscillator and the Hartmann ring-shaped potential are considered as particular cases. When $\alpha = \beta = \gamma = 0$, the discrete energy spectrum, the normalized wave function of the spherical oscillator and the Coulomb potential of a hydrogen-like ion, for a state of orbital quantum number $l \geq 0$, are recovered.

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1. Introduction

Noncentral generalizations of the Coulomb, the harmonic oscillator and radial barrier potentials in three spatial dimensions are very useful for describing nonspherically symmetric systems in quantum physics and chemistry. As examples, we can quote two axially symmetric potentials. The first example has been the ring-shaped potential proposed by Hartmann in 1972 and the second by Quesne in 1988, used as models for describing of the ring-shaped molecules [1–4]. The Hartmann and Quesne potentials have been solved by using various approaches such as the path integral technique [5–7] and the algebraic method [8–13]. These two models provide analytical observations which belong to a set of three-dimensional potentials investigated by Smorodinsky, Winternitz and coworkers [14] and was later revived in 1990 by Evans [15–17] who presented a list of these Smorodinsky-Winternitz potentials including the corresponding functionally independent integrals and all separating coordinate systems. Furthermore, the path integral method and the algebraic approach so(2,1) have also been used for the quantum mechanical treatment of these potentials [18, 19].

The harmonic oscillator, the Coulomb and the ring-shaped

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potentials are particular cases of the noncentral electric potential which can be given as follows:

$$V(r,\theta) = V(r) + \frac{1}{r^2}V(\theta), \tag{1}$$

where $V(r)$ is a radial potential and the $\theta-$dependent part is defined by

$$V(\theta) = \frac{\hbar^2}{2\mu}\frac{\gamma + \beta\sin^2\theta + \alpha\sin^4\theta}{\sin^2\theta\cos^2\theta}, \tag{2}$$

where α, β and γ are real constants. Without loss of generality, note that we can make the change of variable $\theta \to (\theta/2)$ in $V(\theta)$. The potential described in (1) can be added to the list of the Smorodinsky-Winternitz potentials allowing exact solutions of the corresponding quantum mechanical problem. Like the ring-shaped potentials, $V(r,\theta)$ plays an important role in all situations where axial symmetry is relevant. As an example; it is of interest for the study of the rotational-vibrational states of diatomic molecules.

Recently, this potential [20] and a spherically harmonic oscillatory ring-shaped potential [21] have been treated through the resolution of Schrödinger's equation within the framework of the Nikiforov-Uvarov method, but the coefficients A, B and L in Ref. [20] and $\tilde{a}$, $\tilde{b}$ and Λ in Ref. [21] have not been given correctly. When $V(\theta) = 0$, it is clear that their solutions are not valid for all states, i.e., for an orbital quantum number $l = 0$, since according to these works, the possible values of l are defined as $l = 1 + 2\nu + |m|$, $(\nu, |m| \in \mathbb{N})$ in Ref. [20] and $l = 1 + 2n_r + |m|$, $(n_r, |m| \in \mathbb{N})$ in Ref. [21]. More recently, a class of noncentral potentials has been discussed by using the Laplace transform approach [22]. We therefore think that it is worthwhile to give the rigorous solution to the compound potential described in (1) using the path integral approach.

The plan of the present study is as follows: in Section 2, we shall construct the Green's function for the potential (1) with a general radial dependence $V(r)$ by path integration. After separating and carrying out the integration over the variables ϕ_j, it is shown that the propagator with the (r,θ) part can be decomposed into two independent kernels, one associated with the radial variable and the other with the polar variable owing to the procedure of reparametrization of the paths. The angular part is brought back to the Pöschl-Teller potential propagator previously presented by many authors [23–25]. In Section 3, we shall consider the case where the radial potential is a harmonic oscillator. The energy spectrum as well as the wave functions of the bound states are calculated. In Section 4, the case where the radial dependence of (1) is an attractive Coulomb potential will be discussed. The energy spectrum and the normalized wave functions are also found. The ring-shaped oscillator and the Hartmann ring-shaped potential are studied as particular cases in Section 5. We will provide our conclusions in Section 6.

2. The Green's function for a general radial potential $V(r)$ plus the $V(\theta)-$potential

In spherical coordinates, the time-sliced representation of the Feynman propagator takes the form

$$\begin{aligned} K(\vec{r}'', \vec{r}'; T) \;=\; & \lim_{N\to\infty}\int \left(\frac{\mu}{2i\pi\hbar\varepsilon}\right)^{\frac{3}{2}N}\prod_{j=1}^{N-1} r_j^2 \sin\theta_j dr_j d\theta_j d\phi_j \\ & \times \exp\left(\frac{i}{\hbar}\sum_{j=1}^{N} S(j,j-1)\right), \end{aligned} \tag{3}$$

where

$$\begin{aligned} S(j,j-1) \;=\; & \frac{\mu}{2\varepsilon}[r_j^2 + r_{j-1}^2 - 2r_j r_{j-1}(\cos\theta_j\cos\theta_{j-1} \\ & + \sin\theta_j \sin\theta_{j-1}\cos(\triangle\phi_j)] \\ & -\varepsilon\left(V(\tilde{r}_j) + \frac{1}{\tilde{r}_j^2}V(\tilde{\theta}_j)\right) \end{aligned} \tag{4}$$

represents the action in the short time interval $[t_{j-1}, t_j]$, with the usual notations $\varepsilon = t_j - t_{j-1}$, $T = N\varepsilon = t'' - t'$, $\vec{r}'' = \vec{r}(t'')$, $\vec{r}' = \vec{r}(t')$, and for any variable u: $u_j = u(t_j)$, $\tilde{u}_j = (u_j + u_{j-1})/2$, $\triangle u_j = u_j - u_{j-1}$.

Our first task is to separate the $\phi-$dependent part in the path integral. To do this, we use the Fourier expansion (see formula (8.511.4) p. 973 in Ref. [26])

$$\exp[u\cos(\triangle\phi)] = \sum_{m=-\infty}^{+\infty} I_m(u)\exp[im(\triangle\phi)], \tag{5}$$

and the following asymptotic behaviour (see formula (8.451) p. 961 in Ref. [26]) of the modified Bessel functions, for small ε :

$$I_m\left(\frac{\nu}{\varepsilon}\right) \approx \left(\frac{\varepsilon}{2\pi\nu}\right)^{\frac{1}{2}}\exp\left[\frac{\nu}{\varepsilon} - \frac{\varepsilon}{2\nu}\left(m^2 - \frac{1}{4}\right)\right]. \tag{6}$$

After integrating over the angular variables ϕ_j, the propagator will then be decomposed into partial kernels:

$$K\left(\vec{r}'', \vec{r}'; T\right) = \sum_{m=-\infty}^{+\infty}\frac{\exp[im(\phi'' - \phi')]}{2\pi}K_m(r'', \theta'', r', \theta'; T), \tag{7}$$

where

$$K_m(r'',\theta'',r',\theta';T) = \frac{1}{\left[r''^2 r'^2 \sin\theta'' \sin\theta'\right]^{\frac{1}{2}}} \lim_{N\to\infty} \int \left(\frac{\mu}{2i\pi\hbar\varepsilon}\right)^N \prod_{j=1}^{N} \left(r_j r_{j-1}\right)^{\frac{1}{2}} \prod_{j=1}^{N-1} dr_j d\theta_j \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N} A(j,j-1)\right\}, \quad (8)$$

with

$$\begin{aligned} A(j,j-1) &= \frac{\mu}{2\varepsilon}\left[\left(\triangle r_j\right)^2 + 4r_j r_{j-1}\sin^2\left(\frac{\triangle\theta_j}{2}\right)\right] + \varepsilon\left(\frac{\hbar^2\alpha}{2\mu\widetilde{r}_j^2} - V(\widetilde{r}_j)\right) \\ &-\varepsilon\frac{\hbar^2}{8\mu\widetilde{r}_j^2}\left(\frac{4\gamma + m^2 - \frac{1}{4}}{\sin^2\left(\widetilde{\theta}_j/2\right)} + \frac{4(\alpha+\beta+\gamma) + m^2 - \frac{1}{4}}{\cos^2\left(\widetilde{\theta}_j/2\right)}\right). \end{aligned} \quad (9)$$

Let us symmetrize this expression (8) around the mid-point of the interval $\left[t_{j-1}, t_j\right]$ and let us expand the measure up to order 2 in $\triangle r_j$,

$$\prod_{j=1}^{N}\left(r_j r_{j-1}\right)^{\frac{1}{2}}\prod_{j=1}^{N-1} dr_j d\theta_j \approx \prod_{j=1}^{N}\widetilde{r}_j\left(1 - \frac{\left(\triangle r_j\right)^2}{8\widetilde{r}_j^2}\right)\prod_{j=1}^{N-1} dr_j d\theta_j, \quad (10)$$

together with the action $A(j,j-1)$ up to order 4 in $\triangle u_j$,

$$\begin{aligned} A(j,j-1) &\approx \frac{\mu}{2\varepsilon}\left[\left(\triangle r_j\right)^2 + \widetilde{r}_j^2\left(\triangle\theta_j\right)^2\right] - \frac{\mu}{8\varepsilon}\left(\left(\triangle r_j\right)^2\left(\triangle\theta_j\right)^2 + \widetilde{r}_j^2\frac{\left(\triangle\theta_j\right)^4}{3}\right) + \varepsilon\left(\frac{\hbar^2\alpha}{2\mu\widetilde{r}_j^2} - V(\widetilde{r}_j)\right) \\ &-\varepsilon\frac{\hbar^2}{8\mu\widetilde{r}_j^2}\left(\frac{4\gamma + m^2 - \frac{1}{4}}{\sin^2\left(\widetilde{\theta}_j/2\right)} + \frac{4(\alpha+\beta+\gamma) + m^2 - \frac{1}{4}}{\cos^2\left(\widetilde{\theta}_j/2\right)}\right). \end{aligned} \quad (11)$$

By means of the Mc Laughlin and Schulman procedure [27, 28], let us introduce a pure quantum correction, by making the following substitutions:

$$\left(\triangle r_j\right)^2 \to \frac{i\hbar\varepsilon}{\mu}; \left(\triangle r_j\right)^2\left(\triangle\theta_j\right)^2 \to \frac{1}{\widetilde{r}_j^2}\left(\frac{i\hbar\varepsilon}{\mu}\right)^2; \left(\triangle\theta_j\right)^4 \to \frac{3}{\widetilde{r}_j^4}\left(\frac{i\hbar\varepsilon}{\mu}\right)^2. \quad (12)$$

The partial kernel (8) then becomes

$$K_m(r'',\theta'',r',\theta';T) = \frac{1}{\left[r''^2 r'^2 \sin\theta'' \sin\theta'\right]^{\frac{1}{2}}} \lim_{N\to\infty} \int \left(\frac{\mu}{2i\pi\hbar\varepsilon}\right)^N \prod_{j=1}^{N}\widetilde{r}_j \prod_{j=1}^{N-1} dr_j d\theta_j \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\widetilde{A}(j,j-1)\right\}, \quad (13)$$

where

$$\begin{aligned} \widetilde{A}(j,j-1) &\approx \frac{\mu}{2\varepsilon}\left[\left(\triangle r_j\right)^2 + \widetilde{r}_j^2\left(\triangle\theta_j\right)^2\right] + \varepsilon\left[\frac{\hbar^2}{2\mu\widetilde{r}_j^2}\left(\alpha + \frac{1}{4}\right) - V(\widetilde{r}_j)\right] \\ &-\varepsilon\frac{\hbar^2}{8\mu\widetilde{r}_j^2}\left(\frac{4\gamma + m^2 - \frac{1}{4}}{\sin^2\left(\widetilde{\theta}_j/2\right)} + \frac{4(\alpha+\beta+\gamma) + m^2 - \frac{1}{4}}{\cos^2\left(\widetilde{\theta}_j/2\right)}\right). \end{aligned} \quad (14)$$

At this point, we notice that the radial and angular variables r_j and θ_j are mixed. Our immediate goal is to separate these. For this, let us introduce the energy E with the help of the Green's function (Fourier transform of the propagator):

$$G(\overrightarrow{r}'',\overrightarrow{r}';E) = \int_0^\infty dT \exp\left[\frac{i}{\hbar}ET\right] K(\overrightarrow{r}'',\overrightarrow{r}';T), \quad (15)$$

and let us apply the procedure of path reparametrization [29, 30], performing the time transformation $t \to s$ defined by

$$\frac{dt}{ds} = r^2(s), \tag{16}$$

or, in discrete form

$$\varepsilon = \sigma_j r_j r_{j-1} = \sigma_j \widetilde{r}_j^2 \left(1 - \frac{(\triangle r_j)^2}{4\widetilde{r}_j^2}\right); \qquad \sigma_j = s_j - s_{j-1}. \tag{17}$$

Taking into account the constraint

$$T = \int_0^S r^2(s)ds, \tag{18}$$

the Green's function (15) is then rewritten as

$$G(\overrightarrow{r}'', \overrightarrow{r}'; E) = \sum_{m-\infty}^{+\infty} \frac{\exp\left[im(\phi'' - \phi')\right]}{2\pi} \int_0^\infty dS P_E^m(r'', \theta'', r', \theta'; S), \tag{19}$$

where

$$\begin{aligned} P_E^m(r'', \theta'', r', \theta'; S) \;=\; & [\sin\theta'' \sin\theta']^{-\frac{1}{2}} \lim_{N\to\infty} \int \prod_{j=1}^{N} \left(\frac{\mu}{2i\pi\hbar\sigma_j}\right) \frac{1}{\widetilde{r}_j} \left(1 + \frac{(\triangle r_j)^2}{4\widetilde{r}_j^2}\right) \\ & \times \prod_{j=1}^{N-1} dr_j d\theta_j \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{\mu}{2\sigma_j}\left(1 + \frac{(\triangle r_j)^2}{4\widetilde{r}_j^2}\right)\left(\frac{(\triangle r_j)^2}{\widetilde{r}_j^2} + (\triangle\theta_j)^2\right)\right.\right. \\ & \left.\left. + \sigma_j\left[\frac{\hbar^2}{2\mu}\left(\alpha + \frac{1}{4}\right) - \widetilde{r}_j^2\left(V(\widetilde{r}_j) - E\right)\right] - \frac{\hbar^2\sigma_j}{8\mu}\left(\frac{4\gamma + m^2 - \frac{1}{4}}{\sin^2\left(\widetilde{\theta}_j/2\right)} + \frac{4\left(\alpha + \beta + \gamma\right) + m^2 - \frac{1}{4}}{\cos^2\left(\widetilde{\theta}_j/2\right)}\right)\right]\right\} \end{aligned} \tag{20}$$

is the new kernel.

Let us remove the terms of order 2 and 4 contained in the measure and in the action, again using the Mc Laughlin and Schulman procedure

$$(\triangle r_j)^2 \to \widetilde{r}_j^2\left(\frac{i\hbar\sigma_j}{\mu}\right); (\triangle r_j)^2 (\triangle\theta_j)^2 \to \widetilde{r}_j^2\left(\frac{i\hbar\sigma_j}{\mu}\right)^2; (\triangle\theta_j)^4 \to 3\widetilde{r}_j^4\left(\frac{i\hbar\sigma_j}{\mu}\right)^2. \tag{21}$$

This leads to a pure quantum correction $\left(-\frac{\hbar^2}{4\mu}\sigma_j\right)$. Consequently, the expression (20) admits a separation of the variables r_j and θ_j and decomposes into a product of radial and angular kernels

$$P_E^m(r'', \theta'', r', \theta'; S) = P_E^m(r'', r'; S) P_E^m(\theta'', \theta'; S), \tag{22}$$

with

$$P_E^m(r'', r'; S) \;=\; \lim_{N\to\infty} \int \prod_{j=1}^{N}\left(\frac{\mu}{2i\pi\hbar\sigma_j}\right)^{\frac{1}{2}} \frac{1}{\widetilde{r}_j} \prod_{j=1}^{N-1} dr_j \exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{\mu}{2\sigma_j}\frac{(\triangle r_j)^2}{\widetilde{r}_j^2} + \sigma_j\left[\widetilde{r}_j^2\left(E - V(\widetilde{r}_j)\right) + \frac{\hbar^2}{2\mu}\left(\alpha - \frac{1}{4}\right)\right]\right]\right\}, \tag{23}$$

and

$$\begin{aligned} P_E^m(\theta'',\theta';S) &= [\sin\theta''\sin\theta']^{-\frac{1}{2}} \lim_{N\to\infty}\int\prod_{j=1}^{N}\left(\frac{\mu}{2i\pi\hbar\sigma_j}\right)^{\frac{1}{2}}\prod_{j=1}^{N-1}d\theta_j \\ &\quad\times\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{\mu}{2\sigma_j}\left(\triangle\theta_j\right)^2-\sigma_j\frac{\hbar^2}{8\mu}\left(\frac{4\gamma+m^2-\frac{1}{4}}{\sin^2\left(\widetilde{\theta}_j/2\right)}+\frac{4\left(\alpha+\beta+\gamma\right)+m^2-\frac{1}{4}}{\cos^2\left(\widetilde{\theta}_j/2\right)}\right)\right]\right\} \\ &= \frac{1}{2}[\sin\theta''\sin\theta']^{-\frac{1}{2}}K^{(PT)}(\theta'',\theta';S), \end{aligned} \tag{24}$$

where $K^{(PT)}(\theta'',\theta';S)$ can be identified with the propagator for a particle of mass $M=4\mu$ placed in the Pöschl-Teller potential. The path integral has been evaluated by means of the path integration over the $SU(2)$ group manifold and its solution reads as [23–25]:

$$\begin{aligned} K^{(PT)}(\theta'',\theta';S) &= \lim_{N\to\infty}\int\prod_{j=1}^{N}\left(\frac{M}{2i\pi\hbar\sigma_j}\right)^{\frac{1}{2}}\prod_{j=1}^{N-1}d\left(\frac{\theta_j}{2}\right)\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N}\left[\frac{M}{2\sigma_j}\left(\frac{\triangle\theta_j}{2}\right)^2\right.\right. \\ &\quad\left.\left.-\sigma_j\frac{\hbar^2}{2M}\left(\frac{4\gamma+m^2-\frac{1}{4}}{\sin^2\left(\widetilde{\theta}_j/2\right)}+\frac{4\left(\alpha+\beta+\gamma\right)+m^2-\frac{1}{4}}{\cos^2\left(\widetilde{\theta}_j/2\right)}\right)\right]\right\} \\ &= \sum_{\nu=0}^{+\infty}\exp\left[-i\frac{\hbar}{2M}\left(2\nu+\kappa+\lambda+1\right)^2S\right]\Psi_\nu^{PT}\left(\theta''\right)\Psi_\nu^{PT*}\left(\theta'\right), \end{aligned} \tag{25}$$

where

$$\kappa=\sqrt{4\gamma+m^2},\qquad \lambda=\sqrt{4(\alpha+\beta+\gamma)+m^2}, \tag{26}$$

and the wave functions are given by

$$\Psi_\nu^{PT}(\theta) = \left[2\left(2\nu+\kappa+\lambda+1\right)\frac{\nu!\,\Gamma\left(\nu+\kappa+\lambda+1\right)}{\Gamma\left(\nu+\kappa+1\right)\Gamma\left(\nu+\lambda+1\right)}\right]^{\frac{1}{2}}\sin^{\kappa+\frac{1}{2}}\left(\frac{\theta}{2}\right)\cos^{\lambda+\frac{1}{2}}\left(\frac{\theta}{2}\right)P_\nu^{(\kappa,\lambda)}(\cos\theta). \tag{27}$$

The $P_\nu^{(\kappa,\lambda)}(\cos\theta)$ denote Jacobi polynomials.
Next we insert (27), (25), (24), (23) and (22) in (19) and we come back to the former time variable t using Eq. (16), that is to say, by means of the transformation $s\to t$ defined by $ds/dt=r^{-2}(t)$, taking into account the pure quantum correction calculated according to Mc Laughlin and Schulman's procedure. The Green's function (19) then becomes

$$G(\overrightarrow{r}'',\overrightarrow{r}';E)=\sum_{m=-\infty}^{+\infty}\sum_{\nu=0}^{+\infty}\frac{\exp\left[im\left(\phi''-\phi'\right)\right]}{2\pi}\Phi_\nu^{(\kappa,\lambda)}\left(\theta''\right)\Phi_\nu^{(\kappa,\lambda)*}\left(\theta'\right)G_{m,\nu}(r'',r';E), \tag{28}$$

where the wave functions $\Phi_m(\phi)$ and $\Phi_\nu^{(\kappa,\lambda)}(\theta)$, properly normalized, are

$$\Phi_m(\phi)=\frac{1}{\sqrt{2\pi}}e^{im\phi},\qquad m=0,\pm1,\pm2,..., \tag{29}$$

and

$$\Phi_\nu^{(\kappa,\lambda)}(\theta) = \left[\left(\nu+\frac{\kappa+\lambda+1}{2}\right)\frac{\nu!\,\Gamma\left(\nu+\kappa+\lambda+1\right)}{\Gamma\left(\nu+\kappa+1\right)\Gamma\left(\nu+\lambda+1\right)}\right]^{\frac{1}{2}}\sin^{\kappa}\left(\frac{\theta}{2}\right)\cos^{\lambda}\left(\frac{\theta}{2}\right)P_\nu^{(\kappa,\lambda)}(\cos\theta), \tag{30}$$

and

$$G_{m,\nu}(r'',r';E)=\frac{1}{r''r'}\int_0^{\infty}dT\ \exp\left[\frac{iET}{\hbar}\right]\int Dr(t)\exp\left\{\frac{i}{\hbar}\int_0^T L(r,\dot{r})dt\right\} \tag{31}$$

is the radial Green's function with the effective radial Lagrangian given by

$$L(r,\dot{r})=\frac{\mu}{2}\dot{r}^2-\frac{\hbar^2}{2\mu}\frac{L}{r^2}-V(r). \tag{32}$$

Here the constant L is defined by

$$L=\frac{1}{4}\left(2\nu+\sqrt{4(\alpha+\beta+\gamma)+m^2}+\sqrt{4\gamma+m^2}+1\right)^2-\alpha-\frac{1}{4}. \tag{33}$$

Expression (31) is to be evaluated once $V(r)$ is specified. Note that the normalized wave functions given by Eq. (30) and the constants (26) and (33) are different from the ones of Ref. [20].

3. The harmonic oscillator plus the $V(\theta)-$potential

When $V(r)$ is taken to be the spherical harmonic oscillator potential

$$V(r)=\frac{1}{2}\mu\omega^2r^2, \tag{34}$$

the radial Green's function (31) can be seen to take the form of the radial Green's function for the spherical harmonic oscillator supplemented by a centrifugal barrier which has been calculated [31] with the result

$$G_{m,\nu}(r'',r';E)\ =\ \frac{\mu\omega}{i\hbar\sqrt{r''r'}}\int_0^{\infty}dT\ \frac{\exp\left[\frac{iET}{\hbar}\right]}{\sin(\omega T)}I_{\sqrt{L+\frac{1}{4}}}\left(\frac{\mu\omega r''r'}{i\hbar\sin(\omega T)}\right)\exp\left[\frac{i\mu\omega}{2\hbar}\left(r''^2+r'^2\right)\cot(\omega T)\right]. \tag{35}$$

Due to the formula [26],

$$\int_0^{\infty}dq\ \frac{\exp(-2pq)}{\sinh q}\exp\left[-\frac{1}{2}(x+y)\coth q\right]I_{2\gamma}\left(\frac{\sqrt{xy}}{\sinh q}\right)=\frac{\Gamma\left(p+\gamma+\frac{1}{2}\right)}{\sqrt{xy}\,\Gamma(2\gamma+1)}M_{-p,\gamma}(x)\,W_{-p,\gamma}(y)\,, \tag{36}$$

which is valid for Re $\left(p+\gamma+\frac{1}{2}\right)>0,\mathrm{Re}(\gamma)>0$ and $y>x$, the radial Green's function for $r''>r'$, takes the closed form

$$G_{m,\nu}(r'',r';E)\ =\ \frac{1}{i\omega\left(r''r'\right)^{\frac{3}{2}}}\frac{\Gamma\left(p+\frac{1}{2}\sqrt{L+\frac{1}{4}}+\frac{1}{2}\right)}{\Gamma\left(\sqrt{L+\frac{1}{4}}+1\right)}M_{-p,\frac{1}{2}\sqrt{L+\frac{1}{4}}}\left(\frac{\mu\omega}{\hbar}r'^2\right)W_{-p,\frac{1}{2}\sqrt{L+\frac{1}{4}}}\left(\frac{\mu\omega}{\hbar}r''^2\right), \tag{37}$$

where $M_{-p,\frac{1}{2}\sqrt{L+\frac{1}{4}}}\left(\frac{\mu\omega}{\hbar}r'^2\right)$ and $W_{-p,\frac{1}{2}\sqrt{L+\frac{1}{4}}}\left(\frac{\mu\omega}{\hbar}r''^2\right)$ are the standard Whittaker functions and $p=-E/2\hbar\omega$.
The energy spectrum for bound states can be deduced from the poles of the radial Green's function which occur when the argument of the Euler function $\Gamma\left(p+\frac{1}{2}\sqrt{L+\frac{1}{4}}+\frac{1}{2}\right)$ is a negative integer or equal to zero, that is to say, when $p+\frac{1}{2}\sqrt{L+\frac{1}{4}}+\frac{1}{2}=-n_r$, for $n_r=0,1,2,\ldots$. The discrete energy spectrum is then given by

$$E_{n_r,\nu,m}=\hbar\omega\left(2n_r+\sqrt{L+\frac{1}{4}}+1\right). \tag{38}$$

At the poles, the Whittaker functions can be expressed in terms of the confluent hypergeometric functions as (see formulas (9.220.2) and (9.220.4), p. 1059 in Ref. [26]):

$$M_{-p,\frac{1}{2}\sqrt{L+\frac{1}{4}}}\left(\frac{\mu\omega}{\hbar}r'^2\right) = e^{-\frac{\mu\omega}{2\hbar}r'^2}\left(\frac{\mu\omega}{\hbar}r'^2\right)^{\frac{1+\sqrt{L+\frac{1}{4}}}{2}} {}_1F_1\left(-n_r, 1+\sqrt{L+\frac{1}{4}}; \frac{\mu\omega}{\hbar}r'^2\right), \tag{39}$$

$$W_{-p,\frac{1}{2}\sqrt{L+\frac{1}{4}}}\left(\frac{\mu\omega}{\hbar}r''^2\right) = (-1)^{n_r}\frac{\Gamma\left(n_r+\sqrt{L+\frac{1}{4}}+1\right)}{\Gamma\left(\sqrt{L+\frac{1}{4}}+1\right)}M_{-p,\frac{1}{2}\sqrt{L+\frac{1}{4}}}\left(\frac{\mu\omega}{\hbar}r''^2\right). \tag{40}$$

Hence, we have near the poles,

$$G_{n_r,\nu,m}(r'',r';E) \sim \frac{i\hbar}{E-E_{n_r,\nu,m}}R_{n_r,\nu,m}(r'')R_{n_r,\nu,m}(r'), \tag{41}$$

with the normalized radial wave functions

$$\begin{aligned} R_{n_r,\nu,m}(r) &= \left(\frac{\mu\omega}{\hbar}\right)^{\frac{3}{4}}\left[\frac{2\Gamma\left(n_r+\sqrt{L+\frac{1}{4}}+1\right)}{n_r!}\right]^{\frac{1}{2}}\frac{1}{\Gamma\left(\sqrt{L+\frac{1}{4}}+1\right)} \\ &\quad\times\exp\left(-\frac{\mu\omega}{2\hbar}r^2\right)\left(\frac{\mu\omega}{\hbar}r^2\right)^{-\frac{1}{4}+\frac{1}{2}\sqrt{L+\frac{1}{4}}} {}_1F_1\left(-n_r, 1+\sqrt{L+\frac{1}{4}}; \frac{\mu\omega}{\hbar}r^2\right). \end{aligned} \tag{42}$$

4. The Coulomb potential plus the $V(\theta)-$potential

When $V(r)$ is the attractive Coulomb potential of the hydrogen-like ion defined by

$$V(r) = -\frac{Ze^2}{r}, \tag{43}$$

the radial Green's function (31) takes the form

$$G_{m,\nu}(r'',r';E) = \frac{1}{r''r'}\int_0^\infty dT\ \exp\left[\frac{iET}{\hbar}\right]\int Dr(t)\exp\left\{\frac{i}{\hbar}\int_0^T\left(\frac{\mu}{2}\dot{r}^2-\frac{\hbar^2}{2\mu}\frac{L}{r^2}+\frac{Ze^2}{r}\right)dt\right\}. \tag{44}$$

By using the earlier results for the path integration [25] and the final integration over T, we obtain

$$G_{m,\nu}(r'',r';E) = \frac{2}{i\omega r''r'}\frac{\Gamma\left(p+\sqrt{L+\frac{1}{4}}+\frac{1}{2}\right)}{\Gamma\left(2\sqrt{L+\frac{1}{4}}+1\right)}M_{-p,\sqrt{L+\frac{1}{4}}}\left(\frac{\mu\omega}{\hbar}r'\right)W_{-p,\sqrt{L+\frac{1}{4}}}\left(\frac{\mu\omega}{\hbar}r''\right), \tag{45}$$

where $p=-\frac{2Ze^2}{\hbar\omega}$, $\omega=2\sqrt{-\frac{2E}{\mu}}$ and $r''>r'$.
In order to determine the discrete energy spectrum and the radial wave functions, properly normalized, we proceed similarly as in the preceding section and obtain

$$E_{n_r,\nu,m} = -\frac{\mu Z^2e^4}{2\hbar^2\left[n_r+\frac{1}{2}\sqrt{\left(2\nu+\sqrt{4(\alpha+\beta+\gamma)+m^2}+\sqrt{4\gamma+m^2}+1\right)^2-4\alpha+\frac{1}{2}}\right]^2}, \tag{46}$$

$$
\begin{aligned}
R_{n_r,\nu,m}(r) \;=\; & \frac{2}{a\left(n_r+\sqrt{L+\frac{1}{4}}+\frac{1}{2}\right)^2\Gamma\left(2\sqrt{L+\frac{1}{4}}+1\right)}\left[\frac{\Gamma\left(n_r+2\sqrt{L+\frac{1}{4}}+1\right)}{a n_r!}\right]^{\frac{1}{2}} \\
& \times\exp\left(-\frac{r}{a\left(n_r+\sqrt{L+\frac{1}{4}}+\frac{1}{2}\right)}\right)\left(\frac{2r}{a\left(n_r+\sqrt{L+\frac{1}{4}}+\frac{1}{2}\right)}\right)^{-\frac{1}{2}+\sqrt{L+\frac{1}{4}}} \\
& \times {}_1F_1\left(-n_r,2\sqrt{L+\frac{1}{4}}+1;\frac{2r}{a\left(n_r+\sqrt{L+\frac{1}{4}}+\frac{1}{2}\right)}\right), \qquad (47)
\end{aligned}
$$

where $a=\frac{\hbar^2}{\mu Z e^2}$ is Bohr's radius.

5. Special cases

As a check of the correctness of the results given in above with those available in the literature, we can consider three cases.

5.1. First case. The ring-shaped oscillator

By setting $\alpha=\beta=0$, $\gamma\neq 0$ and taking $V(r)=\frac{1}{2}\mu\omega^2r^2$ in the definition (1), we obtain the following potential:

$$
V(r,\theta)=\frac{1}{2}\mu\omega^2r^2+\frac{2\hbar^2}{\mu}\frac{\gamma}{r^2\sin^2\theta},\quad 0<\theta<\pi. \qquad (48)
$$

This potential has been treated, within the circular cylinder coordinates, through the resolution of Schrödinger's equation [4], and via the path integral approach [32].
The parameters (26) and (33) can thus written

$$
\begin{cases}
\kappa=\lambda=\sqrt{4\gamma+m^2}=\eta, \\
L=J(J+1),
\end{cases} \qquad (49)
$$

with $J=\nu+\eta$.
The Green's function associated to the ring-shaped oscillator can be deduced from expressions (37) and (28),

$$
\begin{aligned}
G(\vec{r}'',\vec{r}';E) \;=\; & \frac{1}{2\pi i\omega\left(r''r'\right)^{\frac{3}{2}}}\sum_{m=-\infty}^{+\infty}\sum_{\nu=0}^{+\infty}\left(J+\frac{1}{2}\right)\frac{\Gamma(J+\eta+1)\Gamma\left(p+\frac{J}{2}+\frac{3}{4}\right)}{\Gamma\left(J-\eta+1\right)\Gamma\left(J+\frac{3}{2}\right)} \\
& \times e^{im\left(\phi''-\phi'\right)}P_J^{-\eta}(\cos\theta'')P_J^{-\eta}(\cos\theta')W_{-p,\frac{1}{2}\left(J+\frac{1}{2}\right)}\left(\frac{\mu\omega}{\hbar}r''^2\right)M_{-p,\frac{1}{2}\left(J+\frac{1}{2}\right)}\left(\frac{\mu\omega}{\hbar}r'^2\right), \qquad (50)
\end{aligned}
$$

with $p=-E/2\hbar\omega$.
The discrete energy spectrum can be deduced from expression (38),

$$
E_{n_r,\nu,m}=\hbar\omega\left(2n_r+J+\frac{3}{2}\right). \qquad (51)
$$

The normalized wave functions are given by (29) for the azimuthal part, and can be deduced from expressions (30) and (42) for the polar and radial parts,

$$
\Phi_J^{(\eta,\eta)}(\theta)=\left[\left(J+\frac{1}{2}\right)\frac{\Gamma(J+\eta+1)}{\Gamma(J-\eta+1)}\right]^{\frac{1}{2}}P_J^{-\eta}(\cos\theta), \qquad (52)
$$

and

$$R_{n_r,\nu,m}(r) = \left(\frac{\mu\omega}{\hbar}\right)^{\frac{3}{4}}\left[\frac{2\Gamma\left(n_r+J+\frac{3}{2}\right)}{n_r!}\right]^{\frac{1}{2}}\frac{1}{\Gamma\left(J+\frac{3}{2}\right)}\left(\frac{\mu\omega}{\hbar}r^2\right)^{\frac{J}{2}}\exp\left(-\frac{\mu\omega}{2\hbar}r^2\right)\,{}_1F_1\left(-n_r,J+\frac{3}{2};\frac{\mu\omega}{\hbar}r^2\right). \quad (53)$$

5.2. Second case. The Hartmann ring-shaped potential

For $\alpha=\beta=0$, $\gamma\neq 0$ and $V(r)=-\frac{Ze^2}{r}$, the potential (1) is reduced to the form

$$V(r,\theta)=-\frac{Ze^2}{r}+\frac{2\hbar^2}{\mu}\frac{\gamma}{r^2\sin^2\theta},\quad 0<\theta<\pi. \quad (54)$$

This potential has been proposed by Hartmann as a model for the benzene molecule [1–3].
The Green's function relative to this potential can be deduced from expressions (45) and (28),

$$\begin{aligned} G(\vec{r}'',\vec{r}';E) &= \frac{1}{2\pi i\omega r''r'}\sum_{m=-\infty}^{+\infty}\sum_{\nu=0}^{+\infty}(2J+1)\frac{\Gamma(J+\eta+1)\Gamma\left(p+J+\frac{1}{2}\right)}{\Gamma(J-\eta+1)\Gamma(2J+2)}e^{im(\phi''-\phi')} \\ &\times P_J^{-\eta}(\cos\theta'')P_J^{-\eta}(\cos\theta')W_{-p,J+\frac{1}{2}}\left(\frac{\mu\omega}{\hbar}r''\right)M_{-p,J+\frac{1}{2}}\left(\frac{\mu\omega}{\hbar}r'\right), \end{aligned} \quad (55)$$

where $p=-2Ze^2/\hbar\omega$, and $\omega=2\sqrt{-2E/\mu}$.
The energy spectrum and the radial wave functions can be obtained from expressions (46) and (47),

$$E_{n_r,\nu,m}=-\frac{\mu Z^2e^4}{2\hbar^2\left(n_r+J+1\right)^2}, \quad (56)$$

$$\begin{aligned} R_{n_r,\nu,m}(r) &= \left[\frac{a}{2}\frac{\Gamma(n_r+2J+2)}{n_r!}\right]^{\frac{1}{2}}\frac{1}{\Gamma(2J+2)}\left(\frac{2r}{a\,(n_r+J+1)}\right)^{J}\exp\left(-\frac{r}{a\,(n_r+J+1)}\right) \\ &{}_1F_1\left(-n_r,2J+1;\frac{2r}{a\,(n_r+J+1)}\right). \end{aligned} \quad (57)$$

The wave functions for the azimuthal and polar parts are identical to Eqs. (29) and (52).

5.3. Third case. A central potential

If one gets rid of the $V(\theta)$−potential by setting $\alpha=\beta=\gamma=0$, the potential (1) reduces to a central potential $V(r)$. The κ,λ and L parameters defined by expressions (26) and (33) can be written

$$\kappa=\lambda=\eta=|m|,\quad L=l(l+1), \quad (58)$$

where $l=\nu+|m|$ stands for the orbital quantum number which takes the values $l=0,1,2,\ldots$, depending on the combinations of ν and $|m|$.
In the case where $V(r)$ is the spherical harmonic oscillator, the energy spectrum (38) goes over to the standard formula,

$$E_{n_r,l}=\hbar\omega\left(2n_r+l+\frac{3}{2}\right). \quad (59)$$

The complete well known wave functions can be deduced from equations (29), (30) and (42),

$$\Psi_{n_r,l,m}(r,\theta,\phi) = \frac{1}{\sqrt{2\pi}}\left(\frac{\mu\omega}{\hbar}\right)^{\frac{3}{4}}\left[(l+1)\frac{\Gamma(l+|m|+1)\Gamma\left(n_r+l+\frac{3}{2}\right)}{n_r!\Gamma(l-|m|+1)}\right]^{\frac{1}{2}}\frac{1}{\Gamma\left(l+\frac{3}{2}\right)}\left(\frac{\mu\omega}{\hbar}r^2\right)^{\frac{l}{2}} \times \exp\left(-\frac{\mu\omega}{\hbar}r^2\right)\ {}_1F_1\left(-n_r,l+\frac{3}{2};\frac{\mu\omega}{\hbar}r^2\right)P_l^{-|m|}(\cos\theta)e^{im\phi}. \tag{60}$$

When $V(r)$ is the Coulomb potential of hydrogen-like ion, the energy spectrum and the normalized wave functions of the bound states can be deduced from equations (46), (29), (30) and (47),

$$E_{n_r,l} = -\frac{\mu Z^2 e^4}{2\hbar^2\left(n_r+l+1\right)^2}, \tag{61}$$

$$\Psi_{n_r,l,m}(r,\theta,\phi) = \frac{1}{a\left(n_r+l+1\right)^2}\left[\frac{2l+1)}{\pi a}\frac{\Gamma(l+|m|+1)\Gamma\left(n_r+2l+2\right)}{n_r!\Gamma(l-|m|+1)}\right]^{\frac{1}{2}}\frac{1}{\Gamma\left(2l+2\right)}\left(\frac{2r}{a\left(n_r+l+1\right)}\right)^{l} \times \exp\left(-\frac{r}{a\left(n_r+l+1\right)}\right)\ {}_1F_1\left(-n_r,2l+2;\frac{2r}{a\left(n_r+l+1\right)}\right)P_l^{-|m|}(\cos\theta)e^{im\phi}. \tag{62}$$

6. Conclusion

As we have shown in this paper, the exact solution of a general radial potential $V(r)$ plus the $V(\theta)-$potential turns out to be remarkably simple when using the Feynman path integral approach. The $\theta-$dependent part of the Green's function is brought back to the Pöschl-Teller potential Green's function using the procedure of reparametrization of the paths. Explicit expressions for the $\theta-$dependent wave functions are obtained in terms of Jacobi polynomials. Formula (28) with the parameters κ, λ and L defined by Eqs. (26) and (33) is the main result of the present paper. The energy eigenvalues and the normalized wave functions are obtained for the spherical harmonic oscillator plus the $V(\theta)-$potential and the Coulomb potential plus the $V(\theta)-$ potential. The ring-shaped oscillator, the Hartmann ring-shaped potential, the spherical harmonic oscillator and the Coulomb potential are considered as particular cases. They constitute a proof of the correctness of our results.

It can be seen that if one substitutes $V(r) = \frac{1}{2}M\omega^2r^2$, $\gamma = \eta$, $\beta = A$, $\alpha = B$, $L = \Lambda$, and $\theta \rightarrow \theta + \frac{\pi}{2}$, one obtains the spherically harmonic oscillatory ring-shaped potential solved by Dong and co-workers [21]. Our results in formulas (46) and (38) hold and it is suffice to replace $\sin\theta$ by $\cos\theta$ in the wave functions (30). This leads us to note that the results given in Ref. [21] are also indisputably unsatisfactory.

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