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An eighth-order KdV-type equation in (1+1) and (2+1) dimensions: multiple soliton solutions

Rapid Communication

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Abstract: In this work we study an eighth-order KdV-type equations in (1+1) and (2+1) dimensions. The new equations are derived from the KdV6 hierarchy. We show that these equations give multiple soliton solutions the same as the multiple soliton solutions of the KdV6 hierarchy except for the dispersion relations.

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1. Introduction

A growing interest has been given to the propagation of nonlinear waves in nonlinear dynamical fields. These waves appear in a great array of contexts such as, hydrodynamics, nonlinear optical fibers, fluid dynamics, plasma physics, chemical physics and in engineering applications as well. The existence of soliton solutions implies perfect balance between nonlinearity and dispersion effects which usually requires specific conditions and cannot be established in general [1–12].

The nonlinear equations are classified as integrable and non-integrable equations. Fokas [2] defines a partial differential equation as completely integrable if and only if it possesses infinitely many generalized symmetries. The existence of a sufficiently large number of conservation laws or symmetries guarantees complete integrability for this equation. The study of integrable equations, that possess sufficiently large number of conservation laws or symmetries, and hence give rise to multiple soliton solutions [1–10], is very important in solitary waves theory. Many systematic methods are used for studying the integrability of nonlinear partial differential equations. For any nonlinear equation, the existence of multiple soliton solutions often indicates the integrability of that equation. However, the integrability should be justified by using other methods such as the Lax pairs or the Painlevé property. Many nonlinear equations, both continuous and discrete, give soliton solutions that decay exponentially at spatial infinity.

Several analytic and numerical methods have been developed to investigate nonlinear evolution equations. Some of these efficient methods are the generalized symmetry method, Cole-Hopf transformation method, Darboux transformation method, Jacobi elliptic method, the tanh-coth method, pseudo spectral method, Painlevé analysis, the

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inverse scattering method, the Bäcklund transformation method, the conservation law method, the Hirota bilinear method [6–20], and the simplified Hirota's method [1]. The inverse scattering method and the Hirota's bilinear method are the pioneer dominant methods which are commonly used methods for the determination of multiple soliton solutions. However, the Hirota's bilinear method and the simplified Hirota's method are rather heuristic and present useful features that make it ideal for the determination of multiple soliton solutions.

The KdV equation is the pioneer model, that gives rise to solitons given by

$$u_t + uu_x + u_{3x} = 0, \tag{1}$$

with recursion operator R given by [1–4]

$$R = D^2 + \frac{2}{3}u + \frac{1}{3}u_x D^{-1}, \tag{2}$$

where D denotes the total derivative with respect to x, and D^{-1} is its integration operator.

The modified KdV (mKdV) equation reads

$$u_t + 6u^2 u_x + u_{3x} = 0, \tag{3}$$

with the following recursion operator

$$R = D^2 + 4u^2 + 4u_x D^{-1}(\cdot u). \tag{4}$$

The last term in (4) is the operator which takes a polynomial $P \in R\{u\}$, multiplies it by u, then applies the D^{-1}, and finally multiplies the result by $4u_x$ [3].

The KdV6 equation [4] is given by

$$\begin{aligned} &u_{xxxt} + u_{6x} + 20u_{4x}u_x + 40u_{3x}u_{2x} + 120u_{2x}u_x^2 \\ &\quad + 4u_{2x}u_t + 8u_x u_{xt} = 0. \end{aligned} \tag{5}$$

However, other forms of the KdV-type equations are studied in the literature.

In this work, our main focus will be on a nonlinear eighth-order KdV-type equation in (1+1), developed from the KdV6 hierarchy [5] that reads

$$\begin{aligned} &8u_{xxxt} + u_{8x} + 28u_{6x}u_x + 84u_{5x}u_{2x} + 140u_{4x}u_{3x} \\ &\quad + 280u_{4x}u_x^2 + 1120u_{3x}u_{2x}u_x + 280u_{2x}^3 \\ &\quad + 1120u_{2x}u_x^3 + 32u_{2x}u_t + 64u_x u_{xt} = 0. \end{aligned} \tag{6}$$

Moreover, we will examine another nonlinear eighth-order KdV-type equation in (2+1) of the form

$$\begin{aligned} &8u_{xxxt} + u_{8x} + 28u_{6x}u_x + 84u_{5x}u_{2x} + 140u_{4x}u_{3x} \\ &\quad + 280u_{4x}u_x^2 + 1120u_{3x}u_{2x}u_x + 280u_{2x}^3 \\ &\quad + 1120u_{2x}u_x^3 + 32u_{2x}u_t + 64u_x u_{xt} \\ &\quad + (u_x + u_y)_x = 0, \end{aligned} \tag{7}$$

where the additional terms $(u_x + u_y)_x$ are added to (6).

Eq. (6) was derived in [5], by considering the spectral problem

$$\psi_{xx} = 2(\lambda - u_x)\psi, \tag{8}$$

where $u(x,t)$ is a potential and λ is a constant spectral parameter. By using the Lenard gradients, it was shown that

$$\left(\frac{1}{2}\partial_x^3 + 2\partial_x u_x + 2u_x\partial_x\right)\left(-4u_x - \frac{1}{2}u_t\right) = 0, \tag{9}$$

which gives (6).

The aim of this work is to determine multiple soliton solutions for (6) and (7). We will show that these two equations possess multiple soliton solutions the same as the multiple soliton solutions of the KdV6 hierarchy, but differ only in the dispersion relations.

2. The eighth-order KdV-type equation in (1+1) dimensions

In this section, we will determine multiple soliton solutions for the eighth-order KdV-type equation in (1+1) dimensions. The goal will be achieved by using the simplified Hirota's method [1, 12–20].

Substituting

$$u(x,t) = e^{\theta_i}, \theta_i = k_i x - c_i t, \tag{10}$$

into the linear terms of (6) gives the dispersion relation by

$$c_i = \frac{1}{8}k_i^5, \tag{11}$$

and as a result we set the dispersion variables

$$\theta_i = k_i x - \frac{1}{8}k_i^5 t, \; i = 1, 2, 3. \tag{12}$$

We next use the transformation

$$u(x,t) = (\ln(f(x,t)))_x, \tag{13}$$

where the auxiliary functions $f(x,t)$ for the single soliton solution is given by

$$f(x,t) = 1 + e^{\theta_1} = 1 + e^{k_1 x - \frac{1}{8}k_1^5 t}. \tag{14}$$

Combining (13)–(14) gives the single soliton solution

$$u(x,t) = \frac{k_1\, e^{k_1 x - \frac{1}{8}k_1^5 t}}{1 + e^{k_1 x - \frac{1}{8}k_1^5 t}}. \tag{15}$$

To determine the two-soliton solutions, the auxiliary function reads

$$\begin{aligned} f(x,t) = {} & 1 + e^{k_1 x - \frac{1}{8}k_1^5 t} + e^{k_2 x - \frac{1}{8}k_2^5 t} \\ & + a_{12} e^{(k_1+k_2)x - \frac{1}{8}(k_1^5+k_2^5)\,t}, \end{aligned} \tag{16}$$

where a_{12} is the phase shift. Substituting (16) into (13) and using the obtained result in (6), one obtains the phase shift a_{12} by

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{17}$$

and this can be generalized to

$$a_{ij} = \frac{(k_i-k_j)^2}{(k_i+k_j)^2}, 1 \le i < j \le 3. \tag{18}$$

The two-soliton solutions are obtained by substituting (17) and (16) into (13). It is interesting to point out that equation (6) does not show any resonant phenomenon because the phase shift term a_{12} in (17) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.
To determine the three soliton solutions, we set the auxiliary function by

$$\begin{aligned} f(x,t) = {} & 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12} e^{\theta_1+\theta_2} \\ & + a_{13} e^{\theta_1+\theta_3} + a_{23} e^{\theta_2+\theta_3} + b_{123} e^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{19}$$

where the phase shifts a_{ij} are derived above in (18). Substituting (19) into (6) gives

$$b_{123} = a_{12}a_{13}a_{23}. \tag{20}$$

This shows that the three soliton solutions are obtainable, and can be obtained by substituting (19) into (13). It is obvious that N-soliton solutions can be obtained for finite N, where $N \ge 1$.

3. The eighth-order KdV-type equation in (2+1) dimensions

In this section, we will follow the analysis presented earlier to determine multiple soliton solutions for the eighth-order KdV-type equation in (2+1) dimensions.
Substituting

$$u(x,t) = e^{\theta_i}, \theta_i = k_i x - c_i t, \tag{21}$$

into the linear terms of (7) gives the dispersion relation by

$$c_i = \frac{1}{8}k_i^5 + \frac{k_i + r_i}{8k_i^2}, \tag{22}$$

and as a result we set the dispersion variables

$$\theta_i = k_i x - \left(\frac{1}{8}k_i^5 + \frac{k_i + r_i}{8k_i^2}\right) t,\ i = 1,2,3. \tag{23}$$

We next use the transformation

$$u(x,t) = (\ln(f(x,t))_x, \tag{24}$$

where the auxiliary functions $f(x,t)$ for the single soliton solution is given by

$$f(x,t) = 1 + e^{\theta_1} = 1 + e^{k_1 x - \left(\frac{1}{8}k_1^5 + \frac{k_1+r_1}{8k_1^2}\right)t}. \tag{25}$$

Combining (24)–(25) gives the single soliton solution

$$u(x,t) = \frac{k_1\, e^{k_1 x - \left(\frac{1}{8}k_1^5 + \frac{k_1+r_1}{8k_1^2}\right)t}}{1 + e^{k_1 x - \left(\frac{1}{8}k_1^5 + \frac{k_1+r_1}{8k_1^2}\right)t}}. \tag{26}$$

To determine the two-soliton solutions, the auxiliary function reads

$$\begin{aligned} f(x,t) = {} & 1 + e^{k_1 x - \left(\frac{1}{8}k_1^5 + \frac{k_1+r_1}{8k_1^2}\right)t} + e^{k_2 x - \left(\frac{1}{8}k_2^5 + \frac{k_2+r_2}{8k_2^2}\right)t} \\ & + a_{12} e^{(k_1+k_2)x - \left(\frac{1}{8}(k_1^5+k_2^5) + \frac{k_1+r_1}{8k_1^2} + \frac{k_2+r_2}{8k_2^2}\right)t}, \end{aligned} \tag{27}$$

where a_{12} is the phase shift. Substituting (27) into (24) and using the obtained result in (7), one obtains the phase shift a_{12} by

$$a_{12} = \frac{(k_1-k_2)^2}{(k_1+k_2)^2}, \tag{28}$$

and this can be generalized to

$$a_{ij} = \frac{\left(k_i - k_j\right)^2}{\left(k_i + k_j\right)^2}, 1 \leq i < j \leq 3. \tag{29}$$

The two-soliton solutions are obtained by substituting (28) and (27) into (24). It is interesting to point out that the equation (7) does not show any resonant phenomenon because the phase shift term a_{12} in (28) cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

To determine the three soliton solutions, we set the auxiliary function by

$$\begin{aligned} f(x,t) = {} & 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} \\ & + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + b_{123}e^{\theta_1+\theta_2+\theta_3}. \end{aligned} \tag{30}$$

where the phase shifts a_{ij} are derived above in (29). Substituting (30) into (7) gives

$$b_{123} = a_{12}a_{13}a_{23}. \tag{31}$$

This shows that the three soliton solutions are obtainable, and can be obtained by substituting (30) into (24). It is obvious that N-soliton solutions can be obtained for finite N, where $N \geq 1$.

4. Discussion

An eighth-order KdV-type equation in (1+1) and (2+1) dimensions was studied. We have shown that these equations possess multiple soliton solutions the same as the KdV6 hierarchy equation except for the dispersion relations. The dispersion relations were found to depend on the coefficients of the spatial variables. The phase shifts depend only on the coefficients of the spatial variable x for both cases. The two forms of this equation do not show any resonant phenomenon because the phase shift term a_{12} for both equations cannot be 0 or ∞ for $|k_1| \neq |k_2|$.

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