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Noether's theorem for fractional variational problems of variable order

Research Article

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Abstract: We prove a necessary optimality condition of Euler–Lagrange type for fractional variational problems with derivatives of incommensurate variable order. This allows us to state a version of Noether's theorem without transformation of the independent (time) variable. Considered derivatives of variable order are defined in the sense of Caputo.

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1. Introduction

Fractional calculus is the scientific discipline that deals with integrals and derivatives of arbitrary real (or complex) order. Since the XVIIth century, when fractional integration and differentiation was brought up for the first time, many well-known mathematicians contributed to the theory, among them Euler, Laplace, Fourier, Abel, Liouville and Riemann. For a comprehensive knowledge of fractional calculus we refer the reader to the books [1–4].

The XXth century has shown that fractional order calculus is more adequate to describe real world problems than integer/standard calculus [5, 6]. Therefore, not only mathematicians currently have a strong interest in fractional calculus but also researchers in applied fields such as mechanics, physics, chemistry, biology, economics, control theory and signal processing [7–9].

A generalization of fractional calculus was proposed in 1993 by Samko and Ross [10]. They considered integrals and derivatives of order α, where α is not a constant but a function. Afterwards, several works were dedicated to these operators [11, 12]. Interesting applications of the proposed calculus are found in the theory of viscous flows and mechanics [13–17].

In 1996, Riewe initiated the theory of the fractional calculus of variations by considering problems of mechanics with fractional order derivatives [18, 19]. Nowadays, the fractional calculus of variations is the focus of much re-

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search (see, e.g., [20–29] and references therein). For a current overview on the subject we refer to the recent book [30]. Here we remark that results for problems of calculus of variations with variable order fractional operators are scarce, reducing to those found in [31–33]. In particular, no Noether's symmetry theorem of variable order has been proved yet.

The main aim of the current work is to generalize Noether's theorem. The theorem was originally proved by the German mathematician Emmy Noether in 1918, asserting that invariance properties of integral functionals lead to corresponding conservation laws [34–36]. In contrast with previous works [26, 37–39], where fractional versions of Noether's theorem for a constant non-integer order α are obtained, here we consider more general fractional variational problems with variable order derivatives.

The paper is organized as follows. In Section 2, basic definitions and properties of variable order fractional operators are given. Sections 3, 4 and 5 contain our main results: we prove integration by parts formulas for variable order fractional integrals (Theorem 5) and derivatives (Theorem 6), a necessary optimality condition of the Euler–Lagrange type for fractional variational problems with derivatives of incommensurate variable order (Theorem 10), and a version of Noether's theorem for such fractional variable order problems (Theorem 14). Then, in Section 6, we illustrate our results through an example. We finish with Section 7 of conclusion.

2. Variable order fractional operators

In this section we recall the basic definitions necessary in the sequel, and derive a new interesting mapping property for the fractional integral operators of variable order (Theorem 2). We set $\Delta := \{(t,\tau) \in \mathbb{R}^2 : a \leq \tau < t \leq b\}$.

Definition 1 (Left and right Riemann–Liouville integrals of variable order).
Let $\alpha : \Delta \to (0,1)$ and $f \in L_1[a,b]$. Then,

$$ {}_aI_t^{\alpha(\cdot,\cdot)} f(t) = \int_a^t \frac{1}{\Gamma(\alpha(t,\tau))}(t-\tau)^{\alpha(t,\tau)-1} f(\tau)d\tau, \quad t > a, $$

is called the left Riemann–Liouville integral of variable fractional order $\alpha(\cdot,\cdot)$, while

$$ {}_tI_b^{\alpha(\cdot,\cdot)} f(t) = \int_t^b \frac{1}{\Gamma(\alpha(\tau,t))}(\tau-t)^{\alpha(\tau,t)-1} f(\tau)d\tau, \quad t < b, $$

denotes the right Riemann–Liouville integral of variable fractional order $\alpha(\cdot,\cdot)$.

Theorem 2.
Let $\alpha(t,\tau) = \alpha(t-\tau)$ with $0 < \alpha(t-\tau) < 1 - \frac{1}{n}$ for all $0 < t-\tau \leq b$ and a certain $n \in \mathbb{N}$ greater or equal than two. If $f \in AC[0,b]$, then

$$ {}_0I_t^{1-\alpha(\cdot)} f(t) = \int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))}(t-\tau)^{-\alpha(t-\tau)} f(\tau)d\tau \in AC[0,b]. $$

If $f \in AC[-b,0]$, then

$$ {}_tI_0^{1-\alpha(\cdot)} f(t) = \int_t^0 \frac{1}{\Gamma(1-\alpha(\tau-t))}(\tau-t)^{-\alpha(\tau-t)} f(\tau)d\tau \in AC[-b,0]. $$

Proof. We give the proof for the left integral; the other case being proved similarly. Let $k(s) := \frac{1}{\Gamma(1-\alpha(s))}s^{-\alpha(s)}$. Since $0 < \alpha(s) < 1 - \frac{1}{n}$,

1. for $s \leq 1$ we have $\ln s \geq 0$ and $s^{-\alpha(s)} < 1$,
2. for $s < 1$ we have $\ln s < 0$ and $s^{-\alpha(s)} < s^{\frac{1}{n}-1}$.

Therefore,

$$ \int_0^b |k(s)|\, d\tau = \int_0^b \left| \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} \right| d\tau < \int_0^1 \frac{1}{\Gamma(1-\alpha(s))} ds + \int_1^b \frac{1}{\Gamma(1-\alpha(s))} s^{\frac{1}{n}-1} ds. $$

Using the inequality

$$ \Gamma(x+1) \geq \frac{x^2+1}{x+1}, \tag{1} $$

valid for all $x \in [0,1]$ (see [40]), we obtain that

$$\int_0^1 \frac{1}{\Gamma(1-\alpha(s))} ds + \int_1^b \frac{1}{\Gamma(1-\alpha(s))} s^{\frac{1}{n}-1} ds < \int_0^1 ds + \int_1^b s^{\frac{1}{n}-1} ds = 1 + nb^{\frac{1}{n}} - n < \infty.$$

It means that $k(s) \in L_1[0,b]$. Moreover, the condition $f \in AC[0,b]$ implies that

$$f(t) = \int_0^t g(\tau) d\tau + f(0), \text{ where } g \in L_1[0,b].$$

Define

$$h(t) := \int_0^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} g(t-s) ds$$

and integrate:

$$\int_0^t h(\theta) d\theta = \int_0^t d\theta \int_0^\theta \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} g(\theta - s) ds. \tag{2}$$

Observe that

$$\int_0^t \left(\int_s^t \left| \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} g(\theta - s) \right| d\theta \right) ds = \int_0^t \left(\left| \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} \right| \left(\int_s^t |g(\theta - s)| \, d\theta \right) \right) ds < \|g\|_1 \int_0^b \left| \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} \right| ds < \infty.$$

This means that the assumptions of Fubini's theorem are satisfied, so we can change the order of integration in (2). Hence,

$$\int_0^t h(\theta) d\theta = \int_0^t ds \int_s^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} g(\theta - s) d\theta = \int_0^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} ds \int_s^t g(\theta - s) d\theta.$$

Putting $\xi = \theta - s$, one has $d\xi = d\theta$ and get

$$\int_0^t h(\theta) d\theta = \int_0^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} ds \int_0^{t-s} g(\xi) d\xi.$$

But $\int_0^{t-s} g(\xi) d\xi = f(t-s) - f(0)$ and, therefore, the following equality holds:

$$\int_0^t h(\theta) d\theta = \int_0^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} f(t-s) ds - f(0) \int_0^t \frac{1}{\Gamma(1-\alpha(s))} s^{-\alpha(s)} ds.$$

Putting $\tau = t - s$, $d\tau = -ds$ and

$$\int_0^t h(\theta) d\theta = \int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} (t-\tau)^{-\alpha(t-\tau)} f(\tau) d\tau - f(0) \int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} (t-\tau)^{-\alpha(t-\tau)} d\tau.$$

Both functions on the right-hand side of the equality

$$\int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} (t-\tau)^{-\alpha(t-\tau)} f(\tau) d\tau = \int_0^t h(\theta) d\theta + f(0) \int_0^t \frac{1}{\Gamma(1-\alpha(t-\tau))} (t-\tau)^{-\alpha(t-\tau)} d\tau$$

belong to $AC[0,b]$, hence ${}_0\mathrm{I}_t^{1-\alpha(\cdot)} f \in AC[0,b]$. □

Definition 3 (Left and right Riemann–Liouville derivatives of variable order).
Let $\alpha : \Delta \to (0,1)$. If ${}_a\mathrm{I}_t^{1-\alpha(\cdot,\cdot)} f \in AC[a,b]$, then the left Riemann–Liouville derivative of variable fractional order $\alpha(\cdot,\cdot)$ is defined by

$$ {}_a\mathrm{D}_t^{\alpha(\cdot,\cdot)} f(t) = \frac{d}{dt}\,{}_a\mathrm{I}_t^{1-\alpha(\cdot,\cdot)} f(t) = \frac{d}{dt}\int\limits_a^t \frac{1}{\Gamma(1-\alpha(t,\tau))}(t-\tau)^{-\alpha(t,\tau)} f(\tau)d\tau, \quad t > a, $$

while the right Riemann–Liouville derivative of variable order $\alpha(\cdot,\cdot)$ is defined for functions f such that ${}_t\mathrm{I}_b^{1-\alpha(\cdot,\cdot)} f \in AC[a,b]$ by

$$ {}_t\mathrm{D}_b^{\alpha(\cdot,\cdot)} f(t) = -\frac{d}{dt}\,{}_t\mathrm{I}_b^{1-\alpha(\cdot,\cdot)} f(t) = \frac{d}{dt}\int\limits_t^b \frac{-1}{\Gamma(1-\alpha(\tau,t))}(\tau-t)^{-\alpha(\tau,t)} f(\tau)d\tau, \quad t < b. $$

Definition 4 (Left and right Caputo derivatives of variable fractional order).
Let $0 < \alpha(t,\tau) < 1$ for all $t,\tau \in [a,b]$. If $f \in AC[a,b]$, then the left Caputo derivative of variable fractional order $\alpha(\cdot,\cdot)$ is defined by

$$ {}_a^C\mathrm{D}_t^{\alpha(\cdot,\cdot)} f(t) = \int\limits_a^t \frac{1}{\Gamma(1-\alpha(t,\tau))}(t-\tau)^{-\alpha(t,\tau)} \frac{d}{d\tau} f(\tau)d\tau, \quad t > a, $$

while the right Caputo derivative of variable fractional order $\alpha(\cdot,\cdot)$ is given by

$$ {}_t^C\mathrm{D}_b^{\alpha(\cdot,\cdot)} f(t) = \int\limits_t^b \frac{-1}{\Gamma(1-\alpha(\tau,t))}(\tau-t)^{-\alpha(\tau,t)} \frac{d}{d\tau} f(\tau)d\tau, \quad t < b. $$

3. Variable order fractional integration by parts

In this section we derive the integration by parts formulas that are essential for proving the Euler–Lagrange equations.

Theorem 5 (Integration by parts for variable order fractional integrals).
If $\frac{1}{n} < \alpha(t,\tau) < 1$ for all $(t,\tau) \in \Delta$ and a certain $n \in \mathbb{N}$ greater or equal than two, and $f, g \in C([a,b];\mathbb{R})$, then

$$ \int_a^b g(t)\,{}_a\mathrm{I}_t^{\alpha(\cdot,\cdot)} f(t)dt = \int_a^b f(t)\,{}_t\mathrm{I}_b^{\alpha(\cdot,\cdot)} g(t)dt. $$

Proof. Define

$$ F(\tau,t) := \begin{cases} \left|\frac{1}{\Gamma(\alpha(t,\tau))}(t-\tau)^{\alpha(t,\tau)-1} g(t) f(\tau)\right| & \text{if } \tau < t, \\ 0 & \text{if } \tau \geq t, \end{cases} $$

for all $(\tau,t) \in [a,b]\times[a,b]$. Since f and g are continuous functions on $[a,b]$, they are bounded on $[a,b]$, i.e., there exist $C_1, C_2 > 0$ such that $|g(t)| \leq C_1$ and $|f(t)| \leq C_2$, $t \in [a,b]$. Therefore,

$$ \begin{aligned} \int_a^b \left(\int_a^b F(\tau,t)d\tau\right) dt &= \int_a^b \left(\int_a^t \left|\frac{1}{\Gamma(\alpha(t,\tau))}(t-\tau)^{\alpha(t,\tau)-1} g(t) f(\tau)\right| d\tau\right) dt \\ &\leq C_1 C_2 \int_a^b \left(\int_a^t \left|\frac{1}{\Gamma(\alpha(t,\tau))}(t-\tau)^{\alpha(t,\tau)-1}\right| d\tau\right) dt \\ &= C_1 C_2 \int_a^b \left(\int_a^t \frac{1}{\Gamma(\alpha(t,\tau))}(t-\tau)^{\alpha(t,\tau)-1} d\tau\right) dt. \end{aligned} $$

Because $\frac{1}{n} < \alpha(t,\tau) < 1$,

1. for $1 \leq t-\tau$ we have $\ln(t-\tau) \geq 0$ and $(t-\tau)^{\alpha(t,\tau)-1} < 1$;
2. for $1 > t-\tau$ we have $\ln(t-\tau) < 0$ and $(t-\tau)^{\alpha(t,\tau)-1} < (t-\tau)^{\frac{1}{n}-1}$.

Therefore,

$$C_1 C_2 \int_a^b \left(\int_a^t \frac{1}{\Gamma(\alpha(t,\tau))} (t-\tau)^{\alpha(t,\tau)-1} d\tau \right) dt \;<\; C_1 C_2 \int_a^b \left(\int_a^{t-1} \frac{1}{\Gamma(\alpha(t,\tau))} d\tau + \int_{t-1}^t \frac{1}{\Gamma(\alpha(t,\tau))} (t-\tau)^{\frac{1}{n}-1} d\tau \right) dt.$$

Moreover, by inequality (1), valid for $x \in [0,1]$, one has

$$\begin{aligned} C_1 C_2 \int_a^b & \left(\int_a^{t-1} \frac{1}{\Gamma(\alpha(t,\tau))} d\tau + \int_{t-1}^t \frac{1}{\Gamma(\alpha(t,\tau))} (t-\tau)^{\frac{1}{n}-1} d\tau \right) dt \\ & \leq C_1 C_2 \int_a^b \left(\int_a^{t-1} \frac{\alpha^2(t,\tau)+\alpha(t,\tau)}{\alpha^2(t,\tau)+1} d\tau + \int_{t-1}^t \frac{\alpha^2(t,\tau)+\alpha(t,\tau)}{\alpha^2(t,\tau)+1} (t-\tau)^{\frac{1}{n}-1} d\tau \right) dt \\ & < C_1 C_2 \int_a^b \left(\int_a^{t-1} d\tau + \int_{t-1}^t (t-\tau)^{\frac{1}{n}-1} d\tau \right) dt \\ & = C_1 C_2 (b-a) \left(\frac{b+a}{2} - 1 + n - a \right) \\ & < \infty. \end{aligned}$$

Hence, one can use the Fubini theorem to change the order of integration:

$$\begin{aligned} \int_a^b g(t) {}_a\mathrm{I}_t^{\alpha(\cdot,\cdot)} f(t) dt &= \int_a^b \left(\int_a^t g(t) f(\tau) \frac{1}{\Gamma(\alpha(t,\tau))} (t-\tau)^{\alpha(t,\tau)-1} d\tau \right) dt \\ &= \int_a^b \left(\int_\tau^b g(t) f(\tau) \frac{1}{\Gamma(\alpha(t,\tau))} (t-\tau)^{\alpha(t,\tau)-1} dt \right) d\tau \\ &= \int_a^b f(\tau) {}_\tau\mathrm{I}_b^{\alpha(\cdot,\cdot)} g(\tau) d\tau. \end{aligned}$$

The proof is complete. □

Theorem 6 (Integration by parts for variable order fractional derivatives).

Let $0 < \alpha(t,\tau) < 1 - \frac{1}{n}$ for all $(t,\tau) \in \Delta$ and a certain $n \in \mathbb{N}$ greater or equal than two. If $f \in C^1([a,b];\mathbb{R})$, $g \in C([a,b];\mathbb{R})$, and ${}_t\mathrm{I}_b^{1-\alpha(\cdot,\cdot)} g \in AC[a,b]$, then

$$\int_a^b g(t) {}_a^C\mathrm{D}_t^{\alpha(\cdot,\cdot)} f(t) dt = f(t) {}_t\mathrm{I}_b^{1-\alpha(\cdot,\cdot)} g(t) \Big|_a^b + \int_a^b f(t) {}_t\mathrm{D}_b^{\alpha(\cdot,\cdot)} g(t) dt.$$

If $f \in C^1([a,b];\mathbb{R})$, $g \in C([a,b];\mathbb{R})$, and ${}_a\mathrm{I}_t^{1-\alpha(\cdot,\cdot)} g \in AC[a,b]$, then

$$\int_a^b g(t) {}_t^C\mathrm{D}_b^{\alpha(\cdot,\cdot)} f(t) dt = - f(t) {}_a\mathrm{I}_t^{1-\alpha(\cdot,\cdot)} g(t) \Big|_a^b + \int_a^b f(t) {}_a\mathrm{D}_t^{\alpha(\cdot,\cdot)} g(t) dt.$$

Proof. By Definition 4, it follows that ${}_a^C\mathrm{D}_t^{\alpha(\cdot,\cdot)} f(t) = {}_a\mathrm{I}_t^{1-\alpha(\cdot,\cdot)} \frac{d}{dt} f(t)$. Applying Theorem 5 and integration by parts for classical (integer order) derivatives, we obtain

$$\begin{aligned} \int_a^b g(t) {}_a^C\mathrm{D}_t^{\alpha(\cdot,\cdot)} f(t) dt &= \int_a^b g(t) {}_a\mathrm{I}_t^{1-\alpha(\cdot,\cdot)} \frac{d}{dt} f(t) dt = \int_a^b \frac{d}{dt} f(t) {}_t\mathrm{I}_b^{1-\alpha(\cdot,\cdot)} g(t) dt \\ &= f(t) {}_t\mathrm{I}_b^{1-\alpha(\cdot,\cdot)} g(t) \Big|_a^b - \int_a^b f(t) \frac{d}{dt} {}_t\mathrm{I}_b^{1-\alpha(\cdot,\cdot)} g(t) dt \\ &= f(t) {}_t\mathrm{I}_b^{1-\alpha(\cdot,\cdot)} g(t) \Big|_a^b + \int_a^b f(t) {}_t\mathrm{D}_b^{\alpha(\cdot,\cdot)} g(t) dt. \end{aligned}$$

The second formula is proved in a similar way. □

4. Euler–Lagrange equations for incommensurate fractional variational problems of variable order

Now we prove necessary optimality conditions of Euler–Lagrange type for incommensurate order fractional variational problems. Let $\alpha_i(t,\tau)$, $\beta_i(t,\tau)$, $i=1,\dots,n$, $n\in\mathbb{N}$, satisfy the assumptions of Theorem 6. Consider the following problem.

Problem 7.

Find a function $q=q(t)$ for which the functional

$$\mathcal{J}[q]=\int_a^b L\left(t,q(t),{}_a^C\mathrm{D}_t^{\alpha_1(t,\tau)}q(t),\dots,{}_a^C\mathrm{D}_t^{\alpha_n(t,\tau)}q(t),{}_t^C\mathrm{D}_b^{\beta_1(t,\tau)}q(t),\dots,{}_t^C\mathrm{D}_b^{\beta_n(t,\tau)}q(t)\right)dt \tag{3}$$

attains an extremum on the set

$$\mathcal{D}=\left\{q\in C^1([a,b];\mathbb{R}):q(a)=q_a,q(b)=q_b \text{ and } {}_a^C\mathrm{D}_t^{\alpha_i(t,\tau)}q,{}_t^C\mathrm{D}_b^{\beta_i(t,\tau)}q\in C([a,b];\mathbb{R}),i=1,\dots,n\right\}.$$

We assume that $L\in C^1\left([a,b]\times\mathbb{R}^{2n+1};\mathbb{R}\right)$; $t\mapsto\partial_{i+2}L$ is continuous, has absolutely continuous integral ${}_t\mathrm{I}_b^{1-\alpha_i(t,\tau)}$ and continuous derivative ${}_t\mathrm{D}_b^{\alpha_i(t,\tau)}$ for each $i=1,\dots,n$; $t\mapsto\partial_{n+i+2}L$ is continuous, has absolutely continuous integral ${}_a\mathrm{I}_t^{1-\beta_i(t,\tau)}$ and continuous derivative ${}_a\mathrm{D}_t^{\beta_i(t,\tau)}$ for each $i=1,\dots,n$. For simplicity, we introduce the following notation:

$$\{q,\alpha,\beta\}(t):=\left(t,q(t),{}_a^C\mathrm{D}_t^{\alpha(t,\tau)}q(t),{}_t^C\mathrm{D}_b^{\beta(t,\tau)}q(t)\right),$$

where

$${}_a^C\mathrm{D}_t^{\alpha(t,\tau)}:=\left({}_a^C\mathrm{D}_t^{\alpha_1(t,\tau)},\dots,{}_a^C\mathrm{D}_t^{\alpha_n(t,\tau)}\right),\quad {}_t^C\mathrm{D}_b^{\beta(t,\tau)}:=\left({}_t^C\mathrm{D}_b^{\beta_1(t,\tau)},\dots,{}_t^C\mathrm{D}_b^{\beta_n(t,\tau)}\right).$$

Definition 8.

Let $f\in AC[a,b]$. For ${}_t\mathrm{I}_b^{1-\gamma(t,\tau)}g\in AC[a,b]$ we define the following operator:

$$\mathrm{D}_-^{\gamma(t,\tau)}[f,g]:=-f\,{}_t\mathrm{D}_b^{\gamma(t,\tau)}[g]+g\,{}_a^C\mathrm{D}_t^{\gamma(t,\tau)}[f], \tag{4}$$

while for ${}_a\mathrm{I}_t^{1-\gamma(t,\tau)}g\in AC[a,b]$ we define

$$\mathrm{D}_+^{\gamma(t,\tau)}[f,g]:=-f\,{}_a\mathrm{D}_t^{\gamma(t,\tau)}[g]+g\,{}_t^C\mathrm{D}_b^{\gamma(t,\tau)}[f]. \tag{5}$$

Remark 9.

If $\gamma(t,\tau)\equiv\gamma$ is a constant function, then

$$\lim_{\gamma\to1^-}\mathrm{D}_-^{\gamma}[f,g]=fg'+gf'=\frac{d}{dt}(fg)=\lim_{\gamma\to1^-}\mathrm{D}_-^{\gamma}[g,f]$$

and

$$\lim_{\gamma\to1^-}\mathrm{D}_+^{\gamma}[f,g]=-fg'-gf'=-\frac{d}{dt}(fg)=\lim_{\gamma\to1^-}\mathrm{D}_+^{\gamma}[g,f].$$

Theorem 10.

Let function q be a solution to Problem 7. Then,

$$\partial_2L\{q,\alpha,\beta\}(t)-\sum_{i=1}^n\mathrm{D}_-^{\alpha_i(t,\tau)}[1,\partial_{i+2}L\{q,\alpha,\beta\}(t)]-\sum_{i=1}^n\mathrm{D}_+^{\beta_i(t,\tau)}[1,\partial_{n+2+i}L\{q,\alpha,\beta\}(t)]=0. \tag{6}$$

Proof. Assume that q is an extremizer of $\mathcal{J}$. Consider the value of $\mathcal{J}$ at a nearby function $\hat{q}(t) = q(t) + \varepsilon\eta(t)$, where $\varepsilon \in \mathbb{R}$ is a small parameter and $\eta \in C^1([a,b];\mathbb{R})$ is an arbitrary function satisfying $\eta(a) = \eta(b) = 0$ and such that ${}_a^C\mathrm{D}_t^{\alpha_i(t,\tau)}\eta$ and ${}_t^C\mathrm{D}_b^{\beta_i(t,\tau)}\eta$ are continuous. Let

$$J(\varepsilon) = \mathcal{J}[\hat{q}] = \int_a^b L\left\{\hat{q},\alpha,\beta\right\}(t)dt.$$

A necessary condition for $\hat{q}$ to be an extremizer is given by

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = 0 \Leftrightarrow \int_a^b \left(\partial_2 L\left\{q,\alpha,\beta\right\}(t)\eta(t) + \sum_{i=1}^n \partial_{i+2}L\left\{q,\alpha,\beta\right\}(t){}_a^C\mathrm{D}_t^{\alpha_i(t,\tau)}\eta(t) + \sum_{i=1}^n \partial_{n+2+i}L\left\{q,\alpha,\beta\right\}(t){}_t^C\mathrm{D}_b^{\beta_i(t,\tau)}\eta(t)\right)dt = 0. \quad (7)$$

Using variable order fractional integration by parts formulas (Theorem 6), we obtain that

$$\int_a^b \partial_{i+2}L\left\{q,\alpha,\beta\right\}(t){}_a^C\mathrm{D}_t^{\alpha_i(t,\tau)}\eta(t)dt = \left.\eta(t){}_t\mathrm{I}_b^{1-\alpha_i(t,\tau)}\partial_{i+2}L\left\{q,\alpha,\beta\right\}(t)\right|_a^b + \int_a^b \eta(t){}_t\mathrm{D}_b^{\alpha_i(t,\tau)}\partial_{i+2}L\left\{q,\alpha,\beta\right\}(t)dt, \quad i = 1,\ldots,n,$$

and

$$\int_a^b \partial_{n+i+2}L\left\{q,\alpha,\beta\right\}(t){}_t^C\mathrm{D}_b^{\beta_i(t,\tau)}\eta(t)dt = -\left.\eta(t){}_a\mathrm{I}_t^{1-\beta_i(t,\tau)}\partial_{n+i+2}L\left\{q,\alpha,\beta\right\}(t)\right|_a^b + \int_a^b \eta(t){}_a\mathrm{D}_t^{\beta_i(t,\tau)}\partial_{n+i+2}L\left\{q,\alpha,\beta\right\}(t)dt, \quad i = 1,\ldots,n.$$

Because $\eta(a) = \eta(b) = 0$, (7) simplifies to

$$\int_a^b \eta(t)\left(\partial_2 L\left\{q,\alpha,\beta\right\}(t) + \sum_{i=1}^n {}_t\mathrm{D}_b^{\alpha_i(t,\tau)}\partial_{i+2}L\left\{q,\alpha,\beta\right\}(t) + \sum_{i=1}^n {}_a\mathrm{D}_t^{\beta_i(t,\tau)}\partial_{n+i+2}L\left\{q,\alpha,\beta\right\}(t)\right)dt = 0.$$

Applying the fundamental lemma of the calculus of variations (see, e.g., [36]), we obtain that

$$\partial_2 L\left\{q,\alpha,\beta\right\}(t) + \sum_{i=1}^n {}_t\mathrm{D}_b^{\alpha_i(t,\tau)}\partial_{i+2}L\left\{q,\alpha,\beta\right\}(t) + \sum_{i=1}^n {}_a\mathrm{D}_t^{\beta_i(t,\tau)}\partial_{n+i+2}L\left\{q,\alpha,\beta\right\}(t) = 0.$$

Finally, by Definition 8, we arrive to (6). □

5. Noether's theorem of fractional variable order

Now we show, employing operators (4) and (5), that invariance of (3) leads to a variable order version of a fractional conservation law. To prove this, we borrow the method from [26, 38], where Noether's theorem is stated for fractional derivatives of constant order.

Definition 11.
A function q that is a solution to (6) is said to be a variable order fractional extremal for functional $\mathcal{J}$.

Definition 12.
We say that functional (3) is invariant under an ε-parameter group of infinitesimal transformations

$$\bar{q}(t) = q(t) + \varepsilon\xi(t, q(t)) + o(\varepsilon) \tag{8}$$

if

$$\int_{t_a}^{t_b} L\left(t, q(t), {}_a^C\mathrm{D}_t^{\alpha(t,\tau)} q(t), {}_t^C\mathrm{D}_b^{\beta(t,\tau)} q(t)\right) dt = \int_{t_a}^{t_b} L\left(t, \bar{q}(t), {}_a^C\mathrm{D}_t^{\alpha(t,\tau)} \bar{q}(t), {}_t^C\mathrm{D}_b^{\beta(t,\tau)} \bar{q}(t)\right) dt \tag{9}$$

for any subinterval $[t_a, t_b] \subseteq [a, b]$.

Lemma 13 (Necessary condition of invariance).
If functional (3) is invariant under an ε-parameter group of infinitesimal transformations (8), then

$$\partial_2 L\left\{q, \alpha, \beta\right\}(t)\xi(t, q(t)) + \sum_{i=1}^{n} \partial_{i+2} L\left\{q, \alpha, \beta\right\}(t) {}_a^C\mathrm{D}_t^{\alpha_i(t,\tau)} \xi(t, q(t)) + \sum_{i=1}^{n} \partial_{n+i+2} L\left\{q, \alpha, \beta\right\}(t) {}_t^C\mathrm{D}_b^{\beta_i(t,\tau)} \xi(t, q(t)) = 0. \tag{10}$$

Proof. Since, by hypothesis, condition (9) is satisfied for any subinterval $[t_a, t_b] \subseteq [a, b]$, we have

$$L\left(t, q(t), {}_a^C\mathrm{D}_t^{\alpha(t,\tau)} q(t), {}_t^C\mathrm{D}_b^{\beta(t,\tau)} q(t)\right) = L\left(t, \bar{q}(t), {}_a^C\mathrm{D}_t^{\alpha(t,\tau)} \bar{q}(t), {}_t^C\mathrm{D}_b^{\beta(t,\tau)} \bar{q}(t)\right). \tag{11}$$

Now, differentiating (11) with respect to ε, then putting $\varepsilon = 0$, and applying definitions and properties of variable order Caputo fractional derivatives, we obtain that

$$\begin{aligned}
0 &= \partial_2 L\left\{q, \alpha, \beta\right\}(t)\xi(t, q(t)) \\
&\quad + \sum_{i=1}^{n} \partial_{i+2} L\left\{q, \alpha, \beta\right\}(t) \frac{d}{d\varepsilon}\left[\int_a^t \frac{1}{\Gamma(1-\alpha_i(t,\tau))}(t-\tau)^{-\alpha_i(t,\tau)} \frac{d}{d\tau}\bar{q}(\tau)d\tau\right]_{\varepsilon=0} \\
&\quad + \sum_{i=1}^{n} \partial_{n+i+2} L\left\{q, \alpha, \beta\right\}(t) \frac{d}{d\varepsilon}\left[\int_t^b \frac{-1}{\Gamma(1-\beta_i(t,\tau))}(\tau-t)^{-\beta_i(t,\tau)} \frac{d}{d\tau}\bar{q}(\tau)d\tau\right]_{\varepsilon=0} \\
&= \partial_2 L\left\{q, \alpha, \beta\right\}(t)\xi(t, q(t)) + \sum_{i=1}^{n} \partial_{i+2} L\left\{q, \alpha, \beta\right\}(t) {}_a^C\mathrm{D}_t^{\alpha_i(t,\tau)} \xi(t, q(t)) \\
&\quad + \sum_{i=1}^{n} \partial_{n+i+2} L\left\{q, \alpha, \beta\right\}(t) {}_t^C\mathrm{D}_b^{\beta_i(t,\tau)} \xi(t, q(t)).
\end{aligned}$$

This concludes the proof. □

Theorem 14 (Noether's theorem for variable order fractional variational problems).
If functional (3) is invariant in the sense of Definition 12, then

$$\sum_{i=1}^{n} \mathrm{D}_{-}^{\alpha_i(t,\tau)}[\xi(t,q(t)), \partial_{i+2}L\{q,\alpha,\beta\}(t)] + \sum_{i=1}^{n} \mathrm{D}_{+}^{\beta_i(t,\tau)}[\xi(t,q(t)), \partial_{n+i+2}L\{q,\alpha,\beta\}(t)] = 0, \quad t \in [a,b],$$

along all variable order fractional extremals $q(\cdot)$.

Proof. By Theorem 10 we have

$$\partial_2 L\{q,\alpha,\beta\}(t) = -\sum_{i=1}^{n} {}_t\mathrm{D}_b^{\alpha_i(t,\tau)}\partial_{i+2}L\{q,\alpha,\beta\}(t) - \sum_{i=1}^{n} {}_a\mathrm{D}_t^{\beta_i(t,\tau)}\partial_{n+i+2}L\{q,\alpha,\beta\}(t). \tag{12}$$

Substituting (12) into (10), we obtain

$$-\sum_{i=1}^{n} \xi(t,q(t))_t\mathrm{D}_b^{\alpha_i(t,\tau)}\partial_{i+2}L\{q,\alpha,\beta\}(t) - \sum_{i=1}^{n} \xi(t,q(t))_a\mathrm{D}_t^{\beta_i(t,\tau)}\partial_{n+i+2}L\{q,\alpha,\beta\}(t)$$
$$+\sum_{i=1}^{n} \partial_{i+2}L\{q,\alpha,\beta\}(t){}_a^C\mathrm{D}_t^{\alpha_i(t,\tau)}\xi(t,q(t)) + \sum_{i=1}^{n} \partial_{n+i+2}L\{q,\alpha,\beta\}(t){}_t^C\mathrm{D}_b^{\beta_i(t,\tau)}\xi(t,q(t)) = 0.$$

Finally, by Definition 8, one has

$$\sum_{i=1}^{n} \mathrm{D}_{-}^{\alpha_i(t,\tau)}[\xi(t,q(t)), \partial_{i+2}L\{q,\alpha,\beta\}(t)] + \sum_{i=1}^{n} \mathrm{D}_{+}^{\beta_i(t,\tau)}[\xi(t,q(t)), \partial_{n+i+2}L\{q,\alpha,\beta\}(t)] = 0.$$

The proof is complete. □

6. An illustrative example

Let α and β be two functions such that $0 < \alpha(t,\tau), \beta(t,\tau) < 1 - \frac{1}{l}$ for a certain natural number l greater or equal than two. Consider the following problem:

$$\mathcal{J}[q] = \int_a^b L\left(t, {}_a^C\mathrm{D}_t^{\alpha(t,\tau)}q(t), {}_t^C\mathrm{D}_b^{\beta(t,\tau)}q(t)\right) dt \longrightarrow \text{extremize}$$
$$q(a) = q_a, \quad q(b) = q_b. \tag{13}$$

For the transformation

$$\bar{q}(t) = q(t) + \varepsilon c + o(\varepsilon), \tag{14}$$

where c is a constant, we have

$$\int_{t_a}^{t_b} L\left(t, {}_a^C\mathrm{D}_t^{\alpha(t,\tau)}q(t), {}_t^C\mathrm{D}_b^{\beta(t,\tau)}q(t)\right) dt = \int_{t_a}^{t_b} L\left(t, {}_a^C\mathrm{D}_t^{\alpha(t,\tau)}\bar{q}(t), {}_t^C\mathrm{D}_b^{\beta(t,\tau)}\bar{q}(t)\right) dt$$

for any $[t_a, t_b] \subseteq [a,b]$. Therefore, $\mathcal{J}[q]$ is invariant under (14) and Noether's theorem (Theorem 14) asserts that

$$D_{-}^{\alpha(t,\tau)}\left[c,\partial_2 L\left(t,{}_a^C D_t^{\alpha(t,\tau)}q(t),{}_t^C D_b^{\beta(t,\tau)}q(t)\right)\right]+D_{+}^{\beta(t,\tau)}\left[c,\partial_3 L\left(t,{}_a^C D_t^{\alpha(t,\tau)}q(t),{}_t^C D_b^{\beta(t,\tau)}q(t)\right)\right]=0$$

along any extremal $q(t)$ of (13).

7. Conclusion

Noether's symmetry theorem, establishing that variational invariance implies conservation laws, is one of the most important results in physics and the calculus of variations. Such conservation laws are, however, only valid for conservative systems. Nonconservative forces, like friction, remove energy from the systems and, as a consequence, Noether's conservation laws cease to be valid. In order to cope with dissipative forces that do not store energy, one possibility is to use fractional calculus [18, 19]. The study of problems of the calculus of variations with fractional derivatives of variable order is, however, a rather recent subject, and available results reduce to those available at [31–33]. Here we obtain Euler–Lagrange optimality conditions for variational problems with fractional derivatives of incommensurate variable order and, for such fractional Euler–Lagrange extremals, we provide a notion of conservation law and prove a version of Noether's theorem.

The variable order calculus of variations is underdeveloped and much remains to be done. We mention here one important question that is still open. The Noether type theorem here obtained only includes the terms related to "conservation of momentum". It would be important to provide a more general form of Noether's theorem with "energy" terms.

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References

[1] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations (Elsevier, Amsterdam, 2006) 204
[2] M. Klimek, On solutions of linear fractional differential equations of a variational type (The Publishing Office of Czestochowa University of Technology, Czestochowa, 2009)
[3] I. Podlubny, Fractional differential equations, Mathematics in Science and Engineering (Academic Press, San Diego, CA, 1999) 198
[4] S. G. Samko, A. A. Kilbas, O. I. Marichev, Fractional integrals and derivatives (Gordon and Breach, Yverdon, 1993)
[5] R. Caponetto, G. Dongola, L. Fortuna, I. Petras, Fractional Order Systems: Modeling and Control Applications (World Scientific Company, Singapore, 2010)
[6] F. Jarad, T. Abdeljawad, D. Baleanu, Nonlinear Anal. Real World Appl. 14, 780 (2013)
[7] D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, Fractional calculus (World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012)
[8] M. D. Ortigueira, Fractional calculus for scientists and engineers, Lecture Notes in Electrical Engineering (Springer, Dordrecht, 2011) 84
[9] J. A. Tenreiro Machado, V. Kiryakova, F. Mainardi, Commun. Nonlinear Sci. Numer. Simul. 16, 1140 (2011)
[10] S. G. Samko, B. Ross, Integral Transform. Spec. Funct. 1, 277 (1993)
[11] S. G. Samko, Anal. Math. 21, 213 (1995)
[12] B. Ross, S. G. Samko, Internat. J. Math. Math. Sci. 18, 777 (1995)
[13] C. F. M. Coimbra, Ann. Phys. 12, 692 (2003)
[14] G. Diaz, C. F. M. Coimbra, Nonlinear Dynam. 56, 145 (2009)
[15] C. F. Lorenzo, T. T. Hartley, Nonlinear Dynam. 29, 57 (2002)
[16] L. E. S. Ramirez, C. F. M. Coimbra, Int. J. Differ. Equ. 2010, 16 (2010)
[17] L. E. S. Ramirez, C. F. M. Coimbra, Phys. D 240, 1111 (2011)
[18] F. Riewe, Phys. Rev. E 53, 1890 (1996)
[19] F. Riewe, Phys. Rev. E 55, 3581 (1997)

[20] O. P. Agrawal, S. I. Muslih, D. Baleanu, Commun. Nonlinear Sci. Numer. Simul. 16, 4756 (2011)
[21] R. Almeida, D. F. M. Torres, Appl. Math. Lett. 22, 1816 (2009)
[22] D. Baleanu, T. Avkar, Nuovo Cimento Soc. Ital. Fis. B 119, 73 (2004)
[23] D. Baleanu, S. I. Muslih, Phys. Scripta 72, 119 (2005)
[24] N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres, Discrete Contin. Dyn. Syst. 29, 417 (2011)
[25] J. Cresson, J. Math. Phys. 48, 34 (2007)
[26] G. S. F. Frederico, D. F. M. Torres, Appl. Math. Comput. 217, 1023 (2010)
[27] M. A. E. Herzallah, D. Baleanu, Nonlinear Dynam. 69, 977 (2012)
[28] D. Mozyrska, D. F. M. Torres, Carpathian J. Math. 26, 210 (2010)
[29] D. Mozyrska, D. F. M. Torres, Signal Process. 91, 379 (2011)
[30] A. B. Malinowska, D. F. M. Torres, Introduction to the fractional calculus of variations (Imp. Coll. Press, London, 2012)
[31] T. M. Atanackovic, S. Pilipovic, Fract. Calc. Appl. Anal. 14, 94 (2011)
[32] T. Odzijewicz, A. B. Malinowska, D. F. M. Torres, Variable order fractional variational calculus for double integrals, Proc. 51st IEEE Conference on Decision and Control, Dec. 10-13, 2012, Maui, Hawaii, art. no. 6426489, 6873
[33] T. Odzijewicz, A. B. Malinowska, D. F. M. Torres, In: A. Almeida, L. P. Castro, F.-O. Speck (Eds.), Fractional variational calculus of variable order, Advances in Harmonic Analysis and Operator Theory (Birkhäuser, Basel, 2013) 291
[34] J. Jost, X. Li-Jost, Calculus of variations (Cambridge Univ. Press, Cambridge, 1998)
[35] Y. Kosmann-Schwarzbach, The Noether theorems: Invariance and conservation laws in the twentieth century (Springer, New York, 2011)
[36] B. van Brunt, The calculus of variations (Springer, New York, 2004)
[37] T. M. Atanacković, S. Konjik, S. Pilipović, S. Simić, Nonlinear Anal. 71, 1504 (2009)
[38] G. S. F. Frederico, D. F. M. Torres, J. Math. Anal. Appl. 334, 834 (2007)
[39] G. S. F. Frederico, D. F. M. Torres, Nonlinear Dynam. 53, 215 (2008)
[40] P. Ivady, J. Math. Inequal. 3, 227 (2009)