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Exact solution of D-dimensional Schrödinger equation generated from certain central power-law potentials

Research Article

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Abstract: By applying Extended Transformation method we have generated exact solution of D-dimensional radial Schrödinger equation for a set of power-law multi-term potentials taking singular potentials $V(r) = ar^{-\frac{1}{2}} + br^{-\frac{3}{2}}$, $V(r) = ar^{\frac{2}{3}} + br^{-\frac{2}{3}} + cr^{-\frac{4}{3}}$, $V(r) = ar + br^{-1} + cr^2$ and $V(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6}$ as input reference. The restriction on the parameters of the given potentials and angular momentum quantum number ℓ are obtained. The multiplet structure of the generated exactly solvable potentials are also shown.

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1. Introduction

In quantum mechanics, it is important to seek new exactly solvable potentials of Schrödinger equation as they incorporate new ideas and/or mathematical techniques in different branches of physics such as atomic and molecular physics, nuclear physics, particle physics etc. Also, exactly solvable potentials are essential for successful implementation of approximate methods in the study of practical quantum systems. Starting from the bound state solutions of hydrogen atom ($\frac{1}{r}$ type potential) and harmonic oscillator (r^2 type potential) problems, many researchers have contributed several new exactly solvable potentials of Schrödinger equation. So far, various methods e.g. the traditional method [1, 2], the Nikiforov-Uvarov (NU) method [3], the ansatz for the eigenfunctions method [4, 5], the Laurent series ansatz for the eigenfunctions method [6], the Extended Transformation (ET) method [7], the supersymmetric (SUSY) method [8], the asymptotic iteration method (AIM) [9] etc. have been developed for the exact solution of the quantum mechanical systems.

Considerable effort has been made in the past several decades to solve the time-independent Schrödinger equation for central multi-term potentials containing negative powers of the radial coordinate in two as well as three dimensions [6, 10–25]. However, analysis of such problems in arbitrary D dimensions is limited to the best of our knowledge [22, 23, 26, 27]. In pursuit of this goal, we have used an efficient transformation method called the extended transformation (ET) method [7, 28–31] that generates new exactly solvable quantum systems (QSs) in arbitrary dimensions from already known exactly solvable QS. The ET method is a two step transformation that includes a coordinate transformation followed by a functional trans-

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formation and a set of plausible ansatze. In our present work, we have taken the singular fraction power potentials $V(r) = ar^{-\frac{1}{2}} + br^{-\frac{3}{2}}$ and $V(r) = ar^{\frac{2}{3}} + br^{-\frac{2}{3}} + cr^{-\frac{4}{3}}$, the mixture potential $ar + br^{-1} + cr^2$ and the singular integer even-power potential $V(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6}$ as the input reference potentials. Our main objective is to generate more exactly solvable potentials (ESPs) from these reference potentials and to show their hierarchical connections. In quantum multi-term potentials, exact solvability requires that there should be an interrelation between the parameters and the angular momentum quantum numbers of the potential under consideration. It is noteworthy that under ET the constraint equation becomes connected to the energy eigenvalue expression and that of the energy eigenvalue to the constraint equation. We discuss a procedure to regroup this set of energy-dependent Sturmian QS to a normal/physical QS.
The paper is organized as follows. In Section 2, we briefly review the ET method. In Section 3, we apply ET on the reference potentials for generating new QSs. Second order ET is highlighted to show the multiplet structure of generated ESPs in Section 4. Finally, the important results and the conclusions are discussed in Section 5.

2. Formalism

For a QS, say A-QS, the radial part of the Schrödinger equation for the potential $V_A(r)$ in D_A dimensional Euclidean space (in natural units $\hbar = 1 = 2m$) is

$$\psi_A''(r) + \frac{(D_A-1)}{r}\psi_A'(r) + \left(E_A - V_A(r) - \frac{\ell_A(\ell_A + D_A - 2)}{r^2}\right)\psi_A(r) = 0 \quad (1)$$

where r is a dimensionless spatial coordinate.

The ET includes a coordinate transformation, which is followed by a functional transformation and a set of plausible ansatze to restore the transformed equation to normal standard form of the Schrödinger equation.
We now invoke the coordinate transformation

$$r \to g_B(r) \quad (2)$$

which is followed by the functional transformation

$$\psi_B(r) = f_B^{-1}(r)\psi_A(g_B(r)) \quad (3)$$

where the transformation function $g_B(r)$ and the modulated amplitude $f_B(r)$ have to be specified within the framework of ET. The transformed Schrödinger equation then takes the form:

$$\psi_B''(r) + \left(\frac{d}{dr}\ln\frac{f_B^2 g_B^{D_A-1}}{g_B'}\right)\psi_B'(r) + \left[\left(\frac{d}{dr}\ln f_B\right)\left(\frac{d}{dr}\ln\frac{f_B' g_B^{D_A-1}}{g_B'}\right) + g_B'^2\left(E_A - V_A(g_B(r)) - \frac{\ell_A(\ell_A + D_A - 2)}{g_B^2}\right)\right]\psi_B(r) = 0 \quad (4)$$

where the prime denotes differentiation with respect to the variable r.
The dimension of the Euclidean space of the transformed QS, henceforth called the B-QS, can be chosen arbitrarily. Let it be denoted by D_B. Then,

$$\frac{d}{dr}\ln\frac{f_B^2 g_B^{D_A-1}}{g_B'} = \frac{D_B-1}{r} \quad (5)$$

Expression (5) fixes $f_B(r)$ as a function of $g_B(r)$ and its derivative. We get

$$f_B(r) = N_B g_B'^{1/2} g_B^{-(D_A-1)/2} r^{(D_B-1)/2} \quad (6)$$

where N_B is the normalization constant.

From equations (3) and (6), we find

$$\psi_B(r) = N_B g_B'^{-1/2} g_B^{(D_A-1)/2} r^{-(D_B-1)/2}\psi_A(g_B(r)) \quad (7)$$

where the transformation function $g_B(r)$ is at least three times differentiable.

The corresponding D_B dimensional Schrödinger equation for B-QS can be rewritten as

$$\psi_B''(r)+\frac{D_B-1}{r}\psi_B'(r)+\left[\frac{1}{2}\{g_B,r\}-\frac{D_A-1}{2}\frac{D_A-3}{2}\left(\frac{g_B'}{g_B}\right)^2+\frac{D_B-1}{2}\frac{D_B-3}{2}\frac{1}{r^2}\right]\psi_B(r)$$
$$g_B'^2\left(E_A-V_A(g_B(r))-\frac{\ell_A(\ell_A+\frac{D_A}{2}-1)^2}{g_B^2}+\frac{(D_A-2)^2}{4g_B^2}\right)\psi_B(r)=0 \quad (8)$$

where

$$\{g_B,r\}=\frac{g_B'''(r)}{g_B'(r)}-\frac{3}{2}\frac{g_B''^2(r)}{g_B'^2(r)} \quad (9)$$

In case of multi-term A-QS, we have to select a term of $V_A(g_B(r))$ as a working potential (WP) to implement ET and is designated as $V_A^W(g_B(r))$.
In order to mould equation (8) to the standard form of the Schrödinger equation, the following plausible ansatze have to be made, which are integral part of the transformation method.

$$g_B'^2V_A^W(g_B(r))=-E_B \quad (10)$$

$$V_B^{(1)}(r)=-g_B'^2E_A \quad (11)$$

$$V_B^{(2)}(r)=g_B'^2\left(V_A(g_B(r))-V_A^W(g_B(r))\right) \quad (12)$$

$$\frac{g_B'^2(\ell_A+\frac{D_A}{2}-1)^2}{g_B^2}=\frac{(\ell_B+\frac{D_B}{2}-1)^2}{r^2} \quad (13)$$

We obtain the new potential $V_B(r)$ as

$$V_B(r)=V_B^{(1)}(r)+V_B^{(2)}(r) \quad (14)$$

Finally, the radial Schrödinger equation for B-QS for the potential $V_B(r)$ can be read as

$$\psi_B''(r)+\frac{D_B-1}{r}\psi_B'(r)$$
$$+\left(E_B-V_B(r)-\frac{\ell_B(\ell_B+D_B-2)}{r^2}\right)\psi_B(r)=0 \quad (15)$$

3. Application of ET on the multi-term singular potentials

3.1. Generation of ESPs from the singular one-fraction power potential

The potential has the form

$$V_A(r)=ar^{-\frac{1}{2}}+br^{-\frac{3}{2}} \quad (16)$$

The exact eigenfunctions and the energy eigenvalues of the potential are given as [19, 22]

$$\psi_A(r)=N_Ar^{\ell_A}\exp\left[-(-E_A)^{\frac{1}{2}}r-\frac{ar^{\frac{1}{2}}}{(-E_A)^{\frac{1}{2}}}\right] \quad (17)$$

$$E_A=-\left[\frac{a^2}{4(2\ell_A+D_A-1)}\right]^{\frac{2}{3}} \quad (18)$$

and the constraint condition between the parameters of the potential and ℓ_A is given by

$$\left(\ell_A+\frac{D_A}{2}-\frac{3}{4}\right)\left(8a\left(\ell_A+\frac{D_A}{2}-\frac{1}{2}\right)\right)^{\frac{1}{3}}+b=0 \quad (19)$$

To implement ET on A-QS, choosing $ar^{-\frac{1}{2}}$ as WP and utilizing (10), we find the transformation function as

$$g_B(r)=\left(\frac{-E_B}{a}\right)^{\frac{2}{3}}\left(\frac{3}{4}r\right)^{\frac{4}{3}} \quad (20)$$

We set the integration constant to zero for $g_B(r)$ to satisfy the required condition $g_B(0)=0$ and $g_B(\infty)=\infty$.
Equations (11) and (20) yield

$$V_B^{(1)}(r)=\alpha_{11}r^{\frac{2}{3}} \quad (21)$$

where

$$\alpha_{11}=(\frac{3}{4})^{\frac{2}{3}}(\frac{-E_B}{a})^{\frac{4}{3}}(-E_A)=C_B^2 \quad (22)$$

C_B^2 is the characteristic constant of the B-QS.

Equations (12) and (20) lead to

$$V_B^{(2)}(r) = \alpha_{12}^{(n)} r^{-\frac{4}{3}} \tag{23}$$

where

$$\alpha_{12}^{(n)} = \left(\frac{3}{4}\right)^{-\frac{4}{3}} \left(\frac{-E_B}{a}\right)^{\frac{1}{3}} b \tag{24}$$

Then from equation (14) the B-QS potential is found as

$$V_{B_1}(r) = \alpha_{11} r^{\frac{2}{3}} + \alpha_{12}^{(n)} r^{-\frac{4}{3}} \tag{25}$$

$V_{B_1}(r)$ is the n-dependent potential, i.e. the Sturmian type potential. This special type of energy dependent potential is equipped with only a single normalized eigenstate. To make $V_{B_1}(r)$ normal/physical, we require $\alpha_{12}^{(n)}$ as the n-independent constant. This will be achieved only if $b \to b_n$ of the A-QS parameter such that $b_n = (\frac{3}{4})^{\frac{4}{3}} (\frac{-E_B}{a})^{-\frac{1}{3}} \alpha_{12}$, where α_{12} is n-independent parameter. This will make the potential $V_{B_1}(r)$ a normal/physical potential. As a consequence, the B-QS potential comes out as

$$V_{B_1}(r) = \alpha_{11} r^{\frac{2}{3}} + \alpha_{12} r^{-\frac{4}{3}} \tag{26}$$

Equations (13) and (20) yield the relation between the angular momentum quantum number ℓ_A of A-QS and ℓ_B of B-QS as

$$8\ell_A = 6\ell_B + 3D_B - 4D_A + 2 \tag{27}$$

The quantized energy eigenvalues of B-QS are

$$E_B = -\alpha_{11}^{\frac{3}{4}} \left[4(2\ell_B + D_B - \frac{2}{3})\right]^{\frac{1}{2}} \tag{28}$$

The parameters of the potential and angular momentum quantum number of B-QS are connected by the constraint equation

$$\alpha_{12} + 2\sqrt{2}\alpha_{11}^{\frac{1}{4}} \left(\ell_B + \frac{D_B}{2} - \frac{2}{3}\right) \left(\ell_B + \frac{D_B}{2} - \frac{1}{3}\right)^{\frac{1}{2}} = 0 \tag{29}$$

The corresponding exact energy eigenfunction of the normal B-QS is obtained from equation (7) and it turns out to be

$$\psi_B(r) = N_B r^{\ell_B} \exp\left(-\frac{3}{4}\alpha_{11}^{\frac{1}{2}} r^{\frac{4}{3}} + \frac{3}{4}\alpha_{11}^{-\frac{1}{2}} E_B r^{\frac{2}{3}}\right) \tag{30}$$

where the normalization constant is $N_B = \left[\frac{-E_B}{<ar^{-\frac{1}{2}}>_A}\right]$, which necessarily exists as the denominator and is the expectation value of a part of the A-QS potential.

Proceeding in a similar way and choosing $br^{-\frac{3}{2}}$ as the WP (equation (16)), we can generate one more QS. By using (10), we get the transformation function for B-QS as

$$g_B(r) = \left(-\frac{E_B}{b}\right)^2 \left(\frac{r}{4}\right)^4 \tag{31}$$

Equations (11), (12), (14) and (31) lead to B-QS potential

$$V_{B_2}(r) = \alpha_{21} r^6 + \alpha_{22} r^4 \tag{32}$$

where

$$\alpha_{21} = \left(\frac{1}{4}\right)^6 \left(-\frac{E_B}{b}\right)^4 (-E_A) \tag{33}$$

and

$$\alpha_{22} = \left(\frac{1}{4}\right)^4 \left(-\frac{E_B}{b}\right)^3 a \tag{34}$$

The energy eigenvalues of $V_{B_2}(r)$ are obtained as

$$E_B = \frac{\alpha_{22}}{2\sqrt{\alpha_{21}}}(2\ell_B + D_B) \tag{35}$$

The corresponding energy eigenfunction of the normal B-QS is obtained from equation (7) and it becomes

$$\psi_B(r) = N_B r^{\ell_B} \exp\left[-\frac{1}{4}\alpha_{21}^{\frac{1}{2}} r^4 - \frac{1}{2}\alpha_{21}^{\frac{1}{4}} (2\ell_B + D_B + 2)^{\frac{1}{2}} r^2\right] \tag{36}$$

The constraint relation between the parameters of the potential and angular momentum quantum number is obtained as

$$\alpha_{22} = 2\alpha_{21}^{\frac{3}{4}} (2\ell_B + D_B + 2)^{\frac{1}{2}} \tag{37}$$

Thus, V_{B_1} and V_{B_2} (equations (26) and (32)) are the new potentials which constitute the exact solution of the radial Schrödinger equation (15). Their corresponding wavefunctions, energy eigenvalues and parameter constraints have also been obtained.

3.2. Generation of ESPs from the singular two-fraction power potential

This potential has the form

$$V_A(r) = ar^{\frac{2}{3}} + br^{-\frac{2}{3}} + cr^{-\frac{4}{3}} \tag{38}$$

The exact energy eigenvalues and eigenfunctions of the A-QS potential are given as [12, 19, 22]

$$E_A = 2\sqrt{a}\left[\left(2\ell_A + D_A - \frac{2}{3}\right)\sqrt{a} + b\right]^{\frac{1}{2}} \tag{39}$$

$$\psi_A(r) = N_A r^{\ell_A} \exp\left[-\frac{3}{4}\sqrt{a}r^{\frac{4}{3}} + \frac{3}{4}\frac{E_A}{\sqrt{a}}r^{\frac{2}{3}}\right] \tag{40}$$

and the constraint condition between the parameters of the potential and ℓ_A is given by

$$\left(2\ell_A + D_A - \frac{4}{3}\right)\left[\left(2\ell_A + D_A - \frac{2}{3}\right)\sqrt{a} + b\right]^{\frac{1}{2}} - c = 0 \tag{41}$$

Choosing $ar^{\frac{2}{3}}$ as WP and utilizing equation (10), we find the transformation function as

$$g_B(r) = \left(-\frac{E_B}{a}\right)^{\frac{3}{8}}\left(\frac{4}{3}r\right)^{\frac{3}{4}} \tag{42}$$

Using equations (11), (12), (14) and (42), we get the new generated potential $V_{B_3}(r)$ and is given as

$$V_{B_3}(r) = \beta_{11}r^{-\frac{1}{2}} + \beta_{12}^{(n)}r^{-1} + \beta_{13}^{(n)}r^{-\frac{3}{2}} \tag{43}$$

where

$$\beta_{11} = \left(\frac{4}{3}\right)^{-\frac{1}{2}}\left(-\frac{E_B}{a}\right)^{\frac{3}{4}}(-E_A) = C_B^2 \tag{44}$$

$$\beta_{12}^{(n)} = \left(\frac{4}{3}\right)^{-1}\left(-\frac{E_B}{a}\right)^{\frac{1}{2}} b \tag{45}$$

$$\beta_{13}^{(n)} = \left(\frac{4}{3}\right)^{-\frac{3}{2}}\left(-\frac{E_B}{a}\right)^{\frac{1}{4}} c \tag{46}$$

C_B^2 is the characteristic constant of the B-QS.
$V_{B_3}(r)$ is a Sturmian potential. To make the potential normal, we require $\beta_{12}^{(n)}$ and $\beta_{13}^{(n)}$ in equation (43) as the n-independent constant. This is achieved by making A-QS parameter $b \to b_n = \frac{4}{3}\left(-\frac{E_B}{a}\right)^{-\frac{1}{2}}\beta_{12}$ and $c \to c_n = \left(\frac{4}{3}\right)^{\frac{3}{2}}\left(-\frac{E_B}{a}\right)^{-\frac{1}{4}}\beta_{13}$, where β_{12} and β_{13} are n-independent parameters. The normal B-QS potential turns out to be

$$V_{B_3}(r) = \beta_{11}r^{-\frac{1}{2}} + \beta_{12}r^{-1} + \beta_{13}r^{-\frac{3}{2}} \tag{47}$$

The related energy eigenvalues are given by equations (44), (46) and (41) (the A-QS constraint equation) and is given as

$$E_B = -\left[\frac{\beta_{11}}{4\beta_{13}}(3 - 4\ell_B - 2D_B)\right]^2 \tag{48}$$

The parameters of the potential and the angular momentum quantum number of B-QS are connected by a constraint relation

$$\begin{aligned}4\beta_{13}^2 =& \beta_{12}(4\ell_B + 2D_B - 3)^2 \\ & - \frac{\beta_{11}}{4\beta_{13}}(4\ell_B + 2D_B - 3)^3(2\ell_B + D_B - 1)\end{aligned} \tag{49}$$

The relationship between the angular momentum quantum number ℓ_A of A-QS and ℓ_B of B-QS is given by equation (13) as

$$\left(\ell_A + \frac{D_A}{2} - 1\right) = \frac{4}{3}\left(\ell_B + \frac{D_B}{2} - 1\right) \tag{50}$$

The corresponding exact energy eigenfunction of the normal B-QS can be obtained from equation (7) and is given as

$$\psi_B(r) = N_B r^{\ell_B} \exp\left[-(-E_B)^{\frac{1}{2}}r - \frac{\beta_{11}}{(-E_B)^{\frac{1}{2}}}r^{\frac{1}{2}}\right] \tag{51}$$

In a similar way, by choosing the terms $br^{-\frac{2}{3}}$ and $cr^{-\frac{4}{3}}$ as the WP (equation (38)) and then applying ET (equations (10) to (14)), we can generate two more exactly solved new QSs. The new generated potentials V_{B_3}, V_{B_4} and V_{B_5} as well as their corresponding wavefunctions, energy eigenvalues and parameter constraints are summarized in Table 1.

3.3. Generation of ESPs from the mixture potential

This potential has the form

$$V_A(r) = ar + br^{-1} + cr^2 \tag{52}$$

Table 1. New exactly solvable potentials in D_B dimensional Euclidean spaces generated from the potential $V_A(r) = ar^{\frac{2}{3}} + br^{-\frac{2}{3}} + cr^{-\frac{4}{3}}$.

$V_B(r)$	E_B	$\psi_B(r)$	Parameter constraint
$V_{B_3} = \beta_{11} r^{-\frac{1}{2}} + \beta_{12} r^{-1} + \beta_{13} r^{-\frac{3}{2}}$	$-\left[\frac{\beta_{11}}{4\beta_{13}}(3-4\ell_B-2D_B)\right]^2$	$r^{\ell_B}\exp\left[-(-E_B)^{\frac{1}{2}} r - \frac{\beta_{11}}{(-E_B)^{\frac{1}{2}}} r^{\frac{1}{2}}\right]$	$4\beta_{13}^2 + \frac{\beta_{11}}{4\beta_{13}}(4\ell_B+2D_B-3)^3(2\ell_B+D_B-1)$
$\beta_{11} = (\frac{4}{3})^{-\frac{1}{2}}(-\frac{E_B}{a})^{\frac{3}{4}}(-E_A)$			$-\beta_{12}(4\ell_B+2D_B-3)^2 = 0$
$\beta_{12} = (\frac{4}{3})^{-1}(-\frac{E_B}{a})^{\frac{1}{2}} b$			
$\beta_{13} = (\frac{4}{3})^{-\frac{3}{2}}(-\frac{E_B}{a})^{\frac{1}{4}} c$			
$V_{B_4} = \beta_{21} r + \beta_{22} r^{-1} + \beta_{23} r^2$	$-\left[\frac{\beta_{21}^2}{4\beta_{23}} - (2\ell_B+D_B)\sqrt{\beta_{23}}\right]$	$r^{\ell_B}\exp\left[-\frac{1}{2}\sqrt{\beta_{23}} r^2 - \frac{1}{2}\frac{\beta_{21}}{\sqrt{\beta_{23}}} r\right]$	$\beta_{22} + (2\ell_B+D_B-1)\frac{\beta_{21}}{2\sqrt{\beta_{23}}} = 0$
$\beta_{21} = \frac{2}{3}(-\frac{E_B}{b})^{\frac{3}{2}}(-E_A)$			
$\beta_{22} = \frac{3}{2}(-\frac{E_B}{b})^{\frac{1}{2}} c$			
$\beta_{23} = (\frac{2}{3})^2(-\frac{E_B}{b})^2 a$			
$V_{B_5} = \beta_{31} r^2 + \beta_{32} r^4 + \beta_{33} r^6$	$\frac{\beta_{32}}{2\sqrt{\beta_{33}}}(2\ell_B+D_B)$	$r^{\ell_B}\exp\left[-\frac{1}{4}\sqrt{\beta_{33}} r^4 - \frac{1}{4}\frac{\beta_{32}}{\sqrt{\beta_{33}}} r^2\right]$	$\frac{\beta_{32}^2}{4\beta_{33}} - \sqrt{\beta_{33}}(2\ell_B+D_B+2) - \beta_{31} = 0$
$\beta_{31} = (\frac{1}{3})^2(-\frac{E_B}{c})^2 b$			
$\beta_{32} = (\frac{1}{3})^4(-\frac{E_B}{c})^3(-E_A)$			
$\beta_{33} = (\frac{1}{3})^6(-\frac{E_B}{c})^4 a$			

The exact energy eigenvalues and eigenfunctions of the A-QS potential are given as

$$E_A = -\left[\frac{a^2}{4c} - (2\ell_A + D_A)\sqrt{c}\right] \tag{53}$$

$$\psi_A(r) = N_A r^{\ell_A} \exp\left[-\frac{1}{2}\sqrt{c} r^2 - \frac{1}{2}\frac{a}{\sqrt{c}} r\right] \tag{54}$$

and the constraint condition between the parameters of the potential and ℓ_A is given by

$$b + (2\ell_A + D_A - 1)\frac{a}{2\sqrt{c}} = 0 \tag{55}$$

Choosing ar, br^{-1} and cr^2 as WPs (equation (52)) and then applying ET (equations (10) to (14), we can generate three more exactly solved QSs. To avoid repetition, the new generated potentials V_{B_6}, V_{B_7} and V_{B_8} as well as their corresponding wavefunctions, energy eigenvalues and parameter constraints are summarized in Table 2.

3.4. Generation of ESPs from the singular even power potential

The potential has the form

$$V_A(r) = ar^2 + br^{-2} + cr^{-4} + dr^{-6} \tag{56}$$

The exact energy eigenvalues and eigenfunctions of the A-QS potential are given as [20, 22]

$$E_A = 2\sqrt{a}\left(2 + \frac{c}{2\sqrt{d}}\right) \tag{57}$$

$$\psi_A(r) = N_A r^{2+\frac{c}{2\sqrt{d}}-\frac{D_A}{2}} \exp\left[-\frac{\sqrt{a}}{2} r^2 - \frac{\sqrt{d}}{2} r^{-2}\right] \tag{58}$$

and the parameters of the potential and the angular momentum quantum number of the A-QS are linked by the constraint equation

$$\left(\frac{3}{2} + \frac{c}{2\sqrt{d}}\right)\left(\frac{1}{2} + \frac{c}{2\sqrt{d}}\right) - b - 2\sqrt{ad} - \left[(\ell_A + \frac{D_A - 2}{2})^2 - \frac{1}{4}\right] = 0 \tag{59}$$

Choosing $V_A^W(g_B(r)) = ag_B^2$ as the WP and then substituting it in (10), we find the transformation function as

$$g_B(r) = \left(-\frac{E_B}{a}\right)^{\frac{1}{4}} (2r)^{\frac{1}{2}} \tag{60}$$

We set the integration constant to zero for $g_B(r)$ to satisfy the required condition $g_B(0) = 0$ and $g_B(\infty) = \infty$. Equations (11), (12), (14) and (60) give

$$V_{B_9}(r) = \gamma_{11} r^{-1} + \gamma_{12} r^{-2} + \gamma_{13}^{(n)} r^{-3} + \gamma_{14}^{(n)} r^{-4} \tag{61}$$

and the coefficients γ_{11}, γ_{12}, $\gamma_{13}^{(n)}$ and $\gamma_{14}^{(n)}$ are required for complete specification of $V_B(r)$ and are given as

$$\gamma_{11} = \frac{1}{2}\left(-\frac{E_B}{a}\right)^{\frac{1}{2}}(-E_A) = C_B^2 \tag{62}$$

where C_B^2 is the characteristic constant of B-QS.

$$\gamma_{12} = \frac{b}{4} \tag{63}$$

Table 2. New exactly solvable potentials generated from the mixture potential $ar+br^{-1}+cr^2$.

$V_B(r)$	E_B	$\psi_B(r)$	Parameter constraint
$V_{B_1}=\rho_{11}r^{\frac{2}{3}}+\rho_{12}r^{-\frac{2}{3}}+\rho_{13}r^{-\frac{4}{3}}$ $\rho_{11}=(\frac{3}{2})^{\frac{2}{3}}(-\frac{E_B}{a})^{\frac{4}{3}}c$ $\rho_{12}=(\frac{3}{2})^{-\frac{2}{3}}(-\frac{E_B}{a})^{\frac{2}{3}}(-E_A)$ $\rho_{13}=(\frac{3}{2})^{-\frac{4}{3}}(-\frac{E_B}{a})^{\frac{1}{3}}b$	$2\sqrt{\rho_{11}}\left[(2\ell_B+D_B-\frac{2}{3})\sqrt{\rho_{11}}+\rho_{12}\right]^{\frac{1}{2}}$	$r^{\ell_B}\exp\left[-\frac{3}{4}\sqrt{\rho_{11}}r^{\frac{4}{3}}+\frac{3}{4}\frac{E_B}{\sqrt{\rho_{11}}}r^{\frac{2}{3}}\right]$	$(2\ell_B+D_B-\frac{4}{3})\left[(2\ell_B+D_B-\frac{2}{3})\sqrt{\rho_{11}}+\rho_{12}\right]^{\frac{1}{2}}$ $-\rho_{13}=0$
$V_{B_2}=\rho_{21}r^2+\rho_{22}r^4+\rho_{23}r^6$ $\rho_{21}=\frac{1}{4}(-\frac{E_B}{b})^2(-E_A)$ $\rho_{22}=\frac{1}{16}(-\frac{E_B}{b})^3a$ $\rho_{23}=\frac{1}{64}(-\frac{E_B}{b})^4c$	$\frac{\rho_{22}}{2\sqrt{\rho_{23}}}(2\ell_B+D_B)$	$r^{\ell_B}\exp\left[-\frac{1}{4}\sqrt{\rho_{23}}r^4-\frac{1}{4}\frac{\rho_{22}}{\sqrt{\rho_{23}}}r^2\right]$	$\frac{\rho_{22}^2}{4\rho_{23}}-\sqrt{\rho_{23}}(2\ell_B+D_B+2)-\rho_{21}=0$
$V_{B_3}=\rho_{31}r^{-\frac{1}{2}}+\rho_{32}r^{-1}+\rho_{33}r^{-\frac{3}{2}}$ $\rho_{31}=\frac{1}{\sqrt{2}}(-\frac{E_B}{c})^{\frac{3}{4}}a$ $\rho_{32}=\frac{1}{2}(-\frac{E_B}{c})^{\frac{1}{2}}(-E_A)$ $\rho_{33}=\frac{1}{2\sqrt{2}}(-\frac{E_B}{c})^{\frac{1}{4}}b$	$-\left[\frac{\rho_{31}}{4\rho_{33}}(3-4\ell_B-2D_B)\right]^2$	$r^{\ell_B}\exp\left[-(-E_B)^{\frac{1}{2}}r-\frac{\rho_{31}}{(-E_B)^{\frac{1}{2}}}r^{\frac{1}{2}}\right]$	$4\rho_{33}^2+\frac{\rho_{31}}{4\rho_{33}}(4\ell_B+2D_B-3)^3(2\ell_B+D_B-1)$ $-\rho_{32}(4\ell_B+2D_B-3)^2=0$

$$\gamma_{13}^{(n)}=\frac{1}{8}\left(-\frac{E_B}{a}\right)^{-\frac{1}{2}}c \tag{64}$$

$$\gamma_{14}^{(n)}=\frac{1}{16}\left(-\frac{E_B}{a}\right)^{-1}d \tag{65}$$

$V_{B_9}(r)$ is a Sturmian potential. To make the potential normal, we require $\gamma_{13}^{(n)}$ and $\gamma_{14}^{(n)}$ in equation (61) as the n-independent constant. This can be achieved by making A-QS parameter $c\to c_n=8(-\frac{E_B}{a})^{\frac{1}{2}}\gamma_{13}$ and $d\to d_n=16(-\frac{E_B}{a})\gamma_{14}$, where γ_{13} and γ_{14} are n-independent parameters. The normal B-QS potential becomes

$$V_{B_9}(r)=\gamma_{11}r^{-1}+\gamma_{12}r^{-2}+\gamma_{13}r^{-3}+\gamma_{14}r^{-4} \tag{66}$$

The energy eigenvalues of B-QS are

$$E_B=-\left[\frac{-\gamma_{11}}{(2+\frac{\gamma_{13}}{\sqrt{\gamma_{14}}})}\right]^2 \tag{67}$$

The parameters are connected by the constraint equation

$$\left(\frac{3}{2}+\frac{\gamma_{13}}{\sqrt{\gamma_{14}}}\right)\left(\frac{1}{2}+\frac{\gamma_{13}}{\sqrt{\gamma_{14}}}\right)-4\gamma_{12}+\frac{8\gamma_{11}\gamma_{14}}{(\gamma_{13}+2\sqrt{\gamma_{14}})} -\left[4\left(\ell_B+\frac{D_B-2}{2}\right)^2-\frac{1}{4}\right]=0 \tag{68}$$

The exact energy eigenfunction is obtained from equation (7) as

$$\psi_B(r)=N_Br^{1+\frac{\gamma_{13}}{2\sqrt{\gamma_{14}}}-\frac{D_B-1}{2}}\exp\left[-(-E_B)^{\frac{1}{2}}r-\sqrt{\gamma_{14}}r^{-1}\right] \tag{69}$$

Next, we consider the WP, $V_A^W(g_B(r))=bg_B^{-2}$ (from equation (56)). Utilizing (10), we find the B-QS transformation function as

$$g_B(r)=\exp(-\eta r) \tag{70}$$

where

$$\eta=\left(-\frac{E_B}{b}\right)^{\frac{1}{2}} \tag{71}$$

It is interesting to note that in this case the transformation function $g_B(r)$ is non power-law type. It will lead to a non power-law type B-QS potential. Unlike the power-law potentials, for non power-law potentials exact solutions are available only for s-waves. Equations (11), (12), (14) and (70) give the potential $V_B(r)$ as

$$V_{B_{10}}(r)=\gamma_{21}\exp(-2\eta r)+\gamma_{22}\exp(2\eta r)+\gamma_{23}\exp(-4\eta r)+\gamma_{24}\exp(4\eta r)+\frac{\eta^2}{4} \tag{72}$$

where

$$\gamma_{21}=\eta^2(-E_A) \tag{73}$$

$$\gamma_{22}=c\eta^2 \tag{74}$$

$$\gamma_{23}=a\eta^2 \tag{75}$$

$$\gamma_{24} = d\eta^2 \tag{76}$$

$V_{B_{10}}(r)$ is energy dependent potential and is equipped with only a single normalized eigenstate. This QS, however, remains Sturmian, as the potential is found to be non conformable to system specific conversion techniques that sometimes allow the conversion of Sturmian QS to normal/physical QS.

Taking A-QS parameter $b = 1$ and considering dimension $D_B = 1$ the energy eigenvalues and energy eigenfunction of the B-QS are

$$E_B = -\frac{1}{16}\left(-\frac{\gamma_{21}}{\sqrt{\gamma_{23}}} - \frac{\gamma_{22}}{\sqrt{\gamma_{24}}}\right)^2 \tag{77}$$

$$\psi_B(r) = N_B \exp\left[-\left(1+\frac{1}{2\eta}\frac{\gamma_{22}}{\sqrt{\gamma_{24}}}\right)\eta r\right]\exp\left[-\frac{1}{2}\frac{\sqrt{\gamma_{23}}}{\eta}\exp(-2\eta r) - \frac{1}{2}\frac{\sqrt{\gamma_{24}}}{\eta}\exp(2\eta r)\right] \tag{78}$$

respectively.

The B-QS potential parameters are restricted by

$$\left(\frac{3}{2}+\frac{1}{2\eta}\frac{\gamma_{22}}{\sqrt{\gamma_{24}}}\right)\left(\frac{1}{2}+\frac{1}{2\eta}\frac{\gamma_{22}}{\sqrt{\gamma_{24}}}\right) - \frac{2}{\eta^2}\sqrt{\gamma_{23}\gamma_{24}} - 1 = 0 \tag{79}$$

Similarly, choosing other terms of the potential (equation (56)) as the WP and then applying ET, we can generate some other QSs having different potentials. The new generated potentials V_{B_9}, $V_{B_{10}}$, $V_{B_{11}}$ and $V_{B_{12}}$ as well as their corresponding wavefunctions, energy eigenvalues and parameter constraints are summarized in Table 3.

4. Second order ET: multiplet structure of generated ESPs

Under ET, the daughter along with the parent potential form a multiplet structure. To show the multiplet structure, we choose the singular two-fraction power potential $ar^{\frac{2}{3}} + br^{-\frac{2}{3}} + cr^{-\frac{4}{3}}$ as the parent potential. Applying ET on B-QS potential $\beta_{11}r^{-\frac{1}{2}} + \beta_{12}r^{-1} + \beta_{13}r^{-\frac{3}{2}}$ (first order daughter potential (47)), we can generate one more QS, called C-QS. To get C-QS, we can use the same set of equations (equations (10) to (15)) provided we change the index B to C and the index A to B. By choosing $\beta_{11}r^{-\frac{1}{2}}$, $\beta_{12}r^{-1}$ and $\beta_{13}r^{-\frac{3}{2}}$ as WPs (equation 47) respectively, we can generate three exactly solved C-QSs. The new generated potentials V_{C_1}, V_{C_2} and V_{C_3} as well as their corresponding wavefunctions, energy eigenvalues and parameter constraints are given in Table 4. These potentials satisfy the C-QS radial Schrödinger equation given by

$$\psi_C''(r) + \frac{D_C-1}{r}\psi_C'(r) + \left(E_C - V_C(r) - \frac{\ell_C(\ell_C+D_C-2)}{r^2}\right)\psi_C(r) = 0 \tag{80}$$

Again, applying ET on the B-QS potential $\beta_{21}r + \beta_{22}r^{-1} + \beta_{23}r^2$ (second row in table 1), we get three exactly solved C-QSs similar to the QSs given in Table 2 (we must change the index B to C and parameters of the potential accordingly).

Thus the singular two-fraction power potential $ar^{\frac{2}{3}} + br^{-\frac{2}{3}} + cr^{-\frac{4}{3}}$ along with the potentials $ar^{-\frac{1}{2}} + br^{-1} + cr^{-\frac{3}{2}}$, $ar + br^{-1} + cr^2$ and $ar^2 + br^4 + cr^6$ form a quartet of exactly solved potentials. The advantage of the multiplet structure is that if any one of the QSs of the family is found, then all the others are exactly solvable and their properties can be very easily obtained through ET. When the exponent of the potential of system A (parent) and system B (daughter) are α_i and β_j respectively, then we obtain the following relations:

$$A_1B_1 = 4, \qquad A_1B_2 = 2A_3, \qquad A_1B_3 = 2A_2$$

where $A_i = \alpha_i + 2$ and $B_j = \beta_j + 2$.

These equations specify the B_j of the dual B-QS from the known A_is of A-QS and generalization of the duality of the one-term potentials. In other words, these relations allow us to find the potentials of the QSs that form the multiplet, without any detailed calculations.

5. Discussion and conclusion

In this paper, we have presented a simple and compact mapping procedure (ET method) to solve D-dimensional

Table 3. New exactly solvable potentials generated from the potential $ar^2+br^{-2}+cr^{-4}+dr^{-6}$.

$V_B(r)$	E_B	$\psi_B(r)$	Parameter constraint
$V_{B_9}(r)=\gamma_{11}r^{-1}+\gamma_{12}r^{-2}+\gamma_{13}r^{-3}+\gamma_{14}r^{-4}$ $\gamma_{11}=\frac{1}{2}(-\frac{E_B}{a})^{\frac{1}{2}}(-E_A)$ $\gamma_{12}=\frac{b}{4}$ $\gamma_{13}=\frac{1}{8}(-\frac{E_B}{a})^{-\frac{1}{2}}c$ $\gamma_{14}=\frac{1}{16}(-\frac{E_B}{a})^{-1}d$	$-\left[\frac{-\gamma_{11}}{(2+\frac{\gamma_{13}}{\sqrt{\gamma_{14}}})}\right]^2$	$r^{1+\frac{\gamma_{13}}{2\sqrt{\gamma_{14}}}-\frac{D_B-1}{2}}\exp\left[-(-E_B)^{\frac{1}{2}}r-\sqrt{\gamma_{14}}r^{-1}\right]$	$\left(\frac{3}{2}+\frac{\gamma_{13}}{\sqrt{\gamma_{14}}}\right)\left(\frac{1}{2}+\frac{\gamma_{13}}{\sqrt{\gamma_{14}}}\right)-4\gamma_{12}$ $+\frac{8\gamma_{11}\gamma_{14}}{(\gamma_{13}+2\sqrt{\gamma_{14}})}-\left[4\left(\ell_B+\frac{D_B-2}{2}\right)^2-\frac{1}{4}\right]=0$
$V_{B_{10}}(r)=\gamma_{21}\exp(-2\eta r)+\gamma_{22}\exp(2\eta r)$ $+\gamma_{23}\exp(-4\eta r)+\gamma_{24}\exp(4\eta r)+\frac{\eta^2}{4}$ $\gamma_{21}=\eta^2(-E_A)$ $\gamma_{22}=c\eta^2$ $\gamma_{23}=a\eta^2$ $\gamma_{24}=d\eta^2$	$-\frac{1}{16}\left(-\frac{\gamma_{21}}{\sqrt{\gamma_{23}}}-\frac{\gamma_{22}}{\sqrt{\gamma_{24}}}\right)^2$	$\exp\left[-\left(1+\frac{1}{2\eta}\frac{\gamma_{22}}{\sqrt{\gamma_{24}}}\right)\eta r\right]\times$ $\exp\left[-\frac{1}{2}\frac{\sqrt{\gamma_{23}}}{\eta}\exp(-2\eta r)-\frac{1}{2}\frac{\sqrt{\gamma_{24}}}{\eta}\exp(2\eta r)\right]$	$\left(\frac{3}{2}+\frac{1}{2\eta}\frac{\gamma_{22}}{\sqrt{\gamma_{24}}}\right)\left(\frac{1}{2}+\frac{1}{2\eta}\frac{\gamma_{22}}{\sqrt{\gamma_{24}}}\right)$ $-\frac{2}{\eta^2}\sqrt{\gamma_{23}\gamma_{24}}-1=0$
$V_{B_{11}}=\gamma_{31}r^2+\gamma_{32}r^{-2}+\gamma_{33}r^{-4}+\gamma_{34}r^{-6}$ $\gamma_{31}=d(-\frac{E_B}{c})^2$ $\gamma_{32}=b$ $\gamma_{33}=(-\frac{E_B}{c})^{-1}(-E_A)$ $\gamma_{34}=a(-\frac{E_B}{c})^{-2}$	$2\sqrt{\gamma_{31}}\left(2+\frac{\gamma_{33}}{2\sqrt{\gamma_{34}}}\right)$	$r^{2+\frac{\gamma_{33}}{2\sqrt{\gamma_{34}}}-\frac{D_B}{2}}\exp\left[-\frac{1}{2}\sqrt{\gamma_{31}}r^2-\frac{1}{2}\sqrt{\gamma_{34}}r^{-2}\right]$	$\left(\frac{3}{2}+\frac{\gamma_{33}}{2\sqrt{\gamma_{34}}}\right)\left(\frac{1}{2}+\frac{\gamma_{33}}{2\sqrt{\gamma_{34}}}\right)-\gamma_{32}$ $-2\sqrt{\gamma_{31}\gamma_{34}}-\left[\left(\ell_B+\frac{D_B-2}{2}\right)^2-\frac{1}{4}\right]=0$
$V_{B_{12}}=\gamma_{41}r^{-1}+\gamma_{42}r^{-2}+\gamma_{43}r^{-3}+\gamma_{44}r^{-4}$ $\gamma_{41}=\frac{c}{2}(-\frac{E_B}{d})^{\frac{1}{2}}$ $\gamma_{42}=\frac{b}{4}$ $\gamma_{43}=\frac{1}{8}(-\frac{E_B}{d})^{-\frac{1}{2}}(-E_A)$ $\gamma_{44}=\frac{a}{16}(-\frac{E_B}{d})^{-1}$	$-\left[\frac{-\gamma_{41}}{(2+\frac{\gamma_{43}}{\sqrt{\gamma_{44}}})}\right]^2$	$r^{1+\frac{\gamma_{43}}{2\sqrt{\gamma_{44}}}-\frac{D_B-1}{2}}\exp\left[-(-E_B)^{\frac{1}{2}}r-\sqrt{\gamma_{44}}r^{-1}\right]$	$\left(\frac{3}{2}+\frac{\gamma_{43}}{\sqrt{\gamma_{44}}}\right)\left(\frac{1}{2}+\frac{\gamma_{43}}{\sqrt{\gamma_{44}}}\right)-4\gamma_{42}$ $+\frac{8\gamma_{41}\gamma_{44}}{(\gamma_{43}+2\sqrt{\gamma_{44}})}-\left[4\left(\ell_B+\frac{D_B-2}{2}\right)^2-\frac{1}{4}\right]=0$

Table 4. New ESPs in D_C-dimensional Euclidean spaces generated from the B-QS potential $\beta_{11}r^{-\frac{1}{2}}+\beta_{12}r^{-1}+\beta_{13}r^{-\frac{3}{2}}$.

$V_C(r)$	E_C	$\psi_C(r)$	Parameter constraint
$V_{C_1}=\eta_{11}r^{\frac{2}{3}}+\eta_{12}r^{-\frac{2}{3}}+\eta_{13}r^{-\frac{4}{3}}$ $\eta_{11}=(\frac{3}{4})^{\frac{2}{3}}(-\frac{E_C}{\beta_{11}})^{\frac{4}{3}}(-E_B)$ $\eta_{12}=(\frac{3}{4})^{-\frac{2}{3}}(-\frac{E_C}{\beta_{11}})^{\frac{2}{3}}\beta_{12}$ $\eta_{13}=(\frac{3}{4})^{-\frac{4}{3}}(-\frac{E_C}{\beta_{11}})^{\frac{1}{3}}\beta_{13}$	$2\sqrt{\eta_{11}}\left[(2\ell_C+D_C-\frac{2}{3})\sqrt{\eta_{11}}+\eta_{12}\right]^{\frac{1}{2}}$	$r^{\ell_C}\exp\left[-\frac{3}{4}\sqrt{\eta_{11}}r^{\frac{4}{3}}+\frac{3}{4}\frac{E_B}{\sqrt{\eta_{11}}}r^{\frac{2}{3}}\right]$	$(2\ell_C+D_C-\frac{4}{3})\left[(2\ell_C+D_C-\frac{2}{3})\sqrt{\eta_{11}}+\eta_{12}\right]^{\frac{1}{2}}$ $-\eta_{13}=0$
$V_{C_2}=\eta_{21}r+\eta_{22}r^{-1}+\eta_{23}r^2$ $\eta_{21}=\frac{1}{2}(-\frac{E_C}{\beta_{12}})^{\frac{3}{2}}\beta_{11}$ $\eta_{22}=2(-\frac{E_C}{\beta_{12}})^{\frac{1}{2}}\beta_{13}$ $\eta_{23}=\frac{1}{4}(-\frac{E_C}{\beta_{12}})^2(-E_B)$	$-\left[\frac{\eta_{21}^2}{4\eta_{23}}-(2\ell_C+D_C)\sqrt{\eta_{23}}\right]$	$r^{\ell_C}\exp\left[-\frac{1}{2}\sqrt{\eta_{23}}r^2-\frac{1}{2}\frac{\eta_{21}}{\sqrt{\eta_{23}}}r\right]$	$\eta_{22}+(2\ell_C+D_C-1)\frac{\eta_{21}}{2\sqrt{\eta_{23}}}=0$
$V_{C_3}=\eta_{31}r^2+\eta_{32}r^4+\eta_{33}r^6$ $\eta_{31}=(\frac{1}{4})^2(-\frac{E_C}{\beta_{13}})^2\beta_{12}$ $\eta_{32}=(\frac{1}{4})^4(-\frac{E_C}{\beta_{13}})^3\beta_{11}$ $\eta_{33}=(\frac{1}{4})^6(-\frac{E_C}{\beta_{13}})^4(-E_B)$	$\frac{\eta_{32}}{2\sqrt{\eta_{33}}}(2\ell_C+D_C)$	$r^{\ell_C}\exp\left[-\frac{1}{4}\sqrt{\eta_{33}}r^4-\frac{1}{4}\frac{\eta_{32}}{\sqrt{\eta_{33}}}r^2\right]$	$\frac{\eta_{32}^2}{4\eta_{33}}-\sqrt{\eta_{33}}(2\ell_C+D_C+2)-\eta_{31}=0$

radial Schrödinger equation exactly for various power-law potentials. By taking, singular multi-term potentials $V(r)=ar^{-\frac{1}{2}}+br^{-\frac{3}{2}}$, $V(r)=ar^{\frac{2}{3}}+br^{-\frac{2}{3}}+cr^{-\frac{4}{3}}$, $V(r)=ar+br^{-1}+cr^2$ and $V(r)=ar^2+br^{-2}+cr^{-4}+dr^{-6}$ as input reference, we have generated different exactly solved QSs. We have showed multiplet structure between the singular two-fraction power potential $ar^{\frac{2}{3}}+br^{-\frac{2}{3}}+cr^{-\frac{4}{3}}$ and its daughter potentials. The restriction on the parameters of the potential, the orbital angular momentum quantum number ℓ and the spatial dimension D have been also given.

To implement ET, we have to select a potential term of the original multi-term potential $V_A(r)$, termed as the working potential (WP). The WP eventually specifies the form of basic transformation function $g_B(r)$ which is one of the important steps of ET method. The transformation procedure may be applied repeatedly by selecting WP differently to generate a variety of solved QSs. Successive application of ET to the generated QS will revert it back to the parent QS. For quantum multi-term potentials it is possible to generate a finite number of different exactly solved quantum systems by selecting working potential, as mentioned earlier. We however restrict ourselves to taking one term WP. Two or multi-term WP are usually avoided as they offer practical difficulties, say, the indefinite integral specifying the transformation function $g_B(r)$ cannot be evaluated analytically in most of the cases. Even if such integrals are found, they are of the form $F(g(r)) = r + C$ and the analytical inverse function $F^{-1}(g(r))$ of such a form cannot be determined. It is noteworthy that under ET, the constraint equation is converted to the energy eigenvalue expression and the energy eigenvalue to the constraint equation of the parent and daughter QSs. A very useful property of the ET method is that the wave functions of the generated QSs are almost always normalizable. Explicit expressions for the energy eigenvalues and eigenfunctions of various potentials are found.

In the case of multi-term power-law potential (n-term), successive application of ET produces no other new solved QS, but goes from one potential to another in a group of $(n+1)$ different QSs. The multiplet of ET has the advantage that if any one of the QSs of the family is found to be exactly solved, then all the others are exactly solvable and their properties can be very easily obtained through ET. It has been found that ET does not form a group in the usual sense because for generating third-order exactly solved potential from the initial exactly solved system, we are required to go through second-order generated potentials. This paper is an endeavor to increase the set of D-dimensional exactly solvable power-law potentials.

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