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Solution of the Dirac equation with magnetic monopole and pseudoscalar potentials

Research Article

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Abstract: The Dirac equation in the presence of the Dirac magnetic monopole potential, the Aharonov-Bohm potential, a Coulomb potential and a pseudo-scalar potential, is solved by separation of variables using the spin-weighted spherical harmonics. The energy spectrum and the form of the spinor functions are obtained. It is shown that the number j in spin-weighted spherical harmonics must be greater than $|q| - \frac{1}{2}$.

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Keywords: Dirac equation • Dirac monopole • Aharonov-Bohm potential • spin-weighted spherical harmonics • pseudo-scalar potential

1. Introduction

After the pioneering work of Dirac [1] on the magnetic monopole, much work has been done to investigate the Dirac equation in the presence of a magnetic monopole fields [2–4]. In particular, the Dirac equation has been solved by many different methods in the presence of a magnetic monopole field and the coulomb potential [5–11]. For example, the Dirac equation for spin 1/2 particle in the 't Hooft-Polyakov magnetic monopole has been studied and the corresponding wave function has been directly obtained by solving the associated system of differential equations and the boundary conditions [11]. Also, the problem of a Dirac electron in the presence of a combination of a coulomb field, a 1/r scalar potential, as well as a Dirac magnetic monopole and an Aharonov-Bohm potential, has been investigated by using the algebraic method of separation of variables [5]. Also, the energy spectrum and its dependence on the intensity of the Aharonov-Bohm and the magnetic monopole strengths have been calculated.

Newman and Penrose introduced the spin-weighted spherical harmonics [12]. These functions have attracted considerable attention, in particular, in solving systems of partial differential equations with tensor or spinor fields by separation of variables [13–16]. Torres del Castillo et al [15] have solved the Dirac equation for a particle subject to a Coulomb potential, a $1/r$ scalar potential, and the potential of a magnetic monopole. Here, the Dirac equation in the presence of the Dirac magnetic monopole potential, the Aharonov-Bohm potential, a Coulomb potential and a pseudo-scalar potential, is solved.

The Aharonov-Bohm effect is important in quantum theory. The Dirac equation with Aharonov-Bohm field has been

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solved and studied [17–25]. Meanwhile, the Schrödinger, Klein-Gordon and Dirac equations with the Aharonov-Bohm field in combination with the coulomb field and the magnetic monopole field have been investigated in Refs [5, 6, 9, 26–28].

In this paper, we want to consider a fermion in the presence of the Dirac magnetic monopole potential, the Aharonov-Bohm potential, a Coulomb potential and a pseudo-scalar potential and obtain the energy spectrum by using the spin-weighted spherical harmonics.

The paper is structured as follows: In section 2 the Dirac equation in the presence of the Dirac magnetic monopole potential, the Aharonov-Bohm potential, a Coulomb potential and a pseudo-scalar potential is written and a system of differential equations is obtained. In section 3 we introduce spin-weighted spherical harmonics by using the result of the Refs [12, 15, 29–34] and express the system of differential equations in terms of the lowering and raising operators of spin-weighted spherical harmonics. In section 4 the system of differential equations is solved by using the separation of variables, following which, the energy spectrum is obtained. In section 5 we check numerically that the series (26) are the solutions of the differential systems (25) which are the radial part of the Dirac equation.

2. Dirac equation

The Dirac equation in the presence of an electromagnetic field and a pseudo-scalar potential V_p is given by

$$i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t} = H\psi(\vec{r},t),$$
$$H = -i\hbar c\vec{\alpha}\cdot\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)+(mc^2+V_p)\beta+e\phi \tag{1}$$

where m is the rest-mass, e is the electric charge of the particle, $\vec{A}$ is the vector electromagnetic potential, ϕ is the scalar electromagnetic potential and V_P is a pseudoscalar potential. Also, α_i and β are the following Hermitian matrices

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{2}$$

The spinor function ψ is considered as follows

$$\psi(\vec{r},t) = e^{-i\frac{Et}{\hbar}}\begin{pmatrix} u(\vec{r}) \\ v(\vec{r}) \end{pmatrix} \tag{3}$$

where $u(\vec{r})$ and $v(\vec{r})$ are the two-component spinors and E is the energy eigenvalue. The pseudo-scalar potential is expressed in the spherical coordinates by the following form:

$$V_p = \mu\frac{\gamma_5}{r} \tag{4}$$

where μ is a constant number and $\gamma_5 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Regarding the relations (2), (3) and (4), the equation (1) takes the form:

$$(E-mc^2-e\phi)u = -i\hbar c\vec{\sigma}\cdot\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)v-\frac{\mu}{r}v$$
$$(E+mc^2-e\phi)v = -i\hbar c\vec{\sigma}\cdot\left(\vec{\nabla}-\frac{ie}{\hbar c}\vec{A}\right)u+\frac{\mu}{r}u. \tag{5}$$

The two component spinors can be written down in terms of a basis as [15]

$$u = u_-a+u_+b$$
$$v = v_-a+v_+b \tag{6}$$

where u_-, u_+, v_- and v_+ is complex functions and the complete basis $\{a,b\}$ are given by

$$a = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\varphi}{2}} \end{pmatrix}, \quad b = \begin{pmatrix} -\sin\frac{\theta}{2}e^{-i\frac{\varphi}{2}} \\ \cos\frac{\theta}{2}e^{i\frac{\varphi}{2}} \end{pmatrix}. \tag{7}$$

Using the Pauli matrices in the spherical coordinates, it is shown that

$$\vec{\sigma}\cdot\vec{\nabla}a = \frac{1}{r}a+\frac{\cot\theta}{r}b, \quad \vec{\sigma}\cdot\vec{\nabla}b = \frac{\cot\theta}{r}a-\frac{b}{r}. \tag{8}$$

By expressing the two component spinors u and v in terms of the spinors basis and considering the algebraic equation (8), we have the following systems of differential equations:

$$-\frac{i}{\hbar c}(E-mc^2-e\phi)u_- = -\frac{1}{r}\left(\frac{\partial}{\partial r}-\frac{ie}{\hbar c}A_r\right)(rv_-)$$
$$-\frac{1}{r}\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\cot\theta\right)v_+$$
$$+\frac{ie}{\hbar c}(A_\theta-iA_\varphi)v_++\frac{i\mu}{\hbar c}\frac{v_-}{r}$$
$$-\frac{i}{\hbar c}(E-mc^2-e\phi)u_+ = +\frac{1}{r}\left(\frac{\partial}{\partial r}-\frac{ie}{\hbar c}A_r\right)(rv_+)$$
$$-\frac{1}{r}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\cot\theta\right)v_-$$
$$+\frac{ie}{\hbar c}(A_\theta+iA_\varphi)v_-+\frac{i\mu}{\hbar c}\frac{v_+}{r},$$

$$-\frac{i}{\hbar c}(E+mc^2-e\phi)v_- = -\frac{1}{r}\left(\frac{\partial}{\partial r}-\frac{ie}{\hbar c}A_r\right)(ru_-) - \frac{1}{r}\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\cot\theta\right)u_+ + \frac{ie}{\hbar c}(A_\theta - iA_\varphi)u_+ - \frac{i\mu}{\hbar c}\frac{u_-}{r}$$
$$-\frac{i}{\hbar c}(E+mc^2-e\phi)v_+ = +\frac{1}{r}\left(\frac{\partial}{\partial r}-\frac{ie}{\hbar c}A_r\right)(ru_+) - \frac{1}{r}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{1}{2}\cot\theta\right)u_- + \frac{ie}{\hbar c}(A_\theta + iA_\varphi)u_- - \frac{i\mu}{\hbar c}\frac{u_+}{r} \quad (9)$$

where A_r, A_θ and A_φ are the components of the vector potential in terms of the spherical basis $\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\varphi\}$. Now in what follows we consider the special case where:

$$A_r = A_\theta = 0$$
$$A_\varphi = A_\varphi^{DM} + A_\varphi^{AB} = \frac{g(1-\cos\theta)}{r\sin\theta} + \frac{F}{r\sin\theta}. \quad (10)$$

In other words, we consider the Dirac monopole (A_φ^{DM}) and the Aharonov-Bohm (A_φ^{AB}) potentials. Moreover, g is the magnetic charge of a Dirac monopole and F is an arbitrary constant. The scalar electromagnetic potential is

$$\phi = -\frac{Ze}{r} \quad (11)$$

where $-Ze$ is the electric charge. By substituting the relations (10) and (11) in the relation (9), we obtain the following system of differential equations

$$-\frac{i}{\hbar c}(E-mc^2+\frac{Ze^2}{r})u_- = -\frac{1}{r}\frac{\partial}{\partial r}(rv_-) - \frac{1}{r}\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\left(q+\frac{1}{2}\right)\cot\theta - \frac{l+q}{\sin\theta}\right)v_+ \frac{i\mu}{\hbar c}\frac{v_-}{r}$$
$$-\frac{i}{\hbar c}(E-mc^2+\frac{Ze^2}{r})u_+ = +\frac{1}{r}\frac{\partial}{\partial r}(rv_+) - \frac{1}{r}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}-\left(q-\frac{1}{2}\right)\cot\theta + \frac{l+q}{\sin\theta}\right)v_- + \frac{i\mu}{\hbar c}\frac{v_+}{r},$$
$$-\frac{i}{\hbar c}(E+mc^2+\frac{Ze^2}{r})v_- = -\frac{1}{r}\frac{\partial}{\partial r}(ru_-) - \frac{1}{r}\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\left(q+\frac{1}{2}\right)\cot\theta - \frac{l+q}{\sin\theta}\right)u_+ + \frac{i}{\hbar c}\left(mc^2+\frac{ze^2}{r}\right)v_- - \frac{i\mu}{\hbar c}\frac{u_-}{r}$$
$$-\frac{i}{\hbar c}(E+mc^2+\frac{Ze^2}{r})v_+ = +\frac{1}{r}\frac{\partial}{\partial r}(ru_+) - \frac{1}{r}\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}-\left(q-\frac{1}{2}\right)\cot\theta + \frac{l+q}{\sin\theta}\right)u_- - \frac{i\mu}{\hbar c}\frac{u_+}{r} \quad (12)$$

where $q = \frac{eg}{\hbar c}$ and $l = \frac{eF}{\hbar c}$. According to Dirac's quantization, the dimensionless parameter q can only take the values $q = 0, \pm 1/2, \pm 1, \ldots$.

3. Spin-weighted spherical harmonics

The spin weighted spherical harmonics are introduced, up to a normalization factor, by [12, 15]:

$$\overline{\gamma}_s\gamma_s({}_sY_{jm}) = [s(s+1)-j(j+1)]_sY_{jm}$$
$$-i\frac{\partial}{\partial\varphi}{}_sY_{jm} = m_sY_{jm} \quad (13)$$

where the rising and lowering operators γ_s and $\overline{\gamma_s}$ are defined by

$$\gamma_{s\,s}Y_{jm} = -\left(\frac{\partial}{\partial\theta}+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}-s\cot\theta\right){}_sY_{jm}$$
$$\overline{\gamma}_{s\,s}Y_{jm} = -\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+s\cot\theta\right){}_sY_{jm} \quad (14)$$

Meanwhile, due to the normalization of the spin-weighted spherical harmonics and considering equation (14), the following equations are satisfied

$$\gamma_{s\,s}Y_{jm} = \sqrt{(j-s)(j+s+1)}_{s+1}Y_{jm}$$
$$\overline{\gamma}_{s\,s}Y_{jm} = -\sqrt{(j+s)(j-s+1)}_{s-1}Y_{jm} \quad (15)$$

and ${}_0Y_{jm} = Y_{jm}$. The indices s, j and m in ${}_sY_{jm}$ can take the following values:

$$\begin{aligned} s = &\ 0 \quad , \pm\frac{1}{2}, \pm 1 \cdots , \\ j = &\ s \quad , |s|+1, |s|+2, \cdots , \\ m = &- j, -j+1, \cdots .j-1, j \end{aligned} \tag{16}$$

It is clear that if s is integer (half integer), then j and m will be integer (half integer). Considering the form of the raising and lowering operators, one can present the equations (12) in terms of these operators as:

$$\begin{aligned} -\frac{i}{\hbar c}\left(E - mc^2 + \frac{ze^2}{r}\right) u_- &= -\frac{1}{r}\frac{\partial}{\partial r}(rv_-) + \frac{1}{r}\bar{\gamma}_{q+\frac{1}{2}} v_+ + \frac{q+l}{r\sin\theta} v_+ + \frac{i\mu}{\hbar c}\frac{v_-}{r}, \\ -\frac{i}{\hbar c}\left(E - mc^2 + \frac{ze^2}{r}\right) u_+ &= +\frac{1}{r}\frac{\partial}{\partial r}(rv_+) + \frac{1}{r}\gamma_{q-\frac{1}{2}} v_- - \frac{q+l}{r\sin\theta} v_- + \frac{i\mu}{\hbar c}\frac{v_+}{r}, \\ -\frac{i}{\hbar c}\left(E + mc^2 + \frac{ze^2}{r}\right) v_- &= -\frac{1}{r}\frac{\partial}{\partial r}(ru_-) + \frac{1}{r}\bar{\gamma}_{q+\frac{1}{2}} u_+ + \frac{q+l}{r\sin\theta} u_+ - \frac{i\mu}{\hbar c}\frac{u_-}{r}, \\ -\frac{i}{\hbar c}\left(E + mc^2 + \frac{ze^2}{r}\right) v_+ &= +\frac{1}{r}\frac{\partial}{\partial r}(ru_+) + \frac{1}{r}\gamma_{q-\frac{1}{2}} u_- - \frac{q+l}{r\sin\theta} u_- - \frac{i\mu}{\hbar c}\frac{u_+}{r}. \end{aligned} \tag{17}$$

4. Separation of variables

For the equations (17), one may suggest separable solutions like:

$$\begin{aligned} v_+ &= A(r)_{q+\frac{1}{2}} Y_{jm} e^{i(q+l)\varphi}, \\ v_- &= B(r)_{q-\frac{1}{2}} Y_{jm} e^{i(q+l)\varphi}, \\ u_+ &= C(r)_{q+\frac{1}{2}} Y_{jm} e^{i(q+l)\varphi}, \\ u_- &= D(r)_{q-\frac{1}{2}} Y_{jm} e^{i(q+l)\varphi}, \end{aligned} \tag{18}$$

where $A(r)$, $B(r)$, $C(r)$ and $D(r)$ are radial functions. In fact, nontrivial solutions appear when the quantum number j satisfies $j \geq |q+\frac{1}{2}|$ or $j \geq |q-\frac{1}{2}|$. Substituting equations (18) in equations (17), we have

$$\begin{aligned} \frac{1}{r}\frac{d}{dr}(rB) - \frac{i\mu}{\hbar c}\frac{1}{r}B + \frac{\kappa}{r}A &= +\frac{i}{\hbar c}(E - mc^2)D + \frac{i}{\hbar c}\frac{ze^2}{r}D \\ \frac{1}{r}\frac{d}{dr}(rA) + \frac{i\mu}{\hbar c}\frac{1}{r}A + \frac{\kappa}{r}B &= -\frac{i}{\hbar c}(E - mc^2)C - \frac{i}{\hbar c}\frac{ze^2}{r}C \\ \frac{1}{r}\frac{d}{dr}(rD) + \frac{i\mu}{\hbar c}\frac{1}{r}D + \frac{\kappa}{r}C &= +\frac{i}{\hbar c}(E + mc^2)B + \frac{i}{\hbar c}\frac{ze^2}{r}B \\ \frac{1}{r}\frac{d}{dr}(rC) - \frac{i\mu}{\hbar c}\frac{1}{r}C + \frac{\kappa}{r}D &= -\frac{i}{\hbar c}(E + mc^2)A - \frac{i}{\hbar c}\frac{ze^2}{r}A \end{aligned} \tag{19}$$

where $\kappa = \sqrt{(j+\frac{1}{2})^2 - q^2}$. We assume that the parameter μ is a pure imaginary number i.e. $\mu = i\upsilon$. Now, using the change of variables $rA = L'$, $rB = M'$, $irC = N'$, $irD = O'$ and $\rho = \sqrt{-\eta\zeta}r$ where $\eta = \frac{1}{\hbar c}(E - mc^2)$ and $\zeta = \frac{1}{\hbar c}(E + mc^2)$, the equations (19) are converted to

$$\begin{aligned} \frac{dM'}{d\rho} + \frac{\upsilon}{\hbar c}\frac{M'}{\rho} + \frac{\kappa}{\rho}L' &= +\sqrt{\frac{-\eta}{\zeta}}O' + \frac{z\alpha}{\rho}O' \\ \frac{dL'}{d\rho} - \frac{\upsilon}{\hbar c}\frac{L'}{\rho} + \frac{\kappa}{\rho}M' &= -\sqrt{\frac{-\eta}{\zeta}}N' - \frac{z\alpha}{\rho}N' \\ \frac{dO'}{d\rho} - \frac{\upsilon}{\hbar c}\frac{O'}{\rho} + \frac{\kappa}{\rho}N' &= -\sqrt{\frac{-\zeta}{\eta}}M' - \frac{z\alpha}{\rho}M' \\ \frac{dN'}{d\rho} + \frac{\upsilon}{\hbar c}\frac{N'}{\rho} + \frac{\kappa}{\rho}O' &= +\sqrt{\frac{-\zeta}{\eta}}L' + \frac{z\alpha}{\rho}L' \end{aligned} \tag{20}$$

where $\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant. By investigating the asymptotic behavior when $\rho \to \infty$, the equations (20) become

$$\frac{dM'}{d\rho} = \sqrt{\frac{-\eta}{\zeta}}O', \qquad \frac{dO'}{d\rho} = -\sqrt{-\frac{\zeta}{\eta}}M' \tag{21}$$

and

$$\frac{dL'}{d\rho} = -\sqrt{\frac{-\eta}{\zeta}}N', \qquad \frac{dN'}{d\rho} = \sqrt{-\frac{\zeta}{\eta}}L'. \tag{22}$$

The equations (21) give

$$\frac{d^2O'}{d\rho^2} - O' = 0, \qquad \frac{d^2M'}{d\rho^2} - M' = 0 \tag{23}$$

and the equations (22) yield

$$\frac{d^2L'}{d\rho^2} - L' = 0, \qquad \frac{d^2N'}{d\rho^2} - N' = 0. \tag{24}$$

As a result, the functions L', M', N' and O' should be proportional to $e^{-\rho}$ i.e. $L' = e^{-\rho}L$, $M' = e^{-\rho}M$, $N' = e^{-\rho}N$ and $O' = e^{-\rho}O$. So the equation (20) are change to

$$\begin{aligned}
&\frac{dM}{d\rho} - (-\frac{\upsilon}{\hbar c\rho} + 1)M + \frac{\kappa}{\rho}L = +\sqrt{\frac{-\eta}{\zeta}}O + \frac{z\alpha}{\rho}O \\
&\frac{dL}{d\rho} + (-\frac{\upsilon}{\hbar c\rho} - 1)L + \frac{\kappa}{\rho}M = -\sqrt{\frac{-\eta}{\zeta}}N - \frac{z\alpha}{\rho}N \\
&\frac{dO}{d\rho} + (-\frac{\upsilon}{\hbar c\rho} - 1)O + \frac{\kappa}{\rho}N = -\sqrt{\frac{-\zeta}{\eta}}M - \frac{z\alpha}{\rho}M \\
&\frac{dN}{d\rho} - (-\frac{\upsilon}{\hbar c\rho} + 1)N + \frac{\kappa}{\rho}O = +\sqrt{\frac{-\zeta}{\eta}}L + \frac{z\alpha}{\rho}L
\end{aligned} \tag{25}$$

In order to solve the above system of differential equations, one may use the Frobenius series solutions as

$$\begin{aligned}
M &= \textstyle\sum_{p=0}^{\infty} m_p \rho^{p+s}, \qquad N = \sum_{p=0}^{\infty} n_p \rho^{p+s}, \\
L &= \textstyle\sum_{p=0}^{\infty} l_p \rho^{p+s}, \qquad O = \sum_{p=0}^{\infty} o_p \rho^{p+s}.
\end{aligned} \tag{26}$$

Substituting the above series in the equations (20) and doing some calculations, we obtain the recursion relations

$$\begin{aligned}
m_{p-1} + \sqrt{\frac{-\eta}{\zeta}}o_{p-1} &= m_p\left(p + s + \frac{\upsilon}{\hbar c}\right) - z\alpha o_p + \kappa l_p, \\
l_{p-1} - \sqrt{\frac{-\eta}{\zeta}}n_{p-1} &= l_p\left(p + s - \frac{\upsilon}{\hbar c}\right) + z\alpha n_p + \kappa m_p, \\
o_{p-1} - \sqrt{\frac{-\zeta}{\eta}}m_{p-1} &= o_p\left(p + s - \frac{\upsilon}{\hbar c}\right) + z\alpha m_p + \kappa n_p, \\
n_{p-1} + \sqrt{\frac{-\zeta}{\eta}}l_{p-1} &= n_p\left(p + s + \frac{\upsilon}{\hbar c}\right) - z\alpha l_p + \kappa o_p
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
\left(s + \frac{\upsilon}{\hbar c}\right) m_0 + \kappa l_0 &= +z\alpha o_0, \\
\left(s - \frac{\upsilon}{\hbar c}\right) l_0 + \kappa m_0 &= -z\alpha n_0, \\
\left(s - \frac{\upsilon}{\hbar c}\right) o_0 + \kappa n_0 &= -z\alpha m_0, \\
\left(s + \frac{\upsilon}{\hbar c}\right) n_0 + \kappa o_0 &= z\alpha l_0.
\end{aligned} \tag{28}$$

The equations (28) lead to the following result

$$s = \sqrt{\kappa^2 + \frac{\upsilon^2}{\hbar^2 c^2} - z^2\alpha^2}. \tag{29}$$

where we have considered a positive root in order to avoid a divergent solution at the origin. On the other hand, the recursion relation (27) can be converted to

$$\begin{aligned}
&\left[\sqrt{\frac{-\zeta}{\eta}}\left(p + s + \frac{\upsilon}{\hbar c}\right) + z\alpha\right] m_p + \sqrt{\frac{-\zeta}{\eta}}\kappa l_p = \\
&-\left(p + s - \frac{\upsilon}{\hbar c} - \sqrt{\frac{-\zeta}{\eta}}z\alpha\right) o_p \\
&-\kappa n_p - \left(p + s - \frac{\upsilon}{\hbar c} - \sqrt{\frac{-\eta}{\zeta}}z\alpha\right) l_p - \kappa m_p = \\
&\left[\sqrt{\frac{-\eta}{\zeta}}\left(p + s + \frac{\upsilon}{\hbar c}\right) + z\alpha\right] n_p + \sqrt{\frac{-\eta}{\zeta}}\kappa o_p.
\end{aligned} \tag{30}$$

The series (26) will have good behavior at infinity if they terminate for a finite value N [35, 36]. So we assume

$$n_{X+1} = l_{D+1} = o_{M+1} = m_{N+1} = 0.$$

Also, we assume $N > M > D > X$. So

$$n_N = n_{N+1} = l_N = l_{N+1} = o_N = o_{N+1} = m_{N+1} = 0.$$

Regarding the equation (27), it appears that $m_N = 0$, which is in contradiction with our initial assumption. So, all the series contain an equal number of terms, that is,

$$o_{N+1} = m_{N+1} = l_{N+1} = n_{N+1} = 0.$$

So by using the equations (27), we have

$$\begin{aligned}
m_N &= -\sqrt{\frac{-\eta}{\zeta}}o_N \\
l_N &= \sqrt{\frac{-\eta}{\zeta}}n_N.
\end{aligned} \tag{31}$$

Substituting the relation (31) in (30), one obtains

$$E_N = \frac{mc^2(N+s)}{\sqrt{z^2\alpha^2 + (N+s)^2}} \tag{32}$$

where $N = 0, 1, 2, \ldots$ and s is given by (29).

Now we follow the special case $j = |q| - \frac{1}{2}$. If $q > 0$, then the wave functions are

$$\begin{aligned} v_- &= B(r)_{q-\frac{1}{2}} Y_{q-\frac{1}{2},m} e^{i(q+l)\varphi} \\ u_- &= D(r)_{q-\frac{1}{2}} Y_{q-\frac{1}{2},m} e^{i(q+l)\varphi} \\ u_+ &= v_+ = 0. \end{aligned} \tag{33}$$

If $q < 0$, then the wave functions are

$$\begin{aligned} v_+ &= A(r)_{q+\frac{1}{2}} Y_{q-\frac{1}{2},m} e^{i(q+l)\varphi} \\ u_+ &= C(r)_{q+\frac{1}{2}} Y_{q-\frac{1}{2},m} e^{i(q+l)\varphi} \\ u_- &= v_- = 0. \end{aligned} \tag{34}$$

At the same time, we have

$$\kappa = 0, \qquad s = \sqrt{\frac{\upsilon^2}{\hbar^2 c^2} - z^2\alpha^2}. \tag{35}$$

It is seen that if μ is real then s will not be real, so the special case $j = |q| - 1/2$ is not valid and j must be greater than $|q| - 1/2$.

5. Numerical investigation

In this section we want to verify numerically that the series (26) are the solutions of the differential systems (25).

The dimensionless parameter q can only take the values $q = 0, \ \pm 1/2, \ \pm 1, \ \ldots$ in according to the Dirac quantization requirement. So, for example, we consider $q = 1/2$. A nontrivial solution suggests that the quantum number j in equations (18) must satisfy the inequality $j \geq |q - 1/2|$ or $j \geq |q + 1/2|$ thus $j = |q| - 1/2, \ |q| + 1/2, \ |q| + 3/2, \ldots$ if $q \neq 0$. By considering this condition, we assume $j = 2$ for instance. As a result, the parameter κ is $\kappa = 2.4495$. We consider $z = 1$, $\upsilon = 2 \times 10^{-26}$ and the equations (29) and (32) give the parameters S and E, respectively. According to the relations (27) it is clear that the parameters l_0 and m_0 can be arbitrary constants. Therefore, we choose $l_0 = 2$ and $m_0 = 3$. Also, the parameters n_0 and o_0 are given by $n_0 = -8.6037$ and $o_0 = 11.5495$. So, we can obtain all of the constants l_p, m_p, n_p and o_p for $p = 1, \ 2, \ 3, \ \ldots$ in the series (26) by using the recursion relations (27). Hence, all the terms of the series are obtained and we can plot the functions $L(\rho)$, $M(\rho)$, $N(\rho)$ and $O(\rho)$ in terms of ρ.

At the same time, we can numerically solve the differential systems (25) by using the initial conditions. For example, we consider the initial condition as

$$\begin{pmatrix} L(\rho(0)) \\ M(\rho(0)) \\ N(\rho(0)) \\ O(\rho(0)) \end{pmatrix} = \begin{pmatrix} -2 \times 10^{-4} \\ -10^{-4} \\ 9 \times 10^{-4} \\ -10^{-3} \end{pmatrix}$$

where $\rho(0) = 0.1$. In Figures 1, 2, 3 and 4 we compare the two numerically solutions, series solution and differential systems solutions, and show that they are consistent with each other.

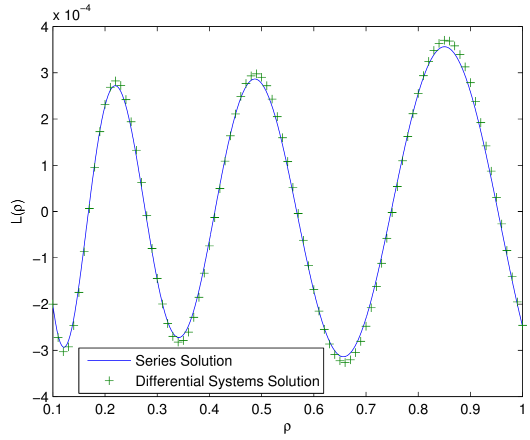

Figure 1. The plot of $L(\rho)$ in terms of ρ when $N = 100$

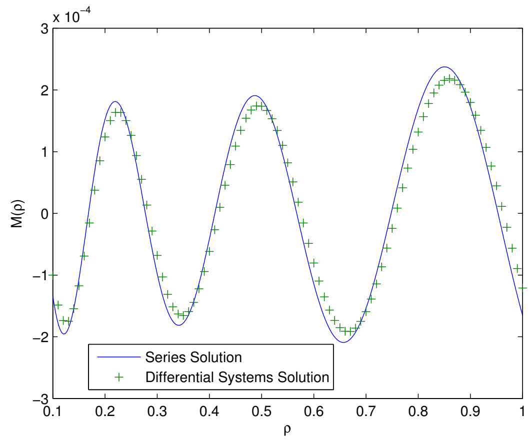

Figure 2. The plot of $M(\rho)$ in terms of ρ when $N = 100$

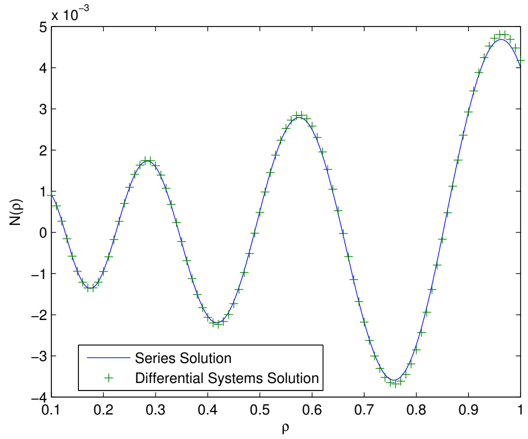

Figure 3. The plot of $N(\rho)$ in terms of ρ when $N = 100$

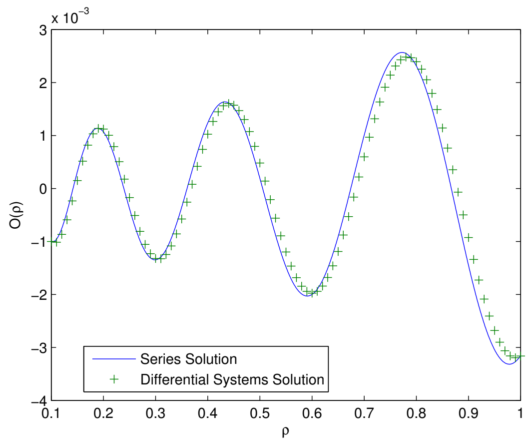

Figure 4. The plot of $O(\rho)$ in terms of ρ when $N = 100$

6. Conclusion

The Dirac equation in the presence of the Aharonov-Bohm field, magnetic monopole field and some other potentials, and for different choices of gauge has been studied by many authors. In this paper, the Dirac equation is solved by the separation of variables using the spin weighted spherical harmonics in the presence of the Dirac magnetic magnetic potential, the Aharonov-Bohm potential, a coulomb potential and a pseudo-scalar potential. It is noticed that the Aharonov-Bohm potential does not play any role in the energy spectrum. However, in the wave function, the angular part is related to the Aharonov-Bohm potential. In the other words, when this potential is not present, it is sufficient that to take $l = 0$ in the wave function. Moreover in the case of $j = |q| - 1/2$, it is seen that s is an imaginary number. However in the Frobenius series, s must be a real number. So, one can conclude that j must be greater than $|q| - 1/2$ in the presence of the magnetic monopole and pseudo-scalar potentials when μ is real.

In the section 5 we numerically investigate whether the series (26) are the solutions of the differential systems (25). We obtain all the terms of the series by using the recursion relation and plot the functions $L(\rho)$, $M(\rho)$, $N(\rho)$ and $O(\rho)$. Moreover, we solve numerically the differential systems, then compare this with the results of the series solution and show that they are consistent with each other.

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