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Bound state of solution of Dirac-Coulomb problem with spatially dependent mass

Research Article

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Abstract: The bound state solution of Coulomb Potential in the Dirac equation is calculated for a position dependent mass function $M(r)$ within the framework of the asymptotic iteration method (AIM). The eigenfunctions are derived in terms of hypergeometric function and function generator equations of AIM.

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1. Introduction

The solution of the relativistic Dirac equation for quantum mechanical systems in cases of both spatially dependent mass and constant mass plays an important role in many branches of physics [1–18]. The Dirac equation with position dependent masshas attracted greater interest as its importance has been recognised [5–15]. This equation has been addressed for solvable potentials by a number of different methods [19–30]. Ciftci et al. [31–33] recently proposed an alternative method, the asymptotic iteration method (AIM) which draws the attention of a many researchers for relativistic equations [34–40].
This method has the advantage of obtaining the solution of an eigenvalue problem without needing to obtain a direct solution to the differential equation. The Dirac equation with first order differential with two dimensional (2D) Coulomb potential for constant mass, [41–47] and for spatially dependent mass, [48] has been solved by using AIM. M. Hamzavi and collaborators [48] have considered the Coulomb potential including a Coulomb-like tensor potential under the pseudospin symmetry limit.
The main purpose of this study is to solve the spectrum of Coulomb potential for position dependent mass in the case of the Dirac equation without Coulomb-like tensor potential. This will be approached by considering the relation between vector and scalar potential as $S(r) = V(r)(b-1)$ where b is an arbitrary parameter [37, 38]. The second section includes the formalism of the Dirac equation with position dependent mass. The asymptotic iteration method is introduced in Section 3. The calculation of eigenvalues and eigenfunctions of Coulomb potential is then outlined in the subsequent section. The last section devotes to conclusions.

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2. Formalism of the Dirac equation

The Dirac equation for a central field in 3-dimensions is written for spherically symmetric vector potential $V(r)$ and spherically symmetric scalar potential $S(r)$ by using the parameters $\hbar = c = 1$ as

$$E_{nl}\ \Psi = \left[\sum_{j=1}^{3} \alpha_j p_j + \beta(m + S(r)) + V(r)\right]\Psi, \qquad (1)$$

where m is the mass of the particle, $S(r)$ is a spherically symmetric scalar potential, $V(r)$ is a spherically symmetric vector potential, α and β are the usual Dirac matrices satisfying anticommutation relations, and E_{nl} is the corresponding set of eigenvalues. After some algebraic calculations, one obtains the following first-order linear coupled differential equations (1).
Without any approximation, the Dirac equation for a central field in spherical coordinates can be separated into the variables. Thus the eigenfunction of the orbital and spin angular momentum can be found as

$$\begin{aligned}\frac{dF_{nk}(r)}{dr} + \frac{k}{r}F_{nk}(r) &= (E_{nl} + M(r) \\ &- V(r) + S(r))\,Q_{nk}(r) \qquad (2)\\ \frac{dQ_{nk}(r)}{dr} - \frac{k}{r}Q_{nk}(r) &= -(E_{nl} - M(r) \\ &- V(r) - S(r))\,F_{nk}(r), \qquad (3)\end{aligned}$$

where $k = -(l+1)$ for the total angular momentum $j = l + 1/2$, and l is angular momentum quantum number. $F_{nk}(r)$ and $Q_{nk}(r)$ are the radial wave function of the upper and the lower-spinor components respectively, and the general form of two second-order differential equations for corresponding eigenfunctions are obtained by eliminating wave function $F_{nk}(r)$ in Eq. (2) and $Q_{nk}(r)$ in Eq. (3) we get

$$\left[\frac{d^2}{dr^2} - \frac{k(k+1)}{r^2}\right]F_{nk}(r) - \frac{\left(\frac{dM(r)}{dr} - \frac{d\Delta(r)}{dr}\right)\left(\frac{d}{dr} + \frac{k}{r}\right)F_{nk}(r)}{M(r) + E_{nl} - \Delta(r)} = \left[(M(r) + E_{nl} - \Delta(r))(M(r) - E_{nl} + \sum(r))\right]F_{nk}(r) \qquad (4)$$

$$\left[\frac{d^2}{dr^2} - \frac{k(k-1)}{r^2}\right]Q_{nk}(r) - \frac{\left(\frac{dM(r)}{dr} + \frac{d\Delta(r)}{dr}\right)\left(\frac{d}{dr} - \frac{k}{r}\right)Q_{nk}(r)}{M(r) - E_{nl} + \Delta(r)} = \left[(M(r) + E_{nl} - \Delta(r))(M(r) - E_{nl} + \sum(r))\right]Q_{nk}(r), \qquad (5)$$

where

$$\sum(r) = V(r) + S(r), \text{ and } \Delta(r) = V(r) - S(r).$$

We use the relationship between scalar and vector potentials,

$$S(r) = V(r)(b-1), b \geq 0$$

to define
$\sum(r)$ and $\Delta(r)$.
In this general description of scalar potential, by choosing values for parameter b of 0, 1, and 2, the scalar potential leads to the case of $S(r) = -V(r)$, $S(r) = 0$ (purely vector potential), and $S(r) = V(r)$ respectively. The other choices of b lead to the required condition for the case of $S(r) > V(r)$. This transformation yields to

$$\sum(r) = bV(r), \text{ and } \Delta(r) = (2-b)V(r).$$

3. Asymptotic iteration method

The AIM is proposed to solve the second-order linear differential equation in the for

$$y'' = \lambda_0(x)y' + s_0(x)y, \qquad (6)$$

where the functions $\lambda_0(x)$ and $s_0(x)$ are differentiable with $\lambda_0(x) \neq 0$. The solution of any differential equation which can be written in form of Eq. (6) has a general solution within the framework AIM

$$\begin{aligned} y(x) &= \exp\left(-\int \alpha\, dt\right)\Big[C_2 \\ &+ C_1 \int \exp\left(\int (\lambda_0(\tau) + 2\alpha(\tau))d\tau\right)dt\Big],\end{aligned}$$

where C_i are integral constants. The arbitrary functions for the limit of n are

$$\begin{aligned}\lambda_n(x) &= \lambda'_{n-1}(x) + s_{n-1}(x) + \lambda_0(x)\lambda_{n-1}(x)\\ s_n(x) &= s'_{n-1}(x) + s_0(x)\lambda_{n-1}(x).\end{aligned}$$

with asymptotic expression

$$\frac{s_n(x)}{\lambda_n(x)} = \frac{s_{n-1}(x)}{\lambda_{n-1}(x)} = \alpha(x).$$

And the termination condition is in the form of

$$\begin{aligned}\Delta_k(x) &= \begin{vmatrix} s_n(x) & \lambda_n(x) \\ s_{n-1}(x) & \lambda_{n-1}(x) \end{vmatrix}\\ &= \lambda_{n-1}(x)s_n(x) - \lambda_n(x)s_{n-1}(x), \qquad (7)\\ & \quad k = 1, 2, 3, \ldots .\end{aligned}$$

which gives the solution for physical systems.

4. Solution of the Dirac–Coulomb problem

The Dirac-Coulomb potential is considered by proposing the Coulomb as vector potential and scalar potential

$$V(r) = \frac{V_0}{r}, \text{ yields to } S(r) = \frac{V_0(b-1)}{r}, \qquad (8)$$

where V_0 is an arbitrary real constant. The Eq. (4) can not be solved analytically because of the effect of the last term $(\frac{dM(r)}{dr} - \frac{d\Delta(r)}{dr})$. Therefore, we calculate the required mass function that satisfies the equality $(\frac{dM(r)}{dr} - \frac{d\Delta(r)}{dr} = 0)$ to eliminate this effect. Thus, using this equality condition, the mass function is obtained as the following function

$$M(r) = \frac{(2-b)V_0}{r} + m_0, \qquad (9)$$

where m_0 is the rest mass of the fermionic particle and $(2-b)V_0$ is the perturbed mass [49]. By substituting the potential functions in Eq. (8) and variable mass function in Eq. (9) into Eq. (4), we obtain

$$\frac{-k(k+1)}{r^2}F_{nk}(r) - (E_{nl} + m_0)\left(-E_{nl} + \frac{2V_0(E_{nl}+m_0)}{r} + m_0\right)F_{nk}(r) + \frac{d^2}{dr^2}F_{nk}(r) = 0. \qquad (10)$$

At this point, if the following notations are made

$$-E_{nl}^2 + m_0^2 = \varepsilon_{nl}^2, k(k+1) = A(A+1), -2(E_{nl}+m_0)V_0 = B,$$

the eigenvalues equation transforms to

$$\left(-\varepsilon_{nl}^2 - \frac{A(A+1)}{r^2} + \frac{B}{r}\right)F_{nk}(r) + \frac{d^2F_{nk}(r)}{dr^2} = 0. \qquad (11)$$

The wavefunction may be written by using AIM as

$$F_{nk}(r) = r^{A+1}\exp[-\varepsilon_{nl}]\chi(r). \qquad (12)$$

Then, by substituting this wavefunction into Eq. (11), we obtain

$$(B - 2(1+A)\varepsilon_{nl})\chi(r) + 2(1 + A - \varepsilon_{nl}r)\chi'(r) + r\chi''(r) = 0.$$

Thus, $\chi''(r)$ becomes

$$\chi''[r] = \frac{-2(1+A-\varepsilon_{nl}r)}{r}\chi'[r] + \frac{-B+2(1+A)\varepsilon_{nl}}{r}\chi[r]. \qquad (13)$$

This equation is now amenable to apply the AIM. After comparing Eq. (13) with the second-order differential equation Eq. (6), we get the arbitrary functions $\lambda_0(r)$ and $s_0(r)$ as

$$\lambda_0(r) = \frac{-2(1+A-\varepsilon_{nl}r)}{r}, s_0(r) = \frac{-B+2(1+A)\varepsilon_{nl}}{r}. \qquad (14)$$

By using the termination condition for energy, the general form of eigenvalue is calculated

$$\varepsilon_{nl} = \frac{B}{2(n+1+A)}. \qquad (15)$$

Returning to the parameters definition, we have

$$-E_{nl}^2 - m_0^2 = \varepsilon_{nl}^2 = -\left(\frac{B}{2(n+1+A)}\right)^2 \qquad (16)$$

and this yields

$$-E_{nl}^2 = m_0^2 - \left(\frac{B}{2(n+1+A)}\right)^2. \qquad (17)$$

In order to find the corresponding energy eigenfunctions with AIM, we may use the following energy eigenfunction generator

$$\chi(r) = exp\left(-\int^r \frac{s_k(r)}{\lambda_k(r)} dr\right). \qquad (18)$$

By applying the function generator, the $f_n(r)$ functions can be written in series expansion by hypergeometric functions with constant $(B+n+1)^n$ and $\prod_{k=0}^{(n-1)}(B+2+k)$. By generalizing these expansions, we get

$$\chi(r) = (B+n+1)^n \left[\prod_0^{n-1}(2B+2+k)\right] x_1F_1(-n, 2B+2; 2\varepsilon_{nl}r). \qquad (19)$$

The upper spinor component of the radial wave function may be written as

$$F_{nk}(r) = r^{A+1}e^{(-\varepsilon_{nl}r)}(B+n+1)^n \left[\prod_0^{n-1}(2B+2+k)\right] x_1F_1(-n, 2B+2; 2\varepsilon_{nl}r). \qquad (20)$$

The lower spinor wave function can be obtained via a similar algebraic calculation. The mass function for lower spinor is calculated as $M(r) = \frac{(b-2)V_o}{r} + m_0$. After an equivalent algebraic procedure, we get the same results for eigenfunctions and eigenvalues with different parameters as

$$G_{nk}(r) = r^{A+1}e^{(-\varepsilon_{nl}r)}(B+n+1)^n \left[\prod_0^{n-1}(2B+2+k)\right] x_1F_1(-n, 2B+2; 2\varepsilon_{nl}r),$$

where

$$\begin{aligned} E_{nl}^2 - m_0^2 &= -\varepsilon_{nl}^2, & (21)\\ k(k-1) + 4V_0^2(b^2-3b+2) &= A(A+1), & (22)\\ 2(E_{nl} + (2b-3)m_0)V_0 &= B. & (23) \end{aligned}$$

These eigenfunctions and eigenvalues agree with the results in [48] after mapping the corresponding parameters with those used in [48].

5. Conclusion

The spectrum of the position dependent mass Dirac equation for Coulomb potential has been obtained within the framework of the AIM method without solving the differential equation. The mass function is considered in the form which satisfies the equality condition, $\frac{dM(r)}{dr} - \frac{dV(r)}{dr} = 0$. When the vector potential $V(r)$ is considered equal (state $b = 2$) to the spherical Scalar potential, the mass function is reduced to a constant mass situation. In the upper spinor wavefunction if $b > 2$ ($S(r) > V(r)$), the perturbed mass term results in a negative effect in $M(r)$. But in the lower spinor wavefunction, this condition yields a positive effect in $M(r)$. Therefore, by adjusting the parameter b, the bound-state solutions for spinor wavefunctions may be calculated by applying AIM and comparing with the corresponding results in [48].

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References

[1] W. Greiner, Relativistic Quantum Mechanics Wave Equation, 3rd edition (Freankfert, Germen, 1987)
[2] A. D. Saavedra, F. Boronat, J. Polls, A. Fabrocini,

Phys. Rev. B 50, 4248 (1994)
[3] T. Gora, F. Williams, Phys. Rev. 177, 1179 (1969)
[4] A. D. Alhaidari, Phys. Rev. A 75, 042707 (2007)
[5] A. D. Alhaidari, H. Bahlouli, A. Al Hasan, M. S. Abdelmonem, Phys. Rev. A 75, 062711 (2007)
[6] A. D. Alhaidari, Phys. Lett. A 322, 72 (2004)
[7] O. Panella, S. Biondini, A. Arda, J. Phys. A Math. Theor. 43, 325302 (2010)
[8] A. De Souza Dutra, C. S. Jia, Phys. Lett. 352, 484 (2010)
[9] I. O. Vakarchuk, J. Phy. A Math. Gen. 38, 4727 (2005)
[10] S. M. Ikhadair, R. Sever, Appl. Math. Comput. 216, 911 (2010)
[11] C. S. Jia, T. Chen, L. G. Cui, Phys. Lett. A 373, 1621 (2009)
[12] C. S. Jia, A. de Souza Dutra, Ann. Phys. 323, 566 (2008)
[13] L. Dekar, L. Chetouani, T. F. Hammann, J. Math. Phys. 39, 2551 (1998)
[14] X. L. Peng, J. Y. Liu, C. S. Jia, Phys. Lett. A 352, 478 (2006)
[15] D. Agboola, math-ph/arxiv:1011, 2368v1
[16] J. Yu, S. H. Dong, G. H. Sun, Phys. Lett. A 322, 290 (2004)
[17] J. Yu, S. H. Dong, Phys. Lett. A 325, 194 (2004)
[18] S. H. Dong, J. J. Pena, C. P. Garcia, J. G. Ravelo, Mod. Phys. Lett. A 22, 1039 (2007)
[19] J. Wu., Y. Alhassid, J. Math. Phys. 31, 557 (1990)
[20] R. A. Swainson, G. W. F. Drake, J. Phys. A: Math. Gen. 24, 79 (1991)
[21] G. Chen, Phys. Lett. A 326, 55 (2004)
[22] L. Infeld, T. E. Hull, Rev. Mod. Phys. 23, 21 (1951)
[23] R. P. Feynman, A. R. Hibbs, Quantum Mechanics and Path Integrals, (McGraw-Hill, New York, 1965)
[24] C. Grosche, J. Phys. A. Math. Gen. 28, 5889 (1995)
[25] E. Witten, Nucl. Phys. B 188, 513 (1981)
[26] A. Comtet, A. Bandrank, D. K. Campbell, Phys. Lett. B 150, 159 (1985)
[27] S. H. Dong, X. Y. Gu, Z. Q. Ma, S. Dong, Int. J. Mod. Phys. E. 11, 483 (2002)
[28] S. H. Dong, G. H. Sun, D. Popov, J. Math. Phys. 44, 4467 (2003)
[29] S. H. Dong, M. L. Cassou, Phys. Lett. A 330, 168 (2004)
[30] G. de A. Marques, V. B. Bezerra, S. H. Dong, Mod. Phys. Lett. A 28, 1350137 (2013)
[31] H. Ciftci, R. L. Hall, N. Saad, J. Phys. A: Math. Gen. 36, 11807 (2003)
[32] H. Ciftci, R. L. Hall, N. Saad, J. Phys. A: Math. Gen. 38, 1147 (2005)
[33] F. M. Fernandez, J. Phys. A. Math. Gen. 37, 617332 (2004)
[34] T. Barakat, K. Abodayeh, A. Mukheimer, J. Phys. A. Math. Gen. 38, 1299 (2005)
[35] T. Barakat, Phys. Lett. A 334, 411 (2005)
[36] E. Olğar, R. Koç, H. Tütüncüler, Chin. Phys. Lett. 23, 539 (2006)
[37] E. Olğar, Chin. Phys. Lett. 26, 020302 (2009)
[38] E. Olğar, H. Mutaf, Commun. Theor. Phys. 53, 1043 (2010)
[39] E. Olğar, Chin. Phys. Lett. 25, 1939 (2008)
[40] E. Olğar, R. Koç, H. Tütüncüler, Phys. Scr. 78, 015011 (2008)
[41] R. E. Moss, Am. J. Phys. 55, 397 (1987)
[42] H. Galic, Am. J. Phys. 56, 312 (1988)
[43] S. H. Dong, Z. Q. Ma., Phys. Lett. 312, 78 (2003)
[44] D. Agboola, Int. J. Quantum Chem. 112, 1029 (2012)
[45] D. Agboola, Few Body Syst. 52, 31 (2012)
[46] D. Agboola, Pramana. Phys. 76, 875 (2011)
[47] G. Esposito, P. Santorelli, J. Phys. A. Gen. 32, 5643 (1999)
[48] M. Hamzavi, A. A. Rajabi, H. Hassanabadi, Phys. Lett. A 374, 4303 (2010)
[49] S. M. Ikhdair, R. Sever, Appl. Math. Phys. 316, 545 (2010)