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MAXIMUM PRINCIPLE AND EXISTENCE OF SOLUTIONS FOR NON NECESSARILY COOPERATIVE SYSTEMS INVOLVING SCHRÖDINGER OPERATORS

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ABSTRACT. In this paper, we obtain the necessary and sufficient conditions for having the maximum principle and existence of positive solutions for some cooperative systems involving Schrödinger operators defined on unbounded domains. Then, we deduce the existence of solutions for semi-linear systems. Finally we discuss the generalized maximum principle (ϕ_q-positivity) for non cooperative systems.

§1. Introduction

We consider the following semilinear system

$$\begin{cases} LY + QY = Ag(x)Y + F(x,Y) & \text{in} \quad \Omega \\ Y \to 0 \quad \text{as} \quad |x| \to \infty \\ Y = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{S}$$

where Ω is an unbounded domain of $\mathbb{R}^N$, L is an $n \times n$ diagonal matrix of Laplace operators, Q is an $n \times n$ diagonal matrix of potential functions $q_i (1 \le i \le n)$, $g(x)$ is a weight function tending to zero at infinity, F is a given n-vector function and $A = (a_{ij})$ is a constant $n \times n$ cooperative matrix such that:

$$a_{ij} \ge 0 \qquad \text{for all} \quad i \ne j \tag{1}$$

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It is well known that the maximum principle plays an important role in the theory of partial differential equations. An analogous theory has appeared for semilinear systems in [1], [6]–[10], [12], [14], [17]. In [9], [10], the authors studied system (S) with $q_i = 0$ and $g(x) = 1$, defined on bounded domains with Dirichlet conditions. The problem with $q_i = 0$ defined on the whole space $\mathbb{R}^n$ has been established in [15], [16]. The system with equal potentials defined on $\mathbb{R}^n$ has been considered in [2], [4].

Here, we extend these results to system (S). In section two, we obtain necessary or sufficient conditions for having the maximum principle and existence of positive solutions for cooperative linear systems. Then, we study semilinear systems in section three; we adapt the method of sub and super solutions for proving the existence of nonnegative solutions. Finally, in section four, we study the generalized maximum principle (ϕ_q-positivity) for non cooperative systems.

To prove our theorems, we make use of an earlier results by Djellit and Yechoui [11] who have proved that for $N > 2$, $q > 0$ it holds:
if there exist $\alpha > 0$, $\beta \geq 1$, $\alpha > \beta$ and k, $c > 0$ such that

$$0 < g(x) \leq \frac{k}{(1+|x|^2)^\alpha}, \qquad 0 < q(x) \leq \frac{c}{(1+|x|^2)^\beta}, \tag{2}$$

then the eigenvalue problem:

$$\begin{cases} (-\Delta + q)y = \lambda g(x) y & \text{in} \quad \Omega \\ y \to 0 & \text{as} \quad |x| \to \infty \\ y = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{E}$$

has simple principal eigenvalue (λ_q^+) which is associated to a positive eigenfunction φ_q on $V(\Omega)$.

Moreover (λ_q^+) is characterized by:

$$\lambda_q^+ \int_\Omega g(x)|y|^2 \leq (|\nabla y|^2 + q|y|^2), \qquad \text{for all} \quad y \in V(\Omega) \tag{3}$$

where $V(\Omega) = \{ y \in D'(\Omega) : \ (1+|x|^2)^{-\frac{1}{2}} y \in L^2(\Omega), \ \nabla y \in L^2(\Omega) \}$ is a Hilbert space with an inner product $(y, \psi)_v = \int_\Omega \big(\nabla y \cdot \nabla \psi + \frac{1}{1+|x|^2} y \cdot \psi \big) \, \mathrm{d}x$ and a norm

$$\|y\|_V = \bigg(\int_\Omega \bigg(|\nabla y|^2 + \frac{1}{1+|x|^2} |y|^2 \bigg) \, \mathrm{d}x \bigg)^{\frac{1}{2}}$$

which is equivalent to

$$\|y\|_q = \bigg(\int_\Omega (|\nabla y|^2 + q|y|^2) \, \mathrm{d}x \bigg)^{\frac{1}{2}}.$$

We also introduce the Hilbert space

$$\mathscr{H} = \Big\{ y:\ \Omega \to \mathbb{R},\ \int_\Omega g y^2 \,\mathrm{d}x < \infty \Big\} = L^2_g(\Omega)$$

with an inner product

$$(y,\psi)_g = \int_\Omega g y \psi \,\mathrm{d}x.$$

§2. Cooperative linear systems

In this section, we study the maximum principle and existence of positive solutions for system (S) when F is a linear function.

Definition. We say that the *maximum principle* holds for system (S) if $F \geq 0$ and Y is a solution for (S) imply $Y \geq 0$.

Definition. A non singular square matrix $B = (b_{ij})$ is a *M-matrix* if $b_{ij} \leq 0$ for $i \neq j$, $b_{ii} > 0$ and if all principal minors extracted from B are positive.

The ith equation of system (S) can be written as:

$$\begin{cases} (-\Delta + q_i) y_i = g(x) \displaystyle\sum_{j=1}^{n} a_{ij} y_j + f_i & \text{in} \quad \Omega \\ y_i \to 0 \qquad \text{as} \quad |x| \to \infty \\ y_i = 0 \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{S$_i$}$$

Theorem 1. *Assume that* (1) *and* (2) *hold. For $f_i \geq 0$, system* (S) *satisfies the maximum principle if:*

(4) *The matrix $(\Lambda^+_{q_i} - A)$ is a non singular M-matrix.*

(5) *Moreover, if the maximum principle holds for system* (S)*, then the matrix $(\Lambda^+_q - A)$ is a non singular M-matrix, where*

$$q = \max\{q_i\}, \qquad \Lambda^+_{q_i} = \begin{pmatrix} \lambda^+_{q_1} & 0 & \dots & 0 \\ 0 & \lambda^+_{q_2} \dots & 0 & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda^+_{q_n} \end{pmatrix}.$$

Proof.

(i) Assume that $f_i \geq 0$ and $(y_i)_{i=1}^n \in \prod_{i=1}^{n} V_{q_i}$ is a solution of (S). Multiplying (S$_i$) by $\overline{y}_i = \max\{-y_i, 0\}$ and integrating over Ω, we obtain by Green's formula:

$$\int_\Omega \nabla y_i \cdot \nabla y_i^- + \int_\Omega q_i y_i y_i^- = \sum_{j=1}^{n} a_{ij} \int_\Omega g(x) y_j y_i^- + \int_\Omega f_i y_i^-$$

i.e.

$$\int_\Omega |\nabla y_i^-|^2 + \int_\Omega q_i |y_i^-|^2 = a_{ii} \int_\Omega g(x)|y_i^-|^2 - \sum_{j\neq i}^{n} a_{ij} \int_\Omega g(x) y_j y_i^- - \int_\Omega f_i y_i^-,$$

by (3), we get

$$\lambda_{q_i}^+ \int_\Omega g(x)|y_i^-|^2 \le a_{ii} \int_\Omega g(x)|y_i^-|^2 - \sum_{j\neq i}^{n} a_{ij} \int_\Omega g(x) y_j y_i^- - \int_\Omega f_i y_i^-,$$

then

$$\big(\lambda_{q_i}^+ - a_{ii}\big) \int_\Omega |\sqrt{g} y_i^-|^2 \le \sum_{j\neq i}^{n} a_{ij} \int_\Omega g(x) y_j^- y_i^-,$$

by Cauchy Schwartz inequality, we have

$$\big(\lambda_{q_i}^+ - a_{ii}\big) \left(\int_\Omega |\sqrt{g} y_i^-|^2 \right)^{\frac{1}{2}} - \sum_{j\neq i}^{n} a_{ij} \left(\int_\Omega |\sqrt{g} y_j^-|^2 \right)^{\frac{1}{2}} \le 0$$

$$\Longrightarrow \begin{bmatrix} \lambda_{q_1}^+ - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda_{q_2}^+ - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \dots & \lambda_{q_n}^+ - a_{nn} \end{bmatrix} \begin{bmatrix} \left(\int |\sqrt{g} y_1^-|^2\right)^{\frac{1}{2}} \\ \vdots \\ \vdots \\ \left(\int |\sqrt{g} y_n^-|^2\right)^{\frac{1}{2}} \end{bmatrix} \le 0$$

by (4), $y_1^- = y_2^- = \dots = y_n^- = 0$ and hence $y_1, y_2, \dots, y_n \ge 0$.

(ii) Assume that $0 \le f_1 \in L^2_{\frac{1}{g}}(\Omega)$ and that the maximum principle holds for system (S).

We rewrite (S_i) as follows

$$\begin{cases} (-\Delta + q) y_i = g(x) \sum_{j=1}^{n} a_{ij} y_j + H_i & \text{in} \quad \Omega \\ y_i \to 0 \qquad \text{as} \quad |x| \to \infty \\ y_i = 0 \qquad \text{on} \quad \partial\Omega, \end{cases}$$

where $0 \le H_i = (q - q_i) y_i + f_i \in L^2_{\frac{1}{g}}(\Omega)$.

Multiplying by φ_q (the eigenfunction corresponding to λ_q^+) and integrating over Ω, we get:

$$\int_\Omega (-\Delta + q) y_i \varphi_q = \sum_{j=1}^{n} a_{ij} \int_\Omega g(x) y_j \varphi_q + \int_\Omega H_i \varphi_q.$$

By Green's formula and (3), we obtain

$$(\lambda_q^+ - a_{ii}) \int_\Omega g(x)\varphi_q y_i - \sum_{j \neq i}^{n} a_{ij} \int_\Omega g(x) y_j \varphi_q = \int_\Omega H_i \varphi_q. \tag{6}$$

(6) is a Cramer system in $X_i = \int_\Omega g(x)\varphi_q y_i$, $1 \leq i \leq n$. Since the right hand side is nonnegative as well as X_i, we obtain

$$\begin{vmatrix} \lambda_q^+ - a_{11} & -a_{12} & \dots & -a_{1n} \\ -a_{21} & \lambda_q^+ - a_{22} & \dots & -a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ -a_{n1} & -a_{n2} & \dots & \lambda_q^+ - a_{nn} \end{vmatrix} = |\Lambda_q^+ - A| > 0.$$

The condition $\lambda_q^+ - a_{ii} > 0$ is satisfied from the scalar case (see [16]) since the functions g, y_i, H_i and the coefficients a_{ij} for all$i \neq j$ are non negative. □

Remark. If $q_i = q$ $(1 \leq i \leq n)$, then condition (5) is the necessary and sufficient condition for having the maximum principle for system (S).

Theorem 2. *If* (1) *and* (2) *are satisfied, then for* $f_i \geq 0$*, system* (S) *has a unique positive solution if condition* (4) *is satisfied.*

Proof. We consider the bilinear form $a\colon \prod_{i=1}^{n} V_{q_i} \times \prod_{i=1}^{n} V_{q_i} \to \mathbb{R}$ defined by

$$\begin{aligned} a(Y, \Psi) &= a((y_1, y_2, \dots, y_n), (\psi_1, \psi_2, \dots, \psi_n)) \\ &= \sum_{i=1}^{n} \int_\Omega \nabla y_i \nabla \psi_i + \sum_{i=1}^{n} \int_\Omega q_i y_i \psi_i - \sum_{j \neq i}^{n} \int_\Omega g(x) a_{ij} y_j \psi_i - \sum_{i=1}^{n} \int_\Omega g(x) a_{ii} y_i \psi_i \end{aligned}$$

We choose $m \geq 0$ so that $m + a_{ii} > 0$. Then, we have

$$\begin{aligned} a(Y, Y) &= \sum_{i=1}^{n} \int_\Omega [(\nabla y_i)^2 + q_i y_i^2] - \sum_{j \neq i}^{n} \int_\Omega g(x) a_{ij} y_i y_j - \sum_{i=1}^{n} \int_\Omega g(x) a_{ii} y_i^2 \\ &= \sum_{i=1}^{n} \int_\Omega [(\nabla y_i)^2 + (q_i + mg) y_i^2] - \sum_{j \neq i}^{n} \int_\Omega g(x) a_{ij} y_i y_j \\ &\qquad - \sum_{i=1}^{n} (m + a_{ii}) \int_\Omega g(x) y_i^2 \end{aligned}$$

By Cauchy Schwartz inequality and by (3), we get

$$a(Y,Y) \geq \sum_{i=1}^{n} \int_\Omega \left[(\nabla y_i)^2 + (q_i + mg)y_i^2\right] - \sum_{j\neq i}^{n} a_{ij} \left[\int_\Omega \left(\sqrt{g} y_i\right)^2\right]^{\frac{1}{2}} \left[\int_\Omega (\sqrt{g} y_j)^2\right]^{\frac{1}{2}} - \sum_{i=1}^{n} (m + a_{ii}) \int_\Omega g(x) y_i^2$$

$$\geq \sum_{i=1}^{n} \left(1 - \frac{m + a_{ii}}{m + \lambda_{q_i}^+}\right) \int_\Omega \left[(\nabla y_i)^2 + (q_i + mg)y_i^2\right]$$

$$- \sum_{j\neq i}^{n} \frac{a_{ij}}{\sqrt{(m + \lambda_{q_i}^+)(m + \lambda_{q_j}^+)}} \left(\int_\Omega \left[(\nabla y_i)^2 + (q_i + mg)y_i^2\right]\right)^{\frac{1}{2}} \cdot$$

$$\cdot \left(\int_\Omega \left[(\nabla y_j)^2 + (q_j + mg)y_j^2\right]\right)^{\frac{1}{2}}.$$

By (4)

$$a(Y,Y) \geq C \sum_{i=1}^{n} \int_\Omega \left[(\nabla y_i)^2 + (q_i + mg)(y_i)^2\right]$$

$$\geq C \sum_{i=1}^{n} \int_\Omega \left[(\nabla y_i)^2 + q_i (y_i)^2\right] = C \sum_{i=1}^{n} \|y_i\|_{qi}^2, \qquad C > 0.$$

Then by Lax Milgram lemma, since a is a continuous coercive bilinear form, there exists a unique solution $Y = (y_i)_{i=1}^n \in \prod_{i=1}^{n} V_{q_i}$. This solution is nonnegative by the maximum principle. □

§3. Semilinear systems

In this section, we adapt the method of sub and super solutions, to prove the existence of solutions for semilinear cooperative system (S). The proof here is analogous to that of [2], [16].

We assume that $f_i(x, Y) = f_i(x, y_1, y_2, \ldots, y_n)$ is a Caratheodary function such that:

$$0 \leq F(x,Y) = (f_1(x,Y), f_2(x,Y), \ldots, f_n(x,Y)) \leq (Ng(x), \ldots, Ng(x)) + h' \quad \text{for all} \quad Y \geq 0, \ \ x \in \Omega, \tag{7}$$

where N is a positive constant and $0 < h' = (h, h, \dots, h)$ is a bounded function in $\left(L^2_{\frac{1}{g}}(\Omega)\right)^n$.

Theorem 3. *If* (1), (2) *and* (7) *are satisfied, then there exists a positive solution for system* (S) *if:*

(8) *the matrix* $(\Lambda_q^+ - (A + NI))$ *is a non singular* M*-matrix.*

Proof.

(i) Construction of sub-super solutions:

$$Y^o = (y_1^o, y_2^o, \dots, y_n^o) = (0, 0, \dots, 0)$$

is a sub solution of (S):

$$LY^o + QY^o - g(x)AY^o - F(x, Y^o) \le 0.$$

Consider the following system

$$(L + Q)Y = g(x)(A + NI)Y + h' \qquad \text{in} \quad \Omega. \tag{9}$$

From theorem (2), under condition (8), (9) has a unique positive solution

$$Y^* = (y_1^*, y_2^*, \dots, y_n^*).$$

By (7)

$$0 \le (L + Q)Y^* - g(x)AY^* - F(x, Y^*) \tag{10}$$

i.e. $Y^* = (y_1^*, y_2^*, \dots, y_n^*)$ is a super solution of (S).

(ii) Definition of a compact operator:

We introduce $T\colon Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathscr{H}^n \to \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix} = TY \in \prod_{i=1}^n V_{q_i}$, where Ψ is the unique solution of the following system:

$$(L + Q + mg(x)I)\Psi = (mI + A)g(x)Y + F(x, Y) \qquad \text{in} \quad \Omega. \tag{11}$$

Equation (11) can be rewritten as

$$(L + Q)\Psi = -mg(x)\Psi + \overline{F}, \qquad \overline{F} = (mI + A)g(x)Y + F(x, Y) > 0,$$

then (11) has a solution $\Psi \in \prod_{i=1}^n V_{q_i}$ and therefore T is well defined.

(iii) $K = [Y^o, Y^*]$ is invariant by T, i.e. $T(K) \subset K$:

For $0 \le Y \in \prod_{i=1}^n V_{q_i}$, $\Psi \ge 0$. We show now that if $Y \le Y^*$, then $\Psi \le \Psi^*$. Subtract (10) from (11) and we obtain

$$(L+Q+mg(x)I)(Y^* - \Psi) = (mI+A)g(x)(Y^* - Y) + F(x, Y^*) - F(x, Y) = \tau > 0$$

then

$$(L+Q)(T^*-\Psi)=-mg(x)(Y^*-\Psi)+\tau$$

therefore $Y^*-\Psi$ is a positive solution and hence $\Psi\le Y^*$.

(iv) Finally, we show that T is a completely continuous operator: We first prove the continuity of T: Let $Y_k\to Y$ in $\mathscr{H}^n$. By (7), we get

$$F(x,Y_k)\to F(x,Y)\qquad \text{in}\quad (\mathscr{H}')^n.$$

Let us denote by Ψ_k the sequences associated with Y_k, by (11) it follows that:

$$(L+Q+mg(x)I)(\Psi-\Psi_k)=(mI+A)g(x)(Y-Y_k)+F(x,Y)-F(x,Y_k).$$

Multiplying by $(\Psi-\Psi_k)$ and integrating over Ω, we get

$$\int_\Omega (L+Q+mg(x)I)(\Psi-\Psi_k)\cdot(\Psi-\Psi_k)$$
$$=(mI+A)\int_\Omega g(x)(Y-Y_k)\cdot(\Psi-\Psi_k)+\int_\Omega [F(x,Y)-F(x,Y_k)]\cdot(\Psi-\Psi_k)$$

using Green formula

$$\int_\Omega (\nabla|\Psi-\Psi_k|)^2+\int_\Omega Q|\Psi-\Psi_k|^2+m\int_\Omega g(x)|\Psi-\Psi_k|^2$$
$$=(mI+A)\int_\Omega g(x)(Y-Y_k)\cdot(\Psi-\Psi_k)+\int_\Omega [F(x,Y)-F(x,Y_k)]\cdot(\Psi-\Psi_k)$$

by Cauchy-Schwarz inequality, we get

$$\int_\Omega (\nabla|\Psi-\Psi_k|)^2+\int_\Omega Q|\Psi-\Psi_k|^2+m\int_\Omega g(x)|\Psi-\Psi_k|^2$$
$$\le(mI+A)\|Y-Y_k\|_{\mathscr{H}}\cdot\|\Psi-\Psi_k\|_{\mathscr{H}}+\|F(x,Y)-F(x,Y_k)\|_{\mathscr{H}'}\cdot\|\Psi-\Psi_k\|_{\mathscr{H}}$$

therefore

$$\begin{aligned}\|\Psi-\Psi_k\|_q^2\le{}&(mI+A)\|Y-Y_k\|_{\mathscr{H}}\cdot\|\Psi-\Psi_k\|_{\mathscr{H}}\\&+\|F(x,Y)-F(x,Y_k)\|_{\mathscr{H}'}\cdot\|\Psi-\Psi_k\|_{\mathscr{H}}.\end{aligned}$$

Since $\|Y-Y_k\|_{\mathscr{H}}\to 0$, then $\|\Psi-\Psi_k\|_q^2\to 0$. Now, we prove the compactness of T: Multiplying (11) by Ψ and integrating over Ω, we get

$$\int_\Omega |\nabla\Psi|^2+\int_\Omega Q|\Psi|^2+m\int_\Omega g(x)|\Psi|^2=(mI+A)\int_\Omega g(x)Y\Psi+\int_\Omega F(x,Y)\Psi$$

from (4)

$$\begin{aligned}\int_\Omega |\nabla\Psi|^2 + \int_\Omega Q|\Psi|^2 + m\int_\Omega g(x)|\Psi|^2 \\ &\le (mI+A)\int_\Omega g(x)Y\Psi + N\int_\Omega g(x)Y\Psi + \int_\Omega \sqrt{g}\Psi\frac{h}{\sqrt{g}} \\ &= (mI+A+N)\int_\Omega g(x)Y\Psi + \int_\Omega \sqrt{g}\Psi\frac{h}{\sqrt{g}}\end{aligned}$$

by Cauchy-Schwarz inequality, we obtain

$$\|\psi\|_q \le C\big(\|Y\|_{\mathscr{H}} + \|h\|_{\mathscr{H}'}\big) \le C\big(\|Y\|_q + \|h\|_{\mathscr{H}'}\big).$$

Therefore if Y_k is a bounded sequence in $\prod_{i=1}^n V_{q_i}$, the associated sequence ψ_k is bounded in $\prod_{i=1}^n V_{q_i}$. We show now that ψ_k is a Cauchy sequence in $\prod_{i=1}^n V_{q_i}$.

Suppose that $\|Y_k\|_q^2 \le E$, E is a constant.

Choose R large enough such that

$$(1+R^2)g(R) < \frac{\epsilon}{8\gamma E}, \qquad \epsilon > 0 \text{ fixed}, \tag{12}$$

where γ is given by

$$\int_\Omega \big(1+|x|^2\big)^{-1} y^2 \le \gamma \int_\Omega |\nabla y|^2. \tag{13}$$

Let $B = \big\{x \in \Omega : \ |x| < R\big\}$ and $B' = \big\{x \in \Omega : \ |x| > R\big\}$.

Since Y_k is bounded in $\prod_{i=1}^n V_{q_i}$, Y_k is bounded in $(H^1(B))^n$; but B is bounded and therefore the embedding $(H^1(B))^n$ into $(L^2(B))^n$ is compact; hence there exists a convergent subsequence still denoted by $(Y_k)_{k\in\mathbb{N}}$, which is a Cauchy sequence and we can choose j and k large enough so that:

$$\int_B g(x)|Y_k - Y_j|^2\,\mathrm{d}x \le \int_B |Y_k - Y_j|^2\,\mathrm{d}x < \frac{\epsilon}{2}.$$

Moreover using (11) and (12), we obtain:

$$\begin{aligned}\int_{B'} g(x)|Y_k - Y_j|^2\,\mathrm{d}x &= \int_{B'} \big(1+|x|^2\big)g(x)\frac{1}{1+|x|^2}|Y_k - Y_j|^2\,\mathrm{d}x \\ &\le \frac{\epsilon}{8\gamma E}\int_{B'} \frac{1}{1+|x|^2}|Y_k - Y_j|^2\,\mathrm{d}x \\ &\le \frac{\epsilon}{8\gamma E}\gamma\|Y_k - Y_j\|_V^2,\end{aligned}$$

so $\int_{B'} g(x)|Y_k - Y_j|^2\,\mathrm{d}x < \frac{\epsilon}{2}$.

Then Ψ_k is a Cauchy sequence in $\prod_{i=1}^{n} V_{q_i}$. Hence it converges toward Ψ and therefore T is compact in $\prod_{i=1}^{n} V_{q_i}$. By Schauder fixed point theorem, there exists at least one positive solution $Y = (y_i)_{i=1}^n \in \prod_{i=1}^{n} V_{q_i}$ of system (S) satisfying $Y^o \le Y \le Y^*$. □

§4. Non cooperative systems

In this section, we study the generalized maximum principle (ϕ_q-positivity) for $n \times n$ non cooperative systems.

Definition. System (S) (with $q_i = q$) satisfies the *generalized maximum principle* if there holds: if $F \ge 0$ and $Y = (y_1, y_2, \dots, y_n)$ is a solution for (S), then there exists $C > 0$ such that:

$$y_i \ge C\phi_q \qquad \text{for all} \quad i = 1, 2, \dots, n \quad \text{in } \Omega.$$

We start with 2×2 non cooperative systems:

$$\begin{cases} L_q y_1 = a_{11} g(x) y_1 + a_{12} g(x) y_2 + f_1 & \text{in } \Omega \\ L_q y_2 = a_{21} g(x) y_1 + a_{22} g(x) y_2 + f_2 & \text{in } \Omega \\ y_1, y_2 \to 0 \quad \text{as} \quad |x| \to \infty \\ y_1 = y_2 = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{S$_2$}$$

As in [3], we can prove:

Theorem 4. *Assume that $a_{12} < 0$, $a_{21} > 0$, $a_{11} > a_{22}$, $f_1 - \xi_2 f_2 \ge 0$ and $(a_{11} - a_{22})^2 + 4a_{12}a_{21} \ge 0$ are satisfied. The generalized maximum principle holds for system* (S) *if:*

$$a_{11} - \lambda_q < r < \lambda_q - a_{22}.$$

Now, let us consider the following 3×3 non cooperative system:

$$\begin{cases} L_q y_1 = a_{11} g(x) y_1 + a_{12} g(x) y_2 + a_{13} g(x) y_3 + f_1 & \text{in} \quad \Omega \\ L_q y_2 = a_{21} g(x) y_1 + a_{22} g(x) y_2 + a_{23} g(x) y_3 + f_2 & \text{in} \quad \Omega \\ L_q y_3 = a_{31} g(x) y_1 + a_{32} g(x) y_2 + a_{33} g(x) y_3 + f_3 & \text{in} \quad \Omega \\ y_1, y_2, y_3 \to 0 \qquad \text{as} \quad |x| \to \infty \\ y_1 = y_2 = y_3 = 0 \qquad \text{on} \quad \partial\Omega. \end{cases} \tag{S$_3$}$$

Assume that:

$$a_{12}, a_{13} < 0, \qquad a_{21}, a_{23}, a_{31}, a_{32} \geq 0. \tag{14}$$

We insert our 3×3 non-cooperative system into a 4×4 cooperative one. We introduce the following fourth equation of a new variable $y_4 = y_1 - \xi_2 y_2 - \xi_3 y_3$:

$$L_q y_4 = (a_{11} - \xi_2 a_{21} - \xi_3 a_{31} - s) g(x) y_1 + (a_{12} - \xi_2 a_{22} - \xi_3 a_{32} + s\xi_2) g(x) y_2 \\ + (a_{13} - \xi_2 a_{23} - \xi_3 a_{33} + s\xi_3) g(x) y_3 + s g(x) y_4 + f_4$$

where ξ_2 and ξ_3 are positive numbers and $f_4 = f_1 - \xi_2 f_2 - \xi_3 f_3$.

Then system (S$_3$) can be changed into system of the form:

$$\begin{cases} L_q Y = \mathscr{N} g(x) Y + F & \text{in} \quad \Omega \\ Y \to 0 \qquad \text{as} \quad |x| \to \infty \\ Y = 0 \qquad \text{on} \quad \partial\Omega, \end{cases} \tag{S$'_3$}$$

where

$$\mathscr{N} = \begin{pmatrix} a_{11}-r & a_{12}+r\xi_2 & a_{13}+r\xi_3 & r \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{11}-\xi_2 a_{21}-\xi_3 a_{31}-s & a_{12}-\xi_2 a_{22}-\xi_3 a_{32}+s\xi_2 & a_{13}-\xi_2 a_{23}-\xi_3 a_{33}+s\xi_3 & s \end{pmatrix},$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \quad F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \text{ and } L_q = \begin{pmatrix} -\Delta - q & 0 & 0 & 0 \\ 0 & -\Delta - q & 0 & 0 \\ 0 & 0 & -\Delta - q & 0 \\ 0 & 0 & 0 & -\Delta - q \end{pmatrix}.$$

For the cooperativeness of system (S$'_3$), we can choose r, s, ξ_2 and ξ_3 such that

$$r > 0, \qquad a_{12} + r\xi_2 = 0, \qquad a_{13} + r\xi_3 = 0 \tag{15}$$

$$a_{11} - \xi_2 a_{21} - \xi_3 a_{31} - s = 0 \tag{16}$$

$$a_{12} - \xi_2 a_{22} - \xi_3 a_{32} + s\xi_2 = 0 \tag{17}$$

$$a_{13} - \xi_2 a_{23} - \xi_3 a_{33} + s\xi_3 = 0. \tag{18}$$

Using (15), we get

$$\xi_3 = \frac{-a_{13}}{r} > 0 \qquad \text{and} \qquad \xi_2 = \frac{-a_{12}}{r} > 0.$$

Now, system (S_3') satisfies the maximum principle if the matrix

$$\begin{pmatrix} \lambda_q-a_{11}+r & a_{12}+r\xi_2 & a_{13}+r\xi_3 & r \\ a_{21} & \lambda_q-a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & \lambda_q-a_{33} & 0 \\ a_{11}-\xi_2a_{21}-\xi_3a_{31}-s & a_{12}\xi_2a_{22}-\xi_3a_{32}+s\xi_2 & a_{13}-\xi_2a_{23}-\xi_3a_{33}+s\xi_3 & \lambda_q-s \end{pmatrix}$$

is a non-singular M-matrix which means that:

(i) $\lambda_q - a_{11} + r > 0$

(ii) $\begin{vmatrix} \lambda_q - a_{11} + r & -a_{12} - r\xi_2 \\ -a_{21} & \lambda_q - a_{22} \end{vmatrix} > 0$

(iii) $\begin{vmatrix} \lambda_q - a_{11} + r & a_{12} + r\xi_2 & -a_{13} - r\xi_3 \\ -a_{21} & \lambda_q - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda_q - a_{33} \end{vmatrix} > 0$

(iv) $\begin{vmatrix} \lambda_q-a_{11}-r & a_{12}+r\xi_2 & a_{13}+r\xi_3 & r \\ a_{21} & \lambda_q-a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & \lambda_q-a_{33} & 0 \\ a_{11}-\xi_2a_{21}-\xi_3a_{31}-s & a_{12}-\xi_2a_{22}-\xi_3a_{32} & a_{13}-\xi_2a_{23}-\xi_3a_{33}+s\xi & \lambda_q-s \end{vmatrix} > 0$

Using (15)–(18), we obtain:

$$\lambda_q - a_{11} + r > 0, \qquad (\lambda_q - a_{11} + r)(\lambda_q - a_{22}) > 0,$$
$$(\lambda_q - a_{11} + r)[(\lambda_q - a_{22})(\lambda_q - a_{33}) - a_{23}a_{32}] > 0$$

and

$$(\lambda_q - a_{11} + r)[(\lambda_q - a_{22})(\lambda_q - a_{33})(\lambda_q - s) - a_{23}a_{32}(\lambda_q - s)] > 0,$$

which implies that

$\lambda_q-a_{11}+r > 0$, $\lambda_q-a_{22} > 0$, $(\lambda_q-a_{22})(\lambda_q-a_{33}) > a_{23}a_{32}$ and $\lambda_q-s > 0$.

From (14), we obtain:

$$(\lambda_q - a_{22})(\lambda_q - a_{33}) > 0$$

then

$$a_{11} - r < \lambda_q, \quad a_{22} < \lambda_q, \quad a_{33} < \lambda_q, \quad s < \lambda_q.$$

From (17)

$$a_{12} - \xi_3a_{32} = \xi_2(a_{22} - s) \implies a_{22} + r + \frac{\xi_3}{\xi_2}a_{32} = s \tag{19}$$

also from (18)

$$a_{13} - \xi_2a_{23} = \xi_3(a_{33} - s) \implies a_{33} + r + \frac{\xi_2}{\xi_3}a_{23} = s. \tag{20}$$

Since $s < \lambda_q$, then from (19) and (20) we obtain:

$$\lambda_q > a_{22} + r + \frac{\xi_3}{\xi_2} a_{32}, \qquad \lambda_q > a_{33} + r + \frac{\xi_2}{\xi_3} a_{23}$$
$$\implies \lambda_q > a_{22} + r, \qquad \lambda_q > a_{33} + r$$

i.e.

$$r < \lambda_q - a_{22} \qquad \text{and} \qquad r < \lambda_q - a_{33} \qquad \text{also} \qquad r > a_{11} - \lambda_q.$$

Then we have:

Theorem 5. *Assume that* (14) *holds, Then for* $0 \le \xi_3 f_3 \le \xi_2 f_2 \le f_1$, *system* ($S_3$) *satisfies the generalized maximum principle if the following inequalities are satisfied*

$$a_{11} - \lambda_q < r < \lambda_q - a_{22}$$
$$a_{11} - \lambda_q < r < \lambda_q - a_{33}.$$

For non cooperative system (S), we set the following condition

$$\begin{aligned} &a_{1j} < 0 \qquad \text{for} \quad j = 2, 3, \dots, n, \\ &a_{ij} > 0 \qquad \text{for} \quad i = 2, 3, \dots, n, \quad j = 1, 2, 3, \dots, n, \quad i \neq j. \end{aligned} \tag{21}$$

Similarly, to construct $(n+1) \times (n+1)$ cooperative system, we introduce the following equation of a new variable $y_{n+1} = y_1 - \sum_{i=2}^{n} \xi_i y_i$, where ξ_2 to ξ_n are positive numbers and we have:

Theorem 6. *Assume that* (21) *holds, Then for* $0 \le \xi_n f_n \le \dots \le \xi_2 f_2 \le f_1$, *system* (S) *satisfies the generalized maximum principle if:*

$$a_{11} - \lambda_q < r < \lambda_q - a_{ii}, \qquad i = 2, 3, \dots, n.$$

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