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EXTENSIONS OF HOMOGENEOUS POLYNOMIALS ON $c_0(l_2^i)$

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ABSTRACT. We show that a 2-homogeneous polynomial on the complex Banach space $c_0(l_2^i)$ is norm attaining if and only if it is finite (i.e, depends only on finite coordinates). As the consequence, we show that there exists a unique norm-preserving extension for norm-attaining 2-homogeneous polynomials on $c_0(l_2^i)$.

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1. Introduction

It is well known that if a Banach space E is an M-ideal in its bidual E'', then every continuous linear functional φ on E has a unique Hahn-Banach extension to E''. In general, this result is not true for n-homogeneous polynomials. In [1] Aron–Boyd–Choi showed that in the case of complex space c_0 (it is an M-ideal in l_∞), any 2-homogeneous polynomial on c_0, which attains its norm can be uniquely extended to its bidual l_∞. It was also shown there that for n-homogeneous polynomials, where $n \geq 3$, it is impossible in general. A similar problem was considered by Choi–Han–Song in [2] for Lorentz sequences spaces, later on it was generalized by Kaminska–Lee in [4] for Banach function spaces, rearrangement invariant (r.i.) sequence spaces and Marcinkiewicz sequence spaces.

In this paper, we investigate the analogous problem in the complex Banach space $c_0(l_2^i)$ which is an M-ideal in its bidual. In fact, we will show that there exists a unique norm-preserving extension for norm-attaining 2-homogeneous polynomials on $c_0(l_2^i)$. We observe that this result is not a immediate consequence of the previously known results in [4].

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We refer to [3] for notation and results regarding homogeneous polynomials.

Let us denote by l_2^n the space $\mathbb{C}^n$ with the Euclidean norm. In this work we will consider the Banach space

$$X = c_0\big(l_2^i\big) = \Big\{(\lambda^j)_{j\in\mathbb{N}}:\ \lambda^j \in l_2^j \text{ and } \big(\|\lambda^j\|_2\big)_{j\in\mathbb{N}} \in c_0\Big\}$$

and its bidual

$$X'' = l_\infty\big(l_2^i\big) = \Big\{(\lambda^j)_{j\in\mathbb{N}}:\ \lambda^j \in l_2^j \text{ and } (\|\lambda^j\|_2)_{j\in\mathbb{N}} \in l_\infty\Big\},$$

where, in both cases, we will work with the norm $\|(\lambda^j)\| = \sup_j \|\lambda^j\|_2$.

The relevance of such space is because X'' was the first example for which the Dunford-Pettis property fails, but its predual $X' = l_1\big(l_2^i\big)$ has it. This example was given by C. Stegall in [5].

Let us denote by (e_j^i), $j \le i$, the unit sequence $(e_j^i) = (0, 0, \dots, \overbrace{\lambda^i}^{i\text{th}}, 0, \dots)$, where $\lambda^i = (0, \dots, \overbrace{1}^{j\text{th}}, \dots, 0) \in l_2^i$. Such sequence is a Schauder basis for X. By B_X and S_X we will denote the closed unit ball and unit sphere of X, respectively. In some cases, it is useful to write the elements of those spaces as an infinite matrix:

$$x = (\lambda^1, \lambda^2, \lambda^3, \dots) = \begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \cdots \\ & \lambda_2^2 & \lambda_2^3 & \cdots \\ & & \lambda_3^3 & \cdots \end{pmatrix},$$

where the ith column is an element of l_2^i.

2. Results

We will denote by $P(^mX)$ the Banach space of all continuous m-homogeneous polynomials on X with the norm $\|P\| = \sup_{x\in B_X} |P(x)|$. A polynomial $P \in P(^mX)$ is *norm-attaining* if there exists $x \in B_X$ such that $\|P\| = P(x)$.

Our attention turns to examine a norm-attaining polynomial P in $P(^2X)$. We give a characterization of the norm-attaining 2-homogeneous polynomials on X''. To do this, we employ the similar idea as in paper [1]. For this, we need to give the following definition: A polynomial P defined on X is called *finite* if there exists $n \in \mathbb{N}$ such that

$$P(x^1, x^2, \dots, x^n, x^{n+1}, x^{n+2}, \dots) = P(x^1, x^2, \dots, x^n, 0, 0, \dots),$$

for all $x = (x^1, x^2, \dots) \in X$. With this definition, we can show our result.

Theorem 2.1. *A homogeneous polynomial of degree* 2 *on* X'' *attains the norm in* $x^0 \in B_X$ *if and only if it is finite.*

Proof. If P is finite, then it can be considered as a polynomial in $\bigoplus_{i=1}^{n} l_2^i$, for some n and it is trivial that it attains its norm.

Now, we assume that $P \in P(^2X'')$ is norm-attaining in $x^0 = (\lambda^1, \lambda^2, \ldots) \in S_X$.

Obviously, if $\|P\| = 0$, then P is finite. So, we can assume that $\|P\| = 1$.

Let $J = \{j \in \mathbb{N} : \ \|\lambda^j\|_2 = 1\}$. Since $x^0 \in X$, then J is finite. Without loss of generality, let us assume that $J = \{1, 2, \ldots, n\}$.

For each $j \in J$, $\lambda^j \neq 0$. So, for each $j \in J$, we can take an orthonormal basis $\mathscr{B}_j$ such that all the coordinates of λ^j are not nulls. Now, we write all of the other λ^j in the canonical basis of $\mathbb{C}^j$ and then we can represent x^0 in the following way

$$x^0 = (\lambda^1, \lambda^2, \lambda^3, \ldots) = \begin{pmatrix} \lambda_1^1 & \lambda_1^2 & \lambda_1^3 & \cdots \\ & \lambda_2^2 & \lambda_2^3 & \cdots \\ & & \lambda_3^3 & \cdots \end{pmatrix}.$$

For all $y \in B_{X''}$ being $y = (0, 0, \ldots, y^{n+1}, y^{n+2}, \ldots)$ and for each $\lambda \in \mathbb{C}$ with $|\lambda| = 1 - \sup_{i>n} \|\lambda^i\|_2 > 0$ we get $\|x^0 \pm \lambda y\| \leq 1$. Therefore, $|P(x^0 \pm \lambda y)| \leq 1$, and then

$$|P(x^0) \pm 2\lambda \check{P}(x^0, y) + \lambda^2 P(y)| = |1 \pm 2\lambda \check{P}(x^0, y) + \lambda^2 P(y)| \leq 1, \qquad (*)$$

where $\check{P}$ denotes the symmetric bilinear map associated to P. Adding the equations we get $|1 + \lambda^2 P(y)| \leq 1$. If $P(y) \neq 0$, we could take λ such that $\lambda^2 P(y)$ could have the real part positive, and it will be a contradiction. It follows that $P(y) = 0$, for all y being as above. In particular, $P(0, 0, \ldots, \lambda^{n+1}, \lambda^{n+2}, \ldots) = 0$.

For $(*)$, we get $|1 \pm 2\lambda \check{P}(x^0, y)| \leq 1$. This means that the complex number $2\lambda \check{P}(x^0, y)$ is in the disks with centers ± 1 and radius 1. Then, we will get $2\lambda \check{P}(x^0, y) = 0$, i.e., $\check{P}(x^0, y) = 0$, for all y as above. Taking $y = (0, 0, \ldots, \lambda^{n+1}, \lambda^{n+2}, \ldots)$, we will get

$$P(x^0 - y) = P(x^0) - 2\check{P}(x^0, y) + P(y) = 1,$$

and then $P(\lambda^1, \lambda^2, \ldots, \lambda^n, 0, 0, \ldots) = 1$.

Now, we will prove that for the other points P also depends just on the first n variables. Let $A = \{(i, j) : \ 1 \leq i \leq n \text{ and } 1 \leq j \leq i\}$ and $A^* = A \setminus \{(1, 1)\}$. We define $z_1^1 = (\lambda^1, \lambda^2 \ldots \lambda^n, 0, 0, \ldots)$ and for $(i, j) \in A^*$ we define

$$z_j^i = z_1^1 - m e_j^i,$$

where $m := 1 + 2 + \cdots + n$. For each $x = (x_j^i) \in \bigoplus_{i=1}^{n} l_2^i$ we can write

$$(x^1, x^2, \ldots, x^n, 0, 0, \ldots) = \frac{1}{m} \sum_{(i,j)\in A} \frac{x_j^i}{\lambda_j^i} z_1^1 + \frac{1}{m} \sum_{(i,j)\in A^*} \left(\frac{x_1^1}{\lambda_1^1} - \frac{x_j^i}{\lambda_j^i} \right) z_j^i.$$

For all $y = (0, 0, \ldots, y^{n+1}, y^{n+2}, \ldots)$ with $\|y\| \leq 1$ we have that $\|z_1^1 \pm y\| = 1$, and so $|P(z_1^1 \pm y)| \leq 1$. But

$$|P(z_1^1 \pm y)| \leq 1 \implies |P(z_1^1) \pm 2\check{P}(z_1^1, y) + P(y)| \leq 1 \implies |1 \pm 2\check{P}(z_1^1, y)| \leq 1$$

and therefore $\check{P}(z_1^1, y) = 0$, for all $y = (0, 0, \ldots, \lambda^{n+1}, \lambda^{n+2}, \ldots)$. Hence, for all $(x^1, x^2, \ldots, x^n, y^{n+1}, y^{n+2}, \ldots) \in X$ we can write

$$\begin{aligned}
&P(x^1, \ldots, x^n, y^{n+1}, y^{n+2}, \ldots) \\
&= P\Big(\overbrace{(x^1, \ldots, x^n, 0, 0, \ldots)}^{x} + \overbrace{(0, \ldots, 0, y^{n+1}, y^{n+2}, \ldots)}^{y} \Big) \\
&= P(x) + 2\check{P}(x, y) + P(y) = P(x) + 2\check{P}(x, y) \\
&= P(x) + 2\check{P} \left(\frac{1}{m} \sum_{(i,j)\in A} \frac{x_j^i}{\lambda_j^i} z_1^1 + \frac{1}{m} \sum_{(i,j)\in A^*} \left(\frac{x_1^1}{\lambda_1^1} - \frac{x_j^i}{\lambda_j^i} \right) z_j^i, y \right) \\
&= P(x) + \frac{2}{n} \sum_{(i,j)\in A^*} \left(\frac{x_1^1}{\lambda_1^1} - \frac{x_j^i}{\lambda_j^i} \right) \check{P}(z_j^i, y) \\
&= P(x) + \sum_{(i,j)\in A^*} \left(\frac{x_1^1}{\lambda_1^1} - \frac{x_j^i}{\lambda_j^i} \right) \psi_j^i(y),
\end{aligned}$$

where $\psi_j^i = \frac{2}{n}\check{P}(z_j^i, \cdot) \in X'$. Now we will show that $\psi_j^i(y) = 0$ for all $(i, j) \in A^*$ and $y = (0, 0, \ldots, y^{n+1}, y^{n+2}, \ldots)$. Let us assume $(i, j) = (2, 1)$, since the other indices are analogous.

For each θ, we define $\lambda(\theta) \in \bigoplus_{i=1}^{n} l_2^i$ by

$$\lambda(\theta) = \begin{pmatrix} \lambda_1^1 & \lambda_1^2 e^{i\theta} & \cdots & \lambda_1^n \\ & \lambda_2^2 & \cdots & \lambda_2^n \\ & & \ddots & \vdots \\ & & & \lambda_n^n \end{pmatrix}.$$

Hence, we have that

$$P(\lambda(\theta), y^{n+1}, y^{n+2}, \ldots) = P(\lambda(\theta), 0, 0, \ldots) + (1 - e^{i\theta})\psi_1^2(y).$$

Since the basis $\mathscr{B}_2$ was chosen orthonormal, we get $(\lambda_1^2 e^{i\theta}, \lambda_2^2) \in B_{l_2^2}$ and then, $\lambda(\theta) \in B_X$. As $\|P\| = 1$, we have

$$|P(\lambda(\theta), 0, 0, \ldots) + (1 - e^{i\theta})\psi_1^2(y)| \leq 1,$$

for all $y = (0, 0, \ldots, y^{n+1}, y^{n+2}, \ldots)$ with $\|y\| \leq 1$. Taking $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ such that $P(\lambda(\theta), 0, 0, \ldots)$ and $\lambda(1 - e^{i\theta})\psi_1^2(y)$ have the same argument, then we will get

$$|P(\lambda(\theta), 0, 0, \ldots)| + |(1 - e^{i\theta})||\psi_1^2(y)| = |P(\lambda(\theta), 0, \ldots) + \lambda(1 - e^{i\theta})\psi_1^2(y)| \leq 1.$$

Then, for each y as above we can see that $|\psi_1^2(y)| \leq \frac{1 - f(\theta)}{g(\theta)}$, where $f(\theta) = |P(\lambda(\theta), 0, 0, \ldots)|$ and $g(\theta) = |1 - e^{i\theta}|$.

Since $f(0) = 1$, f is differentiable in $\theta = 0$. As f has a local maximum in $\theta = 0$ consequently $f'(0) = 0$. For $\theta > 0$,

$$g(\theta) = |1 - (\cos\theta + i\sin\theta)| = 2\sin\frac{\theta}{2}.$$

Hence, $g'(\theta) = \cos\frac{\theta}{2}$ and $\lim\limits_{\theta \to 0^+} \cos\frac{\theta}{2} = 1$. Using the L'Hospital rule,

$$|\psi_1^2(y)| \leq \lim_{\theta \to 0^+} \frac{1 - f(\theta)}{g(\theta)} = \lim_{\theta \to 0^+} \frac{-f'(\theta)}{g'(\theta)} = 0.$$

So, $|\psi_1^2(y)| = 0$ for all y as above. Then,

$$P(x^1, x^2, \ldots, x^n, y^{n+1}, y^{n+2}, \ldots) = P(x^1, x^2, \ldots, x^n, 0, 0, \ldots)$$

for all $(x^1, x^2, \ldots, x^n, y^{n+1}, y^{n+2}, \ldots) \in X$ and consequently P depends only on finite coordinates. □

We observe that in the proof of Theorem 2.1 the existence of a basis in $\mathbb{C}^n$ for which the coordinates can be written not vanishing, without changing the norm expression in the space, it was essential. Such characterization is not possible if we replace the Euclidean norm of $\mathbb{C}^n$ for example by the norm $\|\cdot\|_1$. In order to see this, let us consider the 2-homogeneous polynomial defined as

$$P(x) = 2(x_1^2)^2 + x_2^2\left(\sum_{j=3}^{\infty} \frac{1}{2^{j-2}} x_1^j\right),$$

where $x = \begin{pmatrix} x_1^1 & x_1^2 & x_1^3 & \cdots \\ & x_2^2 & x_2^3 & \cdots \\ & & x_3^3 & \cdots \end{pmatrix} \in c_0(l_1^i)$. Certainly P is not finite. We will prove it attains its norm. Let us observe that $P(e_1^2) = 2$. If $\|x\| \leq 1$, we have

$$|P(x)| = \left|2(x_1^2)^2 + x_2^2\left(\sum_{j=3}^{\infty} \frac{1}{2^{j-2}} x_1^j\right)\right| \leq 2\left|(x_1^2)^2\right| + \left|x_2^2\right|.$$

Then, it is enough to show that, for $0 \leq r \leq 1$, and $a, b \geq 0$ with $a + b = r$, we have $2a^2 + b \leq 2$. But, taking $b = r - a$, it is easy to see that $2a^2 + r - a \leq 2$, independent of the r chosen. Hence, P attains its norm in e_1^2.

As a consequence of the Theorem 2.1, we have the uniqueness of the norm-preserving extension for the norm-attaining 2-homogeneous polynomials.

Corollary 2.2. *Let P be a norm-attaining 2-homogeneous polynomial on X. Then, there exists a unique norm-preserving extension of P to X''.*

Proof. Let $\bar{P}$ be a norm-preserving extension of P to X''. Since P is norm-attaining, $\bar{P}$ attains its norm in a point $x^0 \in B_X$. By Theorem 2.1 $\bar{P}$ is finite. Then, another norm-preserving extension of P is finite too and it coincides with $\bar{P}$. □

The Corollary 2.2 is not true for n-homogeneous polynomials if $n \geq 3$. Indeed, by Corollary 2.5 in [4], for each $n \geq 3$, there exists a n-homogeneous polynomial that attains its norm, but it has at least two norm-preserving extensions.

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