KURZWEIL INTEGRAL FOR RIESZ SPACE-VALUED FUNCTIONS: UNIFORM CONVERGENCE THEOREM

G. A. MONTEIRO* — R. FERNANDEZ**

(Communicated by David Buhagiar)

ABSTRACT. Our objective here is to prove that the uniform convergence of a sequence of Kurzweil integrable functions implies the convergence of the sequence formed by its corresponding integrals.

©2012
Mathematical Institute
Slovak Academy of Sciences

1. Introduction

The Kurzweil integral for Riesz space-valued functions has been investigated in various papers (see, for example, [2], [3], [5], [7], [9]). In [9], it was proved that:

If X is a Dedekind σ -complete weakly σ -distributive Riesz space, $f \in X^{[a,b]}$ is a bounded function and $(f_n)_{n \in \mathbb{N}}$ is a sequence of Kurzweil integrable functions on $[a, b]$ such that $f_n \xrightarrow{u} f$, then

$$f \text{ is Kurzweil integrable on } [a, b] \quad \text{and} \quad \int_a^b f_n \xrightarrow{o} \int_a^b f. \quad (1)$$

Our purpose here is to obtain (1) without the assumption that f is bounded (see Theorem 3.3 in the sequel).

2. Notations and definitions

A Riesz space X is a real vector space on which a partial ordering $\leq$ compatible with its vector space structure is defined, that is, $(X, \leq)$ is a lattice. In what follows, X denotes a Riesz space.

A sequence $(x_n)_{n \in \mathbb{N}}$ of elements of X is said to be (o) -convergent to $x \in X$, and we write $x_n \xrightarrow{o} x$, if there exists a nonincreasing sequence $(a_n)_{n \in \mathbb{N}}$ of elements of X such that

$$a_n \searrow 0 \quad \text{and} \quad |x_k - x| \leq a_k, \quad \text{for all } k \in \mathbb{N}.$$

It is possible to prove that $x_n \xrightarrow{o} x$ if and only if $(x_n)_{n \in \mathbb{N}}$ is bounded and

$$x = \bigvee_{m=1}^{\infty} \left(\bigwedge_{i=m}^{\infty} x_i \right) = \bigwedge_{m=1}^{\infty} \left(\bigvee_{i=m}^{\infty} x_i \right).$$

This characterization of order convergence, whose proof can be found in [4], will be useful later.

A sequence $(f_n)_{n \in \mathbb{N}}$ of elements of $X^{[a,b]}$ is said to be uniformly convergent to $f \in X^{[a,b]}$, and we write $f_n \xrightarrow{u} f$, if there exists a nonincreasing sequence $(a_n)_{n \in \mathbb{N}}$ of elements of X such that

$$a_n \searrow 0 \quad \text{and} \quad |f_k(x) - f(x)| \leq a_k, \quad \text{for all } x \in [a, b] \quad \text{and } k \in \mathbb{N}.$$

It is easy to check that a sequence $(f_n)_{n \in \mathbb{N}}$ of elements of $X^{[a,b]}$ which is uniformly convergent is also Cauchy uniformly, that is, there exists a nonincreasing sequence $(b_n)_{n \in \mathbb{N}}$ of elements of X such that $b_n \searrow 0$ and, for each $k \in \mathbb{N}$,

$$|f_i(x) - f_j(x)| \leq b_k, \quad \text{for all } x \in [a, b] \quad \text{and } i, j \geq k.$$

A bounded double sequence $(a_{ij})_{ij}$ of elements of X is said to be a (D) -sequence if, for each $i \in \mathbb{N}$, the sequence $(a_{ij})_{j \in \mathbb{N}}$ is nonincreasing and $a_{ij} \searrow 0$ as $j \rightarrow \infty$. These double sequences, as Riečan has mentioned in [7], can be used to indicate an idea of proximity.

The Riesz space X is said to be a Dedekind σ -complete weakly σ -distributive Riesz space, if it is Dedekind σ -complete Riesz space (i.e., every non-empty, countable subset that has upper bound admits a least upper bound or supremum) and

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i\varphi(i)} \right) = 0, \quad \text{for all } (D)\text{-sequence } (a_{ij})_{ij} \text{ of elements of } X.$$

A function $f \in X^{[a,b]}$, where X is a Dedekind σ -complete weakly σ -distributive Riesz space, is said to be Kurzweil integrable on $[a, b]$ if there exists $L \in X$, satisfying:

there is a (D) -sequence $(a_{ij})_{ij}$ of elements of X such that,
for every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists $\delta = \delta(\varphi) \in \mathbb{R}_+^{[a,b]}$ such that

$$\left| \sum_{\Pi} f - L \right| \leq \bigvee_{i=1}^{\infty} a_{i\varphi(i)}, \quad \text{for all } \Pi \in \mathcal{A}_{[a,b]}(\delta). \quad (2)$$

The set $\mathcal{A}_{[a,b]}(\delta)$ consists of all partitions Π of $[a, b]$, $\Pi := \{(I_k, \alpha_k)\}_{k \in \Lambda}$, $\Lambda \subset \mathbb{N}$ finite and $\alpha_k \in I_k \subset]\alpha_k - \delta(\alpha_k), \alpha_k + \delta(\alpha_k)[$, for all $k \in \Lambda$. The element $\sum_{\Pi} f$ denotes the Riemann sum of f with respect to Π , that is, $\sum_{k \in \Lambda} f(\alpha_k) m(I_k)$, where m is the Lebesgue measure in $\mathbb{R}$.

The element L that satisfies (2) is unique, it is called the Kurzweil integral of f on $[a, b]$ and we denote by $\int_a^b f$.

The linear properties for the Kurzweil integral in this sense can be easily established (see [7, Theorem 4]).

3. Uniform Convergence theorem

The proof of Theorem 3.3 follows closely the ideas presented by Riečan and Vrábelová in [9]. For this, we need the following lemma whose proof can be found in [6, Lema 3.2.2] or [8].

LEMMA 3.1. *Let X be a Dedekind σ -complete weakly σ -distributive Riesz space. If $(a_{nij})_{nij}$ is a triple sequence of elements of X such that, for each $n \in \mathbb{N}$, $(a_{nij})_{ij}$ is a (D) -sequence, then there is a double sequence $(b_{ij})_{ij}$ of elements of X such that, for all $S \in X^+ \setminus \{0\}$ and $\varphi \in \mathbb{N}^{\mathbb{N}}$,*

$$S \wedge \left(\sum_{n=1}^k \bigvee_{i=1}^{\infty} a_{ni\varphi(i+n)} \right) \leq \bigvee_{j=1}^{\infty} (S \wedge b_{j\varphi(j)}), \quad \text{for all } k \in \mathbb{N}.$$

Moreover, $(S \wedge b_{ij})_{ij}$ is a (D) -sequence for all $S \in X^+ \setminus \{0\}$.

In order to prove the Uniform Convergence theorem we will also use the next result, concerning convergence, presented in [6].

PROPOSITION 3.2. *Let X be a Dedekind σ -complete weakly σ -distributive Riesz space and $(f_n)_{n \in \mathbb{N}}$ be a sequence of Kurzweil integrable functions on $[a, b]$. If $(f_n)_{n \in \mathbb{N}}$ is Cauchy uniformly, then there exists $\bigwedge_{m=1}^{\infty} \left(\bigvee_{i=m}^{\infty} \int_a^b f_i \right)$ and*

$$\int_a^b f_n \xrightarrow{o} \bigwedge_{m=1}^{\infty} \left(\bigvee_{i=m}^{\infty} \int_a^b f_i \right).$$

P r o o f. It is sufficient to show that the sequence $\left(\int_a^b f_n \right)_{n \in \mathbb{N}}$ is bounded and

$$\bigwedge_{m=1}^{\infty} \left(\bigvee_{i=m}^{\infty} \int_a^b f_i \right) \leq \bigvee_{m=1}^{\infty} \left(\bigwedge_{i=m}^{\infty} \int_a^b f_i \right), \quad (3)$$

considering that the other inequality always holds for bounded sequences.

Since $(f_n)_{n \in \mathbb{N}}$ is Cauchy uniformly, there exists a nonincreasing sequence $(b_n)_{n \in \mathbb{N}}$ of elements of X such that $b_n \searrow 0$ and, for each $k \in \mathbb{N}$,

$$|f_i(x) - f_j(x)| \leq b_k, \quad \text{for all } x \in [a, b] \text{ and } i, j \geq k.$$

Given $n \in \mathbb{N}$, for $i, j \geq n$, we obtain $f_i(x) \leq f_j(x) + b_n$, $x \in [a, b]$, and then

$$\int_a^b f_i \leq \left(\int_a^b f_j \right) + b_n (b - a), \quad \text{for all } i, j \geq n. \quad (4)$$

In particular, it follows that,

$$\left(\int_a^b f_1 \right) - b_1 (b - a) \leq \int_a^b f_k \leq \left(\int_a^b f_1 \right) + b_1 (b - a), \quad k \in \mathbb{N}$$

what proves that $\left(\int_a^b f_n \right)_{n \in \mathbb{N}}$ is bounded.

By (4), for all $n \in \mathbb{N}$, we have

$$\left(\bigvee_{i=n}^{\infty} \int_a^b f_i \right) \leq \left(\bigwedge_{j=n}^{\infty} \int_a^b f_j \right) + b_n (b - a),$$

which implies that,

$$\bigwedge_{m=1}^{\infty} \left(\bigvee_{k=m}^{\infty} \int_a^b f_k \right) - \bigvee_{m=1}^{\infty} \left(\bigwedge_{k=m}^{\infty} \int_a^b f_k \right) \leq \left(\bigvee_{i=n}^{\infty} \int_a^b f_i \right) - \left(\bigwedge_{j=n}^{\infty} \int_a^b f_j \right) \leq b_n (b-a).$$

Noting that $(b_n (b-a)) \searrow 0$, as $n \rightarrow \infty$, the inequality (3) is proved. $\square$

THEOREM 3.3 (Uniform Convergence theorem). *Let X be a Dedekind σ -complete weakly σ -distributive Riesz space and $(f_n)_{n \in \mathbb{N}}$ be a sequence of Kurzweil integrable functions on $[a, b]$. If $f_n \xrightarrow{u} f$, then f is Kurzweil integrable on $[a, b]$ and*

$$\int_a^b f_n \xrightarrow{o} \int_a^b f.$$

Proof. Since $f_n \xrightarrow{u} f$, there exists a nonincreasing sequence $(u_n)_{n \in \mathbb{N}}$ of elements of X such that

$$u_n \searrow 0 \quad \text{and} \quad |f_k(x) - f(x)| \leq u_k, \quad \text{for all } x \in [a, b] \quad \text{and} \quad k \in \mathbb{N}. \quad (5)$$

By (5), $(f_n)_{n \in \mathbb{N}}$ is Cauchy uniformly and, therefore, by Proposition 3.2, there exists $L \in X$ such that $\int_a^b f_n \xrightarrow{o} L$. Thus, there exists a nonincreasing sequence $(w_n)_{n \in \mathbb{N}}$ of elements of X such that

$$w_n \searrow 0 \quad \text{and} \quad \left| \int_a^b f_k - L \right| \leq w_k, \quad \text{for all } k \in \mathbb{N}. \quad (6)$$

Since f_n is Kurzweil integrable on $[a, b]$, for each $n \in \mathbb{N}$, there is a (D) -sequence $(a_{ij}^{(n)})_{ij}$ of elements of X such that, for every $\psi \in \mathbb{N}^{\mathbb{N}}$, there exists $\delta_n = \delta_n(\psi) \in \mathbb{R}_+^{[a, b]}$ such that

$$\left| \sum_{\Pi} f_n - \int_a^b f_n \right| \leq \bigvee_{i=1}^{\infty} a_{i\psi(i)}^{(n)}, \quad \text{for all } \Pi \in \mathcal{A}_{[a, b]}(\delta_n). \quad (7)$$

Using (5), (6) and (7), we have

$$\begin{aligned} \left| \sum_{\Pi} f - L \right| &\leq \left| \sum_{\Pi} f - \sum_{\Pi} f_n \right| + \left| \sum_{\Pi} f_n - \int_a^b f_n \right| + \left| \int_a^b f_n - L \right| \\ &\leq \left| \sum_{k \in \Lambda} (f(\alpha_k) - f_n(\alpha_k)) m(I_k) \right| + \bigvee_{i=1}^{\infty} a_{i\psi(i)}^{(n)} + w_n \end{aligned}$$

$$\begin{aligned}
&\leq \left(\sum_{k \in \Lambda} |f(\alpha_k) - f_n(\alpha_k)| m(I_k) \right) + \bigvee_{i=1}^{\infty} a_{i\psi(i)}^{(n)} + w_n \\
&\leq \left(\sum_{k \in \Lambda} u_n m(I_k) \right) + \bigvee_{i=1}^{\infty} a_{i\psi(i)}^{(n)} + w_n \\
&= u_n (b - a) + \bigvee_{i=1}^{\infty} a_{i\psi(i)}^{(n)} + w_n,
\end{aligned}$$

that is,

$$\left| \sum_{\Pi} f - L \right| \leq u_n (b - a) + w_n + \bigvee_{i=1}^{\infty} a_{i\psi(i)}^{(n)}, \quad \text{for all } \Pi \in \mathcal{A}_{[a,b]}(\delta_n). \quad (8)$$

Consider a triple sequence $(c_{nij})_{nij}$ of elements of X defined by

$$c_{nij} := \begin{cases} w_j + u_j (b - a) & \text{if } n = 1 \\ a_{ij}^{(n-1)} & \text{if } n \geq 2 \end{cases}, \quad i, j \in \mathbb{N}.$$

Note that, for every fixed $n \in \mathbb{N}$, the bounded double sequence $(c_{nij})_{ij}$ is a (D) -sequence of elements of X because, for each $i \in \mathbb{N}$, $(c_{nij})_{j \in \mathbb{N}}$ is a nonincreasing sequence and $c_{nij} \searrow 0$, as $j \rightarrow \infty$.

Let $S := u_1 (b - a) + w_1 + \bigvee_{i,j=1}^{\infty} a_{ij}^{(1)}$. If $S = 0$, then $u_1 = 0$ and, by (5), we obtain $f \equiv f_k$, for all $k \in \mathbb{N}$. Assuming $S \in X^+ \setminus \{0\}$, there exists, by Lemma 3.1, a double sequence $(b_{ij})_{ij}$ of elements of X such that $(S \wedge b_{ij})_{ij}$ is a (D) -sequence and, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$,

$$S \wedge \left(\sum_{r=1}^n \bigvee_{i=1}^{\infty} c_{ri\varphi(i+r)} \right) \leq \bigvee_{j=1}^{\infty} (S \wedge b_{j\varphi(j)}), \quad \text{for all } n \in \mathbb{N}. \quad (9)$$

Now we will prove that the (D) -sequence $(a_{ij})_{ij} := (S \wedge b_{ij})_{ij}$ satisfies the Kurzweil integrability condition for f .

Given $\varphi \in \mathbb{N}^{\mathbb{N}}$, consider $p := \min_{i \in \mathbb{N}} \varphi(i + 1)$ and let $\psi \in \mathbb{N}^{\mathbb{N}}$ be the function defined by $\psi(j) = \varphi(j + p + 1)$, for $j \in \mathbb{N}$. By (8), for $\Pi \in \mathcal{A}_{[a,b]}(\delta_p)$, $\delta_p = \delta_p(\psi)$, we obtain

$$\left| \sum_{\Pi} f - L \right| \leq u_p (b - a) + w_p + \bigvee_{i=1}^{\infty} a_{i\varphi(i+p+1)}^{(p)}.$$

Note that $\bigvee_{j=1}^{\infty} (u_{\varphi(j+1)}(b-a) + w_{\varphi(j+1)}) = u_p(b-a) + w_p$. Thus we can write

$$\left| \sum_{\Pi} f - L \right| \leq \bigvee_{j=1}^{\infty} (u_{\varphi(j+1)}(b-a) + w_{\varphi(j+1)}) + \bigvee_{i=1}^{\infty} a_{i\varphi(i+p+1)}^{(p)},$$

that is,

$$\left| \sum_{\Pi} f - L \right| \leq \bigvee_{j=1}^{\infty} c_{1j\varphi(j+1)} + \bigvee_{i=1}^{\infty} c_{(p+1)i\varphi(i+p+1)} \leq \sum_{r=1}^{p+1} \bigvee_{i=1}^{\infty} c_{ri\varphi(i+r)}. \quad (10)$$

On the other hand, by (8), for $n = 1$, we have

$$\left| \sum_{\Pi} f - L \right| \leq u_1(b-a) + w_1 + \bigvee_{i,j=1}^{\infty} a_{ij}^{(1)} = S, \quad (11)$$

whenever $\Pi \in \mathcal{A}_{[a,b]}(\delta_1)$, where $\delta_1 \in \mathbb{R}_+^{[a,b]}$.

Assuming that $\delta := \min\{\delta_1; \delta_p\}$, for $\Pi \in \mathcal{A}_{[a,b]}(\delta)$ we have, by (9), (10) and (11),

$$\left| \sum_{\Pi} f - L \right| \leq S \wedge \left(\sum_{r=1}^{p+1} \bigvee_{i=1}^{\infty} c_{ri\varphi(i+r)} \right) \leq \bigvee_{j=1}^{\infty} (S \wedge b_{j\varphi(j)}) = \bigvee_{j=1}^{\infty} a_{j\varphi(j)}.$$

Thus f is Kurzweil integrable on $[a, b]$, and moreover $\int_a^b f_n \xrightarrow{o} L = \int_a^b f$. $\square$

REFERENCES

- [1] BARTLE, R.: *A Modern Theory of Integration*. Grad. Stud. Math., Amer. Math. Soc., Providence, RI, 2001.
- [2] BOCCUTO, A.: *Differential and integral calculus in Riesz space*, Tatra Mt. Math. Publ. **14** (1998), 293–323.
- [3] BOCCUTO, A.—SKVORTSOV, V. A.: *Henstock-Kurzweil type integration of Riesz-space-valued functions and applications to Walsh series*, Real Anal. Exchange **29** (2003-04), 419–438.
- [4] LUXEMBURG, W. A.—ZAAANEN, A. C.: *Riesz spaces I*, North-Holland, Amsterdam, 1971.
- [5] MCGILL, P.: *Integration in vector lattices*, J. London Math. Soc. (2) **11** (1975), 347–360.
- [6] MONTEIRO, G. A.: *Integral de Kurzweil para funções a valores em um espaço de Riesz — uma introdução*. Master Dissertation, Universidade de São Paulo, São Paulo, 2007.
- [7] RIEČAN, B.: *On the Kurzweil integral for functions with values in ordered spaces I*, Acta Math. Univ. Comenian. (N.S.) **56-57** (1990), 75–83.

- [8] RIEČAN, B.—VOLAUF, P.: *On a technical lemma in lattice ordered groups*, Acta Math. Univ. Comenian. (N.S.) **44-45** (1984), 31–35.
- [9] RIEČAN, B.—VRÁBELOVÁ, M.: *On the Kurzweil integral for functions with values in ordered spaces II*, Math. Slovaca **43** (1993), 471–475.

Received 5. 7. 2009

Accepted 5. 2. 2010

**Instituto de Ciências Matemáticas e de Computação
Universidade de São Paulo
caixa postal 668
13560-970 São Carlos – SP
BRAZIL
E-mail: gam@icmc.usp.br*

***Instituto de Matemática e Estatística
Universidade de São Paulo
R. do Matão, 1010
05508-090 São Paulo – SP
BRAZIL
E-mail: roselif@ime.usp.br*