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LATTICE UNIFORMITIES ON PSEUDO-D-LATTICES

Anna Avallone — Paolo Vitolo

Dedicated to Prof. David J. Foulis on the occasion of his 80th birthday

(*Communicated by Sylvia Pulmannová*)

ABSTRACT. Let L be a pseudo-D-lattice. We prove that the lattice uniformities on L which make uniformly continuous the operations of L are uniquely determined by their system of neighbourhoods of 0 and form a distributive lattice.

Moreover we prove that every such uniformity is generated by a family of weakly subadditive $[0, +\infty]$-valued functions on L.

Introduction

Effect algebras (alias D-posets) have been independently introduced in 1994 by D. J. Foulis and M. K. Bennett in [11] and by F. Chovanec and F. Kôpka in [25] for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in Quantum Physics [16] and in Mathematical Economics [12, 19], in particular they are a generalization of orthomodular posets and MV-algebras. After 1994, a great number of papers concerning effect algebras have been published.

In 2001, A. Dvurecenskij and T. Vetterlein in [18] introduced the more general structure of a pseudo-effect algebra, which is a non-commutative generalization of an effect algebra and also generalizes the concept of pseudo-MV-algebra introduced by G. Georgescu and A. Iorgulescu in [21] as a non-commutative generalization of an MV-algebra.

The investigation on these structures is motivated by the non-commutative nature of certain psychological processes and quantum mechanical experiments

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(we refer to [15: pp. 3–4] for a more detailed explanation of these issues). Moreover, there even exists a programming language based on non-commutative logic [9]. For a study, see for example [14, 15, 17, 18, 26, 31] and many others.

Given a pseudo-D-lattice, i.e. a lattice-ordered pseudo-effect algebra, we are interested in the study of modular measures defined on it. To this end, using the same terminology as in the commutative case (see for instance [6]), we have introduced the concept of a D-uniformity, which our object of investigation in this paper.

Starting point for motivating this investigation is observing the key role played by D-uniformities (see for example [2, 4, 5]) in the study of modular measures on D-lattices, i.e. lattice-ordered effect algebras, and the result obtained in [8] establishing that a modular measure on a pseudo-D-lattice generates a D-uniformity. This leads us to believe that also in the context of pseudo-D-lattices the investigation of D-uniformities can play an important role in the study of modular measures.

Our aim in studying modular measures is to extend to pseudo-D-lattices the results we have found in D-lattices (see [3, 5, 7] and many others), which allowed us to develop a common generalization of topological methods that have been employed both in Noncommutative Measure Theory and in Fuzzy Measure Theory. In this perspective the present article is an important step inasmuch as here we prove some basic facts to be used in subsequent papers.

Topological methods in Measure Theory rely upon the so-called Fréchet-Nikodým topologies (briefly, FN-topologies). Here, because of the importance of several concepts — such as exhaustivity — which are uniform in nature, we follow the approach adopted by Weber (see [27–29] and others) and make use of uniformities. A D-uniformity on a pseudo-D-lattice is a uniformity which makes uniformly continuous all the relevant operations, namely the sum, the differences, the join and the meet.

In this paper we prove that a D-uniformity is univocally determined by its induced topology and, in fact, by the neighbourhoods of 0. Moreover, since it has been shown in [8] that any modular measure μ gives rise to a D-uniformity $\mathcal{U}(\mu)$, which is the coarsest uniformity making μ uniformly continuous, it is natural to investigate the possibility of generating a D-uniformity as the weak uniformity with respect to a family of mappings: this is carried out in the final part of the paper.

Here is how the paper is organized. In the first section several well-known facts are presented, to be used in the sequel. In Section 2 we give a description of those filters — called D-filters — which arise as systems of neighbourhoods of 0 in D-uniformities. Further, we prove that there is an order-isomorphism between the lattice of all D-uniformities on any pseudo-D-lattice L and the lattice of all D-filters on L; in particular, every D-uniformity on L is uniquely determined by its system of neighbourhoods of 0. As a consequence, we also obtain that the lattice of all D-uniformities on L is distributive.

These results extend similar ones estabilshed in [6] for lattice-ordered effect algebras, in [30] for orthomodular lattices, and in [10, 22] for MV-algebras. Morevover they give, as a particular case, the order-isomorphism found in [23].

In Section 3 we apply the results of the first part to prove that every D-uniformity on a pseudo-D-lattice L is generated by a family of weakly subadditive $[0, +\infty]$-valued functions on L. This generalizes the well-know fact [13] that an FN-topology on a Boolean algebra is generated by a family of submeasures.

1. Preliminaries

Definition 1.1. A partial algebra $(E, +, 0, 1)$, where $+$ is a partial binary operation and 0, 1 are constants, is called a *pseudo-effect algebra* if, for all $a, b, c \in E$, the following properties hold:

(E1) The sums $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case $(a + b) + c = a + (b + c)$.

(E2) For any $a \in E$, there exist exactly one $d \in E$ and exactly one $e \in E$ such that $a + d = e + a = 1$.

(E3) If $a + b$ exists, there are $d, e \in E$ such that $a + b = d + a = b + e$.

(E4) If $1 + a$ or $a + 1$ exists, then $a = 0$.

If $+$ is commutative, then E is an effect algebra. Given any $a, b \in E$, we write $a \perp b$ to mean that the sum $a + b$ is defined.

If we define $a \leq b$ if and only if there exists $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering on E such that $0 \leq a \leq 1$ for every $a \in E$. If E is a lattice with respect to this order, then we say that E is a *lattice-ordered pseudo-effect algebra*.

If E is a pseudo-effect algebra, we can define two partial binary operations on E such that, for $a, b \in E$, a/b is defined if and only if $b\backslash a$ is defined if and only if $a \leq b$, and in this case we have $(b\backslash a) + a = a + (a/b) = b$. In particular, we set ${}^{\perp}a = 1\backslash a$ and $a^{\perp} = a/1$.

As in the commutative case, one can define the structure of a *pseudo-D-poset*, by endowing a poset with two partial difference operations. As shown in [31], such a structure is equivalent to a pseudo-effect algebra.

Hence, we may (and do) consider a lattice-ordered pseudo-effect algebra to be the same thing as a *pseudo-D-lattice*. In the sequel, we shall always use the latter terminology, for the sake of brevity, and also to be consistent with our previous paper [8].

If L is a pseudo-D-lattice, we set, for every $a, b \in L$,

$$a \,{}^*\!\triangle\, b = (a \vee b)\backslash(a \wedge b) \quad \text{and} \quad a \,\triangle^*\, b = (a \wedge b)/(a \vee b).$$

For basic properties of pseudo-effect algebras we refer to [15, 18, 31]. In particular, we need the following facts.

Proposition 1.2. *Let E be a pseudo-effect algebra. For every $a, b, c \in E$, we have:*

(i) *If $a \leq b$, then $b\backslash(a/b) = (b\backslash a)/b = a$. In particular, for $b = 1$, we have ${}^\perp(a^\perp) = ({}^\perp a)^\perp = a$.*

(ii) *If $b \perp c$, then $a \leq b$ if and only if $a \perp c$ and $a + c \leq b + c$.*

(iii) *If $c \perp b$, then $a \leq b$ if and only if $c \perp a$ and $c + a \leq c + b$.*

(iv) *If $a \leq b \leq c$, then $c\backslash b \leq c\backslash a$, $b/c \leq a/c$, $b\backslash a \leq c\backslash a$ and $a/b \leq a/c$.*

(v) *If $a \leq b \leq c$, then $(c\backslash b)/(c\backslash a) = b\backslash a$ and $(a/c)\backslash(b/c) = a/b$. In particular, for $c = 1$, we have ${}^\perp b/{}^\perp a = b\backslash a$ and $a^\perp\backslash b^\perp = a/b$.*

(vi) *If $a \leq b \leq c$, then $(c\backslash a)\backslash(b\backslash a) = c\backslash b$ and $(a/b)/(a/c) = b/c$.*

Proposition 1.3. *Let L be a D-lattice. For every $a, b, c \in L$, we have:*

(i) *If $a \perp c$ and $b \perp c$, then $a \wedge b \perp c$ and $(a \wedge b) + c = (a + c) \wedge (b + c)$.*

(ii) *If $c \perp a$ and $c \perp b$, then $c \perp a \wedge b$ and $c + (a \wedge b) = (c + a) \wedge (c + b)$.*

(iii) *If $a \perp c$ and $b \perp c$, then $a \vee b \perp c$ and $(a \vee b) + c = (a + c) \vee (b + c)$.*

(iv) *If $c \perp a$ and $c \perp b$, then $c \perp a \vee b$ and $c + (a \vee b) = (c + a) \vee (c + b)$.*

(v) *If $a \leq c$ and $b \leq c$, then $c\backslash(a\wedge b) = (c\backslash a)\vee(c\backslash b)$ and $(a\wedge b)/c = (a/c)\vee(b/c)$. In particular, for $c = 1$, we have ${}^\perp(a\wedge b) = {}^\perp a\vee{}^\perp b$ and $(a\wedge b)^\perp = a^\perp\vee b^\perp$.*

(vi) *If $a \leq c$ and $b \leq c$, then $c\backslash(a\vee b) = (c\backslash a)\wedge(c\backslash b)$ and $(a\vee b)/c = (a/c)\wedge(b/c)$. In particular, for $c = 1$, we have ${}^\perp(a\vee b) = {}^\perp a\wedge{}^\perp b$ and $(a\vee b)^\perp = a^\perp\wedge b^\perp$.*

(vii) *If $c \leq a$ and $c \leq b$, then $(a\wedge b)\backslash c = (a\backslash c)\wedge(b\backslash c)$ and $c/(a\wedge b) = (c/a)\wedge(c/b)$. In particular, for $c = a\wedge b$, we have $\big(a\backslash(a\wedge b)\big)\wedge\big(b\backslash(a\wedge b)\big) = \big((a\wedge b)/a\big)\wedge\big((a\wedge b)/b\big) = 0$.*

(viii) *If $c \leq a$ and $c \leq b$, then $(a\vee b)\backslash c = (a\backslash c)\vee(b\backslash c)$ and $c/(a\vee b) = (c/a)\vee(c/b)$.*

We recall the definition of uniformity, in order to fix notation and terminology. We follow the book [24], to which we also refer the reader for further informations.

Definition 1.4. A *uniformity* on a set S is a filter $\mathcal{V}$ of subsets of $S \times S$ (i.e. relations on X) such that:

(U1) $\forall U \in \mathcal{V} \quad U \supseteq \Delta = \{(a, a) :\ a \in S\}$.

(U2) $\forall U \in \mathcal{V} \quad U^{-1} \in \mathcal{V}$.

(U3) $\forall U \in \mathcal{V} \quad \exists V \in \mathcal{V} \quad V \circ V \subseteq U$.

If $\mathcal{V}$ is a uniformity on S, then all sets of the form

$$U(a) = \{b \in S : (a,b) \in U\},$$

where $U \in \mathcal{V}$ and $a \in S$, give a neighbourhood system on S; the topology obtained in this way is said to be *induced* by $\mathcal{V}$.

On the other hand, if ϱ is a pseudometric on S, the collection of all sets of the form

$$U_\varepsilon = \{(a,b) \in S \times S : \varrho(a,b) < \varepsilon\},$$

where ε is a positive real number, is a base for a uniformity; it is well-know that a uniformity arises in this way (i.e. it is pseudometrizable) if and only if it has a countable base.

Uniformities allow to generalize to abstract spaces several non-purely topological concepts, such as uniform continuity or Cauchy nets.

Also, in topological groups, natural uniformities can be introduced (which induce the group topology); in case of an Abelian group, there is a unique such uniformity.

Notation 1.5. In the sequel we will always denote by L a pseudo-D-lattice.

A uniformity $\mathcal{U}$ on L is called *lattice uniformity* if $\vee$ and $\wedge$ are $\mathcal{U}$-uniformly continuous. A lattice uniformity $\mathcal{U}$ is called *D-uniformity* if the operations $+$, $\backslash$ and $/$ are uniformly continuous, too.

The set of all D-uniformities on L will be denoted by $\mathcal{DU}(L)$. It is easy to see that $\mathcal{DU}(L)$ — ordered by inclusion — is a complete lattice, with the discrete uniformity and the trivial uniformity as greatest and smallest elements, respectively.

Notation 1.6. Given $U, V \subseteq L \times L$, set

$$\begin{aligned}
U \vee V &= \{(a \vee c, b \vee d) : (a,b) \in U, (c,d) \in V\},\\
U \wedge V &= \{(a \wedge c, b \wedge d) : (a,b) \in U, (c,d) \in V\},\\
U \backslash V &= \{(a \backslash c, b \backslash d) : c \le a, d \le b, (a,b) \in U, (c,d) \in V\},\\
U / V &= \{(a / c, b / d) : a \le c, b \le d, (a,b) \in U, (c,d) \in V\},\\
U \,{}^*\!\triangle\, V &= \{(a \,{}^*\!\triangle\, c, b \,{}^*\!\triangle\, d) : (a,b) \in U, (c,d) \in V\},\\
U \triangle^* V &= \{(a \triangle^* c, b \triangle^* d) : (a,b) \in U, (c,d) \in V\},\\
U^\perp &= \{(a^\perp, b^\perp) : (a,b) \in U\},\\
{}^\perp U &= \{({}^\perp a, {}^\perp b) : (a,b) \in U\}.
\end{aligned}$$

It is known (see [27]) that a uniformity $\mathcal{U}$ on L is a lattice uniformity if and only if, for every $U \in \mathcal{U}$, there exists $V \in \mathcal{U}$ such that $V \vee \Delta \subseteq U$ and $V \wedge \Delta \subseteq U$ (where $\Delta = \{(a,a) : a \in L\}$).

Similarly, it has been shown in [8] that a lattice uniformity $\mathcal{U}$ on L is a D-uniformity if and only, for every $U \in \mathcal{U}$, there exist $V, W \in \mathcal{U}$ such that $V \backslash \Delta \subseteq U$, $\Delta \backslash V \subseteq U$ and $W^{\perp} \subseteq U$.

Definition 1.7. If $(G,+)$ is an Abelian group, a function $\mu\colon L \to G$ is called *modular measure*, if for every $a, b \in L$, we have

(M1) $\mu(a) + \mu(b) = \mu(a \vee b) + \mu(a \wedge b)$;

(M2) if $a \perp b$, then $\mu(a+b) = \mu(a) + \mu(b)$.

Modular measures on pseudo-D-lattices generalize measures on Boolean algebras. In fact, it is well-known (and easy to see) that a group-valued function μ on a Boolean algebra satisfies $\mu(0) = 0$ and condition (M1) if and only if it satisfies the analogue of (M2), i.e. it is additive on disjoint elements.

On the other hand, conditions (M1) and (M2) are unrelated in pseudo-D-lattices, and also in D-lattices.

In [8: Theor. 2.9], there is proved the following:

Theorem 1.8. *If G is a topological group, then every modular measure $\mu\colon L \to G$ generates a D-uniformity $\mathcal{U}(\mu)$ having as bases both the collections*

$$\{A_W : W \subseteq G \text{ neighbourhood of } 0\} \quad \text{and} \quad \{B_W : W \subseteq G \text{ neighbourhood of } 0\},$$

where

$$A_W = \{(a,b) \in L \times L : \ \forall c \leq a \,{}^*\!\triangle\, b \ \ \mu(c) \in W\},$$
$$B_W = \{(a,b) \in L \times L : \ \forall c \leq a \triangle^* b \ \ \mu(c) \in W\}.$$

Our interest is in extending topological methods for Measure Theory to the general context of modular measures on pseudo-D-lattices. In this vein, Theorem 1.8 above constitutes a strong motivation for the study of D-uniformities which we are going to carry out.

2. D-uniformities and D-filters

In this section we introduce the concept of D-filter, and show that any D-filter is precisely the neighbourhood system at 0 with respect to (the topology induced by) some D-uniformity; the D-uniformity so obtained is unique, and can be constructed in a natural way.

As a consequence we get that a D-uniformity is univocally determined by its induced topology. Moreover we prove that the complete lattice of all D-uniformities is distributive.

Definition 2.1. A filter $\mathcal{F}$ of subsets of L is called *D-filter* if it satisfies the following conditions:

(F1) For every $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that if $a, b \in G$ and $a \perp b$ then $a + b \in F$.

(F2) For every $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that if $a \in G$ and $c \in L$ then $(a \vee c)\backslash c \in F$.

(F3) For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a, b \in L$, $a \le b$ and $b\backslash a \in G$ then $a/b \in F$.

(F4) For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a, b \in L$, $a \le b$ and $a/b \in G$ then $b\backslash a \in F$.

Note that, if L is a D-lattice, the above definition agrees with [6: Def. 2.1], since conditions (F3) and (F4) become trivial in that case. On the contrary, in our present general situation, adding these two conditions will turn out to be crucial.

Notation 2.2. We denote by $\mathcal{FND}(L)$ the set of all D-filters on L.

As we are going to prove, $\mathcal{FND}(L)$ is in fact a complete distributive lattice, and is isomorphic to the lattice $\mathcal{DU}(L)$.

Proposition 2.3. *A D-filter $\mathcal{F}$ on L satisfies the following conditions:*

(1) *For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a \in G$, $b \in L$ and $b \le a$ then $b \in F$.*

(2) *For every $F \in \mathcal{F}$, there exists $G \in \mathcal{F}$ such that if $a \in G$ and $c \in L$ then $c/(a \vee c) \in F$.*

(3) *For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a\,{}^*\!\triangle\, b \in G$ then $a \triangle^{\!*} b \in F$.*

(4) *For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a \triangle^{\!*} b \in G$ then $a\,{}^*\!\triangle\, b \in F$.*

Proof.

(1) Let $F \in \mathcal{F}$, and let $G \in \mathcal{F}$ satisfying (F2). Given any $a \in G$ and any $b \in L$ with $b \le a$, set $c = b/a$. Since $c \le a$, by Proposition 1.2(i) we get

$$b = a\backslash(b/a) = a\backslash c = (a \vee c)\backslash c \in F.$$

(2) Let $F \in \mathcal{F}$; choose $H \in \mathcal{F}$ satisfying (F3), and $G \in \mathcal{F}$ satisfying (F2) with H in place of F. Given any $a \in G$ and $c \in L$, let $b = a \vee c$. By (F2) we have $b\backslash c = (a \vee c)\backslash c \in H$; hence, by (F3), we get $c/(a \vee c) = c/b \in F$.

(3) and (4) follow from (F3) and (F4), respectively, with $a \wedge b$ in place of a and $a \vee b$ in place of b. □

In order to establish further properties of D-filters, a number of lemmas is needed.

Lemma 2.4.

(a) *If* $c \le a$ *and* $c \le b$, *then* $(a\backslash c) \,{}^*\!\triangle\, (b\backslash c) = a \,{}^*\!\triangle\, b$.

(b) *If* $a \le c$ *and* $b \le c$, *then* $(c\backslash a) \triangle^* (c\backslash b) = a \,{}^*\!\triangle\, b$. *In particular, for* $c = 1$, *we have* ${}^\perp a \triangle^* {}^\perp b = a \,{}^*\!\triangle\, b$.

(c) *If* $c \le a$ *and* $c \le b$, *then* $(c/a) \triangle^* (c/b) = a \triangle^* b$.

(d) *If* $a \le c$ *and* $b \le c$, *then* $(a/c) \,{}^*\!\triangle\, (b/c) = a \triangle^* b$. *In particular, for* $c = 1$, *we have* $a^\perp \,{}^*\!\triangle\, b^\perp = a \triangle^* b$.

(e) *If* $a \perp c$ *and* $b \perp c$, *then* $(a + c) \,{}^*\!\triangle\, (b + c) = a \,{}^*\!\triangle\, b$.

(f) *If* $c \perp a$ *and* $c \perp b$, *then* $(c + a) \triangle^* (c + b) = a \triangle^* b$.

P r o o f. We note that (a) and (b) can be derived from the proof of [8: Theor. 2.9], but we repeat the argument for the sake of completeness.

(a) Set $r = a\backslash c$ and $s = b\backslash c$. Applying Proposition 1.3(vii) and (viii), we get $r \wedge s = (a \wedge b)\backslash c$ and $r \vee s = (a \vee b)\backslash c$. Hence, by Proposition 1.2(vi), we have

$$r \,{}^*\!\triangle\, s = \big((a \vee b)\backslash c\big)\backslash\big((a \wedge b)\backslash c\big) = (a \vee b)\backslash(a \wedge b) = a \,{}^*\!\triangle\, b.$$

(b) Set $r = c\backslash a$ and $s = c\backslash b$. Applying Proposition 1.3(v) and (vi), we get $r \vee s = c\backslash(a \wedge b)$ and $r \wedge s = c\backslash(a \vee b)$. Hence, by Proposition 1.2(v), we have

$$r \triangle^* s = \big(c\backslash(a \vee b)\big)/\big(c\backslash(a \wedge b)\big) = (a \vee b)\backslash(a \wedge b) = a \,{}^*\!\triangle\, b.$$

(c) and (d) can be proved in a similar way as (a) and (b).

(e) Set $r = a + c$ and $s = b + c$. Since $a = r\backslash c$ and $b = s\backslash c$, from (a) we get

$$a \,{}^*\!\triangle\, b = (r\backslash c) \,{}^*\!\triangle\, (s\backslash c) = r \,{}^*\!\triangle\, s = (a + c) \,{}^*\!\triangle\, (b + c).$$

(f) can be proved similarly as (e), using (c) instead of (a). □

Lemma 2.5. *Let* $a, b, c \in L$. *If* $a \vee b \perp c$ *then* $\big((a+c) \vee (b+c)\big)\backslash(a+c) = (a \vee b)\backslash a$. *Similarly, if* $c \perp a \vee b$ *then* $(c + a)/\big((c + a) \vee (c + b)\big) = a/(a \vee b)$.

P r o o f. To prove the first statement, suppose $a \vee b \perp c$ and let $d = (a \vee b)\backslash a$. Applying Proposition 1.3(iii), we have

$$d + (a + c) = (d + a) + c = \Big(\big((a \vee b)\backslash a\big) + a\Big) + c = (a \vee b) + c = (a + c) \vee (b + c),$$

hence $\big((a + c) \vee (b + c)\big)\backslash(a + c) = d = (a \vee b)\backslash a$.

The second statement is proved in an analogous way. □

Lemma 2.6.

(a) *If* $r = a \,{}^*\!\triangle\, b$ *and* $s = \big((a\vee c)\wedge(b\vee c)\big)\backslash(a\wedge b)$, *then* $(r\vee s)\backslash s = (a\vee c) \,{}^*\!\triangle\, (b\vee c)$.

(b) *If* $r = a \triangle^* b$ *and* $s = (a\wedge b)/\big((a\vee c)\wedge(b\vee c)\big)$, *then* $s/(r\vee s) = (a\vee c) \triangle^* (b\vee c)$.

Proof.

(a) First observe that

$$a \vee b \vee c = (a \vee b) \vee \big((a \vee c) \wedge (b \vee c)\big). \tag{2.1}$$

Indeed, being $(a \vee c) \wedge (b \vee c) \geq c$, the left-hand side of (2.1) is less than or equal to the right-hand side. On the other hand, $(a \vee c) \wedge (b \vee c) \leq a \vee c \vee a \vee b = a \vee b \vee c$, whence the reverse inequality follows.

Now, since $r = (a \vee b) \backslash (a \wedge b)$, we have $a \vee b = r + (a \wedge b)$; similarly $(a \vee c) \wedge (b \vee c) = s + (a \wedge b)$. Hence from (2.1), applying Lemma 2.5, we get

$$\begin{aligned}(a \vee c) \mathbin{\triangle^*} (b \vee c) &= (a \vee b \vee c) \backslash \big((a \vee c) \wedge (b \vee c)\big)\\ &= \big((a \vee b) \vee \big((a \vee c) \wedge (b \vee c)\big)\big) \backslash \big((a \vee c) \wedge (b \vee c)\big)\\ &= \big(\big(r + (a \wedge b)\big) \vee \big(s + (a \wedge b)\big)\big) \backslash \big(s + (a \wedge b)\big)\\ &= (r \vee s) \backslash s.\end{aligned}$$

(b) Similar to (a). □

The following properties of D-filters will be useful in the sequel.

Proposition 2.7. *A D-filter $\mathcal{F}$ on L satisfies the following conditions:*

(1) *For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a, b \in G$ then $a \vee b \in F$.*

(2) *For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a, b, c \in L$ and $a \mathbin{{}^*\triangle} b \in G$ then $(a \vee c) \mathbin{{}^*\triangle} (a \vee c) \in F$.*

(3) *For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a, b, c \in L$ and $a \mathbin{\triangle^*} b \in G$ then $(a \vee c) \mathbin{\triangle^*} (a \vee c) \in F$.*

(4) *For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a, b, c \in L$ and $a \mathbin{{}^*\triangle} b \in G$ then $(a \wedge c) \mathbin{{}^*\triangle} (a \wedge c) \in F$.*

(5) *For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a, b, c \in L$ and $a \mathbin{\triangle^*} b \in G$ then $(a \wedge c) \mathbin{\triangle^*} (a \wedge c) \in F$.*

(6) *For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a, b, c \in L$, $a \mathbin{{}^*\triangle} b \in G$ and $b \mathbin{{}^*\triangle} c \in G$ then $a \mathbin{{}^*\triangle} c \in F$.*

(7) *For every $F \in \mathcal{F}$ there exists $G \in \mathcal{F}$ such that if $a, b, c \in L$, $a \mathbin{\triangle^*} b \in G$ and $b \mathbin{\triangle^*} c \in G$ then $a \mathbin{\triangle^*} c \in F$.*

Proof.

(1) Let $F \in \mathcal{F}$; choose $H \in \mathcal{F}$ satisfying (F1), and $G \in \mathcal{F}$ satisfying (F2) with H in place of F. Given any $a, b \in G$, by (F2) we have that both $(a \vee b) \backslash b$ and $b = (b \vee 0) \backslash 0$ belong to H. Hence, from (F1) we obtain $a \vee b = \big((a \vee b) \backslash b\big) + b \in F$.

(2) Given $F \in \mathcal{F}$, choose $G \in \mathcal{F}$ satisfying (F2). Now let $r = a \mathbin{{}^*\triangle} b$ and $s = \big((a \vee c) \wedge (b \vee c)\big) \backslash (a \wedge b)$; recalling Lemma 2.6(a), we have $(a \vee c) \mathbin{{}^*\triangle} (b \vee c) = (r \vee s) \backslash s$. Therefore, as $r \in G$, we get $(a \vee c) \mathbin{{}^*\triangle} (b \vee c) \in F$ by (F2).

(3) Given $F \in \mathcal{F}$, choose $G \in \mathcal{F}$ satisfying Proposition 2.3(2). Now let $r = a \mathbin{\triangle^*} b$ and $s = (a \wedge b)/\big((a \vee c) \wedge (b \vee c)\big)$; recalling Lemma 2.6(b), we have $(a \vee c) \mathbin{\triangle^*} (b \vee c) = s/(r \vee s)$. Therefore, as $r \in G$, we get $(a \vee c) \mathbin{\triangle^*} (b \vee c) \in F$ by Proposition 2.3(2).

(4) Let $F \in \mathcal{F}$, and choose $G \in \mathcal{F}$ satisfying (3). Given $a, b, c \in L$ with $a \mathbin{{}^*\triangle} b \in G$, by Lemma 2.4(b) we have ${}^\perp a \mathbin{\triangle^*} {}^\perp b = a \mathbin{{}^*\triangle} b \in G$. Hence recalling Proposition 1.2(i) and Proposition 1.3(v), we obtain

$$\begin{aligned}(a \wedge c) \mathbin{{}^*\triangle} (b \wedge c) &= {}^\perp(a \wedge c) \mathbin{\triangle^*} {}^\perp(b \wedge c)\\ &= ({}^\perp a \vee {}^\perp c) \mathbin{\triangle^*} ({}^\perp b \vee {}^\perp c) \in F.\end{aligned}$$

(5) Similar to the proof of (4), but using (2) instead of (3).

(6) Let $F \in \mathcal{F}$. By (1), we can choose $F_1 \in \mathcal{F}$ such that

$$a, b \in F_1 \implies a \vee b \in F. \tag{2.2}$$

By Proposition 2.3(1), we can choose $F_2 \in \mathcal{F}$ such that

$$\big[a \in F_2 \text{ and } b \leq a\big] \implies b \in F_1. \tag{2.3}$$

Further, by (F1), we can choose $F_3 \in \mathcal{F}$ such that

$$\big[a \in F_3 \text{ and } a \perp b\big] \implies a + b \in F_2. \tag{2.4}$$

Finally, by (2), we can choose $G \in \mathcal{F}$ such that

$$a \mathbin{{}^*\triangle} b \in G \implies (a \vee c) \mathbin{{}^*\triangle} (b \vee c) \in F_3. \tag{2.5}$$

Now consider $a, b, c \in L$ with $a \mathbin{{}^*\triangle} b \in G$ and $b \mathbin{{}^*\triangle} c \in G$. Let $r = (a \vee b \vee c)\backslash(b \vee c)$ and $s = (b \vee c)\backslash c$. As $r = (a \vee b \vee c) \mathbin{{}^*\triangle} (b \vee c) = \big(a \vee (b \vee c)\big) \mathbin{\triangle^*} \big(b \vee (b \vee c)\big)$ and $a \mathbin{{}^*\triangle} b \in G$, it follows from (2.5) that $r \in F_3$; similarly, as $s = (b \vee c) \mathbin{{}^*\triangle} c = (b \vee c) \mathbin{{}^*\triangle} (c \vee c)$ and $b \mathbin{{}^*\triangle} c \in G$, it follows from (2.5) that $s \in F_3$, too.

From the definition of r and s, we get

$$a \vee b \vee c = r + (b \vee c) = r + (s + c) = (r + s) + c,$$

whence $(a \vee b \vee c)\backslash c = r + s$. Since $r, s \in F_3$, it follows from (2.4) that $(a \vee b \vee c)\backslash c \in F_2$; hence $(a \vee c)\backslash c \in F_1$ by (2.3).

Letting $p = (a \vee b \vee c)\backslash(a \vee b)$, $q = (a \vee b)\backslash a$ and arguing in a similar way as above, we obtain that $(a \vee c)\backslash a \in F_1$. By Proposition 1.3(v) and (2.2), we conclude that

$$a \mathbin{{}^*\triangle} c = (a \vee c)\backslash(a \wedge c) = \big((a \vee c)\backslash a\big) \vee \big((a \vee c)\backslash c\big) \in F.$$

(7) Similar to (6). □

Now we establish the main results of this section.

Theorem 2.8.

(a) *If $\mathcal{U}$ is a D-uniformity on L, the filter $\mathcal{F}_{\mathcal{U}}$ of neighbourhoods of 0 with respect to $\mathcal{U}$ is a D-filter.*

(b) *Let $\mathcal{F}$ be a D-filter on L. For every $F \in \mathcal{F}$, define*

$$^{\triangle}F = \{(a,b) \in L \times L :\ a \,{}^{*}\!\triangle\, b \in F\}, \tag{2.6}$$

$$F^{\triangle} = \{(a,b) \in L \times L :\ a \triangle^{*} b \in F\}. \tag{2.7}$$

Then both $^{\triangle}\mathcal{B} = \{^{\triangle}F :\ F \in \mathcal{F}\}$ and $\mathcal{B}^{\triangle} = \{F^{\triangle} :\ F \in \mathcal{F}\}$ are bases for a D-uniformity whose filter of neighbourhoods of 0 coincides with $\mathcal{F}$.

(c) *If two D-uniformities $\mathcal{V}$ and $\mathcal{W}$ on L have the same filter of neighbourhoods of 0 then $\mathcal{V} = \mathcal{W}$.*

Proof.

(a) Since $+$ is continuous at $(0,0)$ with respect to $\mathcal{U}$, for every $F \in \mathcal{F}_{\mathcal{U}}$ there exists $G \in \mathcal{F}_{\mathcal{U}}$ such that if $(a,b) \in G \times G$ and $a \perp b$ then $a+b \in F$. Thus $\mathcal{F}_{\mathcal{U}}$ verifies (F1).

To prove that $\mathcal{F}_{\mathcal{U}}$ verifies (F2), consider $F \in \mathcal{F}_{\mathcal{U}}$ and choose $U \in \mathcal{U}$ with $U(0) \subseteq F$. As $\mathcal{U}$ is a D-uniformity, there are $V_1, V_2 \in \mathcal{U}$ such that $V_1 \backslash \Delta \subseteq U$ and $V_2 \vee \Delta \subseteq V_1$.

Let $G = V_2(0)$. Given $a \in G$ and $c \in L$, since $(0,a) \in V_2$ we get $(c, a \vee c) \in V_2 \vee \Delta \subseteq V_1$; hence

$$\big(0, (a \vee c)\backslash c\big) = \big(c\backslash c, (a \vee c)\backslash c\big) \in V_1 \backslash \Delta \subseteq U.$$

It follows that $(a \vee c)\backslash c \in U(0) \subseteq F$.

Now we show that $\mathcal{F}_{\mathcal{U}}$ satisfies (F3). Let $F \in \mathcal{F}_{\mathcal{U}}$ and take $U \in \mathcal{U}$ such that $U(0) \subseteq F$. Since $\mathcal{U}$ is a D-uniformity, there are $V_1, V_2 \in \mathcal{U}$ such that $\Delta / V_1 \subseteq U$ and $V_2 + \Delta \subseteq V_1$.

Let $G = V_2(0)$. Given $a, b \in L$ with $a \le b$ and $b \backslash a \in G$, we have $(0, b\backslash a) \in V_2$, so that $(a,b) = \big(0 + a, (b\backslash a) + a\big) \in V_2 + \Delta \subseteq V_1$; hence $(0, a/b) = (a/a, a/b) \in \Delta / V_1 \subseteq U$ and therefore $a/b \in U(0) \subseteq F$.

The proof that $\mathcal{F}_{\mathcal{U}}$ satisfies (F4) is similar.

(b) Clearly, for every $F \in \mathcal{F}$, both $^{\triangle}F$ and $F^{\triangle}$ are symmetric and contain the diagonal Δ. Further, if $F_1, F_2 \in \mathcal{F}$ then

$$(F_1 \cap F_2)^{\triangle} = {F_1}^{\triangle} \cap {F_2}^{\triangle} \quad \text{and} \quad {}^{\triangle}(F_1 \cap F_2) = {}^{\triangle}F_1 \cap {}^{\triangle}F_2.$$

Moreover, if $F \in \mathcal{F}$ and $G \in \mathcal{F}$ satisfies Proposition 2.7(6) or (7), then $G^{\triangle} \circ G^{\triangle} \subseteq F^{\triangle}$ or $^{\triangle}G \circ {}^{\triangle}G \subseteq {}^{\triangle}F$, respectively. Finally if $F \in \mathcal{F}$ and $G \in \mathcal{F}$ satisfies Proposition 2.3(3) or (4), then $^{\triangle}G \subseteq F^{\triangle}$ or $G^{\triangle} \subseteq {}^{\triangle}F$, respectively.

Therefore $^{\triangle}\mathcal{B}$ and $\mathcal{B}^{\triangle}$ are both bases of the same uniformity $\mathcal{U}$. We now show that $\mathcal{U}$ is a D-uniformity.

First, let us prove that $\mathcal{U}$ is a lattice uniformity. To this end we consider $U \in \mathcal{U}$ and show that there is $V \in \mathcal{U}$ with $V \vee \Delta \subseteq U$ and $V \wedge \Delta \subseteq U$.

Given $F \in \mathcal{F}$ such that ${}^{\triangle}F \subseteq U$, take $G \in \mathcal{F}$ satisfying Proposition 2.7(2), and let $V = {}^{\triangle}G$. If $(a,b) \in V \vee \Delta$, choose $c,d,r \in L$ with $(c,d) \in V$ and $(c \vee r, d \vee r) = (a,b)$. Then, as $(c,d) \in V = {}^{\triangle}G$, we have $c \,{}^{*}\!\triangle\, d \in G$ and it follows that

$$a \,{}^{*}\!\triangle\, b = (c \vee r) \,{}^{*}\!\triangle\, (d \vee r) \in F.$$

Hence $(a,b) \in {}^{\triangle}F \subseteq U$. Similarly, if $G \in \mathcal{F}$ satisfies Proposition 2.7(4), then letting $V = {}^{\triangle}G$ one gets $V \wedge \Delta \subseteq U$.

In view of [8: Prop. 2.7], to complete the proof that $\mathcal{U}$ is D-uniformity it suffices to show that for every $U \in \mathcal{U}$ there exist $V, W \in \mathcal{U}$ such that $V \backslash \Delta \subseteq U$, $\Delta \backslash V \subseteq U$ and $W^{\perp} \subseteq U$.

Given $U \in \mathcal{U}$, choose $F \in \mathcal{F}$ with ${}^{\triangle}F \subseteq U$, and let $V = {}^{\triangle}F$. It follows from Lemma 2.4(a) that

$$\begin{aligned} V \backslash \Delta &= \{(a \backslash c, b \backslash c) :\ c \le a,\ c \le b,\ a \,{}^{*}\!\triangle\, b \in F\} \\ &= \{(a \backslash c, b \backslash c) :\ c \le a,\ c \le b,\ (a \backslash c) \,{}^{*}\!\triangle\, (b \backslash c) \in F\} = {}^{\triangle}F \subseteq U. \end{aligned}$$

Similarly one sees that $\Delta \backslash V \subseteq U$, too.

Now, given $U \in \mathcal{U}$, choose $F \in \mathcal{F}$ with ${}^{\triangle}F \subseteq U$, and let $W = F^{\triangle}$. If $(a,b) \in W^{\perp}$, by Proposition 1.2(i) we have $({}^{\perp}a, {}^{\perp}b) \in W = F^{\triangle}$. Thus, applying Lemma 2.4(b), we get

$$a \,{}^{*}\!\triangle\, b = {}^{\perp}a \,\triangle^{*}\, {}^{\perp}b \in F$$

so that $(a,b) \in {}^{\triangle}F \subseteq U$.

It remains to prove that the filter of neighbourhoods of 0 with respect to $\mathcal{U}$ coincides with $\mathcal{F}$.

Note that, for every $F \in \mathcal{F}$,

$${}^{\triangle}F(0) = \{a \in L :\ (0,a) \in {}^{\triangle}F\} = \{a \in L :\ 0 \,{}^{*}\!\triangle\, a \in F\} = F \tag{2.8}$$

and therefore F is a neighbourhood of 0. Conversely, if G is a neighbourhood of 0 then, since ${}^{\triangle}\mathcal{B}$ is a base for $\mathcal{U}$, there exists $F \in \mathcal{F}$ such that ${}^{\triangle}F(0) \subseteq G$, hence $F \subseteq G$ by (2.8); as $\mathcal{F}$ is a filter, we conclude that $G \in \mathcal{F}$.

(c) Denote by $\mathcal{F}$ the filter of neighbourhoods of 0 with respect to both $\mathcal{V}$ and $\mathcal{W}$, and let $\mathcal{U}$ be the D-uniformity constructed from $\mathcal{F}$ according to (b). It suffices to show that $\mathcal{V} = \mathcal{U}$.

Let $F \in \mathcal{F}$. As F is a neighbourhood of 0 in the topology induced by $\mathcal{V}$, we may find $U \in \mathcal{V}$ with $U(0) \subseteq F$. Since by [8: Prop. 2.5] the operation ${}^{*}\!\triangle$ is $\mathcal{V}$-uniformly continuous, there exists $V \in \mathcal{V}$ such that $V \,{}^{*}\!\triangle\, \Delta \subseteq U$. We have $V \subseteq {}^{\triangle}F$ because, if $(a,b) \in V$ then

$$(0, a \,{}^{*}\!\triangle\, b) = (a \,{}^{*}\!\triangle\, a, b \,{}^{*}\!\triangle\, a) \in V \,{}^{*}\!\triangle\, \Delta \subseteq U,$$

whence $a \,{}^*\!\triangle\, b \in U(0) \subseteq F$. It follows that $\mathcal{V}$ is finer than $\mathcal{U}$.

Conversely, let $U \in \mathcal{V}$. Choose a symmetric $V_1 \in \mathcal{V}$ with $V_1 \circ V_1 \subseteq U$, and take $V_2, V_3 \in \mathcal{V}$ such that $V_2 \vee \Delta \subseteq V_1$ and $V_3 + \Delta \subseteq V_2$. Now let $F = V_3(0)$, so that $F \in \mathcal{F}$. We claim that ${}^{\triangle}F \subseteq U$.

Indeed, if $(a, b) \in {}^{\triangle}F$ then $a \,{}^*\!\triangle\, b \in F$, that is $(0, a \,{}^*\!\triangle\, b) \in V_3$. It follows that

$$(a \wedge b, a \vee b) = \big(0 + (a \wedge b), (a \,{}^*\!\triangle\, b) + (a \wedge b)\big) \in V_3 + \Delta \subseteq V_2,$$

whence

$$(a, a \vee b) = \big((a \wedge b) \vee a, (a \vee b) \vee a\big) \in V_2 \vee \Delta \subseteq V_1.$$

Similarly one obtains that $(b, a \vee b) \in V_1$; as V_1 is symmetric, we have $(a \vee b, b) \in V_1$, too. Therefore $(a, b) \in V_1 \circ V_1 \subseteq U$, as claimed.

We conclude that $\mathcal{V} \subseteq \mathcal{U}$, which completes the proof. □

Notation 2.9. Let $\mathcal{U}$ be a D-uniformity. In the sequel, according to Theorem 2.8(a), the filter of neighbourhoods of 0 with respect to $\mathcal{U}$ will be denoted by $\mathcal{F}_{\mathcal{U}}$.

Corollary 2.10. *The mapping $\Psi \colon \mathcal{U} \mapsto \mathcal{F}_{\mathcal{U}}$ is an order-isomorphism of $\mathcal{DU}(L)$ onto $\mathcal{FND}(L)$ (both ordered by inclusion).*

Proof. By Theorem 2.8(a), Ψ maps $\mathcal{DU}(L)$ into $\mathcal{FND}(L)$. Moreover, if $\mathcal{F} \in \mathcal{FND}(L)$ and we denote by $\Phi(\mathcal{F})$ the D-uniformity constructed as in Theorem 2.8(b) then, since $\Psi\big(\Phi(\mathcal{F})\big) = \mathcal{F}$, it follows that Ψ is onto.

If $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{FND}(L)$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2$, we have

$$\{{}^{\triangle}F : \ F \in \mathcal{F}_1\} \subseteq \{{}^{\triangle}F : \ F \in \mathcal{F}_2\},$$

hence $\Phi(\mathcal{F}_1) \subseteq \Phi(\mathcal{F}_2)$. On the other hand, if $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{DU}(L)$ and $\mathcal{U}_1 \subseteq \mathcal{U}_2$, the topology induced by $\mathcal{U}_1$ is coarser than the one induced by $\mathcal{U}_2$; therefore $\Psi(\mathcal{U}_1) \subseteq \Psi(\mathcal{U}_2)$.

It remains to prove that Ψ is one-to-one, but this follows immediately from Theorem 2.8(c). □

Remark 2.11. Theorem 2.8 and Corollary 2.10 extend the result of [6: Theorem 2.4] for D-lattices which, in turn, extended similar results for orthomodular lattices [30: Theor. 1.1] and for MV-algebras [22: Theor. 3.6].

Moreover from Corollary 2.10, by restricting to principal filters (i.e. collections of supersets of a given set) one obtains an order-isomorphism between Riesz congruences and normal Riesz ideals which has been established, using a different approach, in [23: Theor. 3.4(3)].

In analogy with (2.6) and (2.7), we introduce the following:

Notation 2.12. Given $F \subseteq L$, we set:

$$F^+ = \big\{(a,b) \in L \times L : \ (\exists h,k \in F)(a \perp h \ \ \& \ \ b \perp k \ \ \& \ \ a+h = b+k)\big\};$$
$$^+F = \big\{(a,b) \in L \times L : \ (\exists h,k \in F)(h \perp a \ \ \& \ \ k \perp b \ \ \& \ \ h+a = k+b)\big\};$$
$$F^\backslash = \big\{(a,b) \in L \times L : \ (\exists i,j \in F)(i \leq a \ \ \& \ \ j \leq b \ \ \& \ \ a\backslash i = b\backslash j)\big\};$$
$$^/F = \big\{(a,b) \in L \times L : \ (\exists i,j \in F)(i \leq a \ \ \& \ \ j \leq b \ \ \& \ \ i/a = j/b)\big\}.$$

The following result gives alternative ways to construct the D-uniformity which corresponds to a D-filter.

Proposition 2.13. *If $\mathcal{U}$ is a D-uniformity then the following are bases for $\mathcal{U}$:*

(1) $\mathcal{B}^+ = \{F^+ : \ F \in \mathcal{F}_\mathcal{U}\}$;
(2) $^+\mathcal{B} = \{^+F : \ F \in \mathcal{F}_\mathcal{U}\}$;
(3) $\mathcal{B}^\backslash = \{F^\backslash : \ F \in \mathcal{F}_\mathcal{U}\}$;
(4) $^/\mathcal{B} = \{^/F : \ F \in \mathcal{F}_\mathcal{U}\}$.

Proof.

(1) Let $F \in \mathcal{F}$. We show that there exist $F_1, F_2 \in \mathcal{F}$ such that ${F_1}^\triangle \subseteq F^+$ and $F_2^+ \subset F^\triangle$.

Take $F_1 \in \mathcal{F}$ satisfying Proposition 2.3(1). Suppose $(a,b) \in {F_1}^\triangle$, i.e. $a \triangle^* b \in F_1$, and let $h = a/(a \vee b)$, $k = b/(a \vee b)$. By Proposition 1.2(iv) we have

$$h \leq (a \wedge b)/(a \vee b) = a \triangle^* b \in F_1,$$

hence $h \in F$. Similarly one sees that $k \in F$. Moreover

$$a + h = a + \big(a/(a \vee b)\big) = a \vee b = b + \big(b/(a \vee b)\big) = b + k.$$

Therefore $(a,b) \in F^+$.

Now choose $G \in \mathcal{F}$ satisfying Proposition 2.7(1), and take $F_2 \in \mathcal{F}$ satisfying Proposition 2.3(1) with G in place of F. Suppose that $(a,b) \in F_2^+$, so that we find $h,k \in F_2$ such that $a \perp h$, $b \perp k$ and $a + h = b + k$. Since

$$a \vee b \leq (a+h) \vee (b+k) = a + h = b + k,$$

applying Proposition 1.2(iv), we obtain

$$a/(a \vee b) \leq a/(a+h) = h \quad \text{and} \quad b/(a \vee b) \leq a/(b+k) = k,$$

hence both $a/(a \vee b)$ and $b/(a \vee b)$ belong to G.

By Proposition 1.3(v), we conclude that

$$a \triangle^* b = (a \wedge b)/(a \vee b) = \big(a/(a \vee b)\big) \vee \big(b/(a \vee b)\big) \in F$$

i.e. $(a,b) \in F^\triangle$.

(2) Similar to (1).

(3) Let $F \in \mathcal{F}$. We show that there exist $F_1, F_2 \in \mathcal{F}$ such that ${F_1}^\triangle \subseteq F^\backslash$ and $F_2^\backslash \subset F^\triangle$.

Take $F_1 \in \mathcal{F}$ satisfying Proposition 2.3(1). Suppose $(a,b) \in F_1{}^{\triangle}$, i.e. $a \mathbin{\triangle^*} b \in F_1$, and let $i = (a \wedge b)/a$, $j = (a \wedge b)/b$. By Proposition 1.2(iv) we have

$$i \leq (a \wedge b)/(a \vee b) = a \mathbin{\triangle^*} b \in F_1,$$

hence $i \in F$. Similarly one sees that $j \in F$. Moreover

$$a\backslash i = a\backslash\big((a \wedge b)/a\big) = a \wedge b = b\backslash\big((a \wedge b)/b\big) = b\backslash j.$$

Therefore $(a,b) \in F^{\backslash}$.

Now choose $G \in \mathcal{F}$ satisfying Proposition 2.7(1), and take $F_2 \in \mathcal{F}$ satisfying Proposition 2.3(1) with G in place of F. Suppose that $(a,b) \in F_2^{\backslash}$, so that we find $i, j \in F_2$ such that $i \leq a$, $j \leq b$ and $a\backslash i = b\backslash j$. Since

$$a\backslash i = b\backslash j = (a\backslash i) \wedge (b\backslash j) \leq a \wedge b,$$

applying Proposition 1.2(iv) and (i), we obtain

$$(a \wedge b)/a \leq (a\backslash i)/a = i \quad \text{and} \quad (a \wedge b)/b \leq (b\backslash j)/b = j,$$

hence both $(a \wedge b)/a$ and $(a \wedge b)/b$ belong to G.

By Proposition 1.3(viii), we conclude that

$$a \mathbin{\triangle^*} b = (a \wedge b)/(a \vee b) = \big((a \wedge b)/a\big) \vee \big((a \wedge b)/b\big) \in F$$

i.e. $(a,b) \in F^{\triangle}$.

(4) Similar to (3). □

The last part of this section is devoted to describe the supremum $\mathcal{F} \vee \mathcal{G}$ and the infimum $\mathcal{F} \wedge \mathcal{G}$ of two D-filters $\mathcal{F}$ and $\mathcal{G}$ on L. As a consequence we will show the the lattice of all D-filters and the lattice of all D-uniformities on L are distributive.

Lemma 2.14. *Let $a \perp b$. If $c \leq b$ then $(a+b)\backslash c = a + (b\backslash c)$. Similarly, if $c \leq a$ then $c/(a+b) = (c/a) + b$.*

Proof. See [8: Lemma 3.1]. □

Notation 2.15. Given $F, G \subseteq L$, let

$$F + G = \{a + b :\ a \in F,\ b \in G,\ a \perp b\},$$
$$F \wedge G = \{a \wedge b :\ a \in F,\ b \in G\}.$$

Proposition 2.16.

(a) *If $\mathcal{F}, \mathcal{G} \in \mathcal{FND}(L)$ then $\{F + G :\ F \in \mathcal{F},\ G \in \mathcal{G}\}$ is a base for $\mathcal{F} \wedge \mathcal{G}$.*

(b) *If $\Gamma \subseteq \mathcal{FND}(L)$ then the supremum $\bigvee \Gamma$ of Γ in $\mathcal{FND}(L)$ exists and coincides with the set of all intersections of finite subsets of $\bigcup \Gamma$. In particular $\mathcal{F}_1 \vee \mathcal{F}_2 = \{F_1 \cap F_2 :\ F_1 \in \mathcal{F}_1,\ F_2 \in \mathcal{F}_2\}$ for every $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{FND}(L)$.*

(c) *If $\mathcal{F}, \mathcal{G} \in \mathcal{FND}(L)$ then $\{F \wedge G :\ F \in \mathcal{F}, G \in \mathcal{G}\}$ is a base for $\mathcal{F} \vee \mathcal{G}$.*

P r o o f.

(a) First observe that

$$\forall F \in \mathcal{F} \quad \forall G \in \mathcal{G} \quad F \cup G \subseteq F + G. \tag{2.9}$$

Indeed, since $0 \in G$, we have

$$F = \{a + 0 : \ a \in F\} \subseteq \{a + b : \ a \in F, \ b \in G, \ a \perp b\} = F + G$$

and similarly for G. In particular all sets of the form $F + G$ with $F \in \mathcal{F}$ and $G \in \mathcal{G}$ are non-empty.

Now, given $F_1, F_2 \in \mathcal{F}$ and $G_1, G_2 \in \mathcal{G}$, let $F = F_1 \cap F_2$ and $G = G_1 \cap G_2$. We have

$$\begin{aligned} F + G &= \{a + b : \ a \in F, \ b \in G, \ a \perp b\} \\ &\subseteq \{a + b : \ a \in F_1, \ b \in G_1, \ a \perp b\} = F_1 + G_1 \end{aligned}$$

and, similarly, $F + G \subseteq F_2 + G_2$; hence

$$(F_1 + G_1) \cap (F_2 + G_2) \supseteq F + G.$$

Therefore $\{F + G : \ F \in \mathcal{F}, \ G \in \mathcal{G}\}$ is a base for a filter $\mathcal{H}$. We prove that $\mathcal{H}$ is a D-filter.

Given $H \in \mathcal{H}$, let $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F + G \subseteq H$. Take $F_1 \in \mathcal{F}$ satisfying (F1), and $G_1 \in \mathcal{G}$ satisfying (F1) with G in place of F: in other words

$$F_1 + F_1 \subseteq F \quad \text{and} \quad G_1 + G_1 \subseteq G. \tag{2.10}$$

Choose $F_2 \in \mathcal{F}$ satisfying (F4) with F_1 in place of F; observe that $F_2 \subseteq F_1$. Let $H' = F_2 + G_1$; clearly $H' \in \mathcal{H}$.

If $a, b \in H'$, with $a \perp b$, then we may find $p_1, p_2 \in F_2$ and $q_1, q_2 \in G_1$ such that $a = p_1 + q_1$ and $b = p_2 + q_2$. Thus

$$a + b = (p_1 + q_1) + (p_2 + q_2) = p_1 + (q_1 + p_2) + q_2 = p_1 + k + q_2,$$

where $k = q_1 + p_2$. Now let $\tilde{p}_2 = k \backslash q_1$; since $q_1 / k = p_2 \in F_2$, we get $\tilde{p}_2 = k \backslash q_1 \in F_1$. Hence, taking (2.10) into account,

$$\begin{aligned} a + b &= p_1 + k + q_2 = p_1 + (\tilde{p}_2 + q_1) + q_2 = (p_1 + \tilde{p}_2) + (q_1 + q_2) \\ &\in (F_1 + F_1) + (G_1 + G_1) \subseteq F + G \subseteq H. \end{aligned}$$

This shows that $\mathcal{H}$ satisfies (F1).

We prove that $\mathcal{H}$ satisfies (F2). Given $H \in \mathcal{H}$, let $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F + G \subseteq H$. Take $F' \in \mathcal{F}$ satisfying (F2), and $G' \in \mathcal{G}$ satisfying (F2) with G in place of F. Let $H' = F + G'$; clearly $H' \in \mathcal{H}$.

If $a \in H'$ and $c \in L$ then we may find $p \in F$ and $q \in G$ such that $a = p + q$. Let

$$d = (q \vee c) \backslash c, \quad p' = (p \vee d) \backslash d \quad \text{and} \quad q' = (q \vee c) \backslash c; \tag{2.11}$$

we have $p' \in F$ and $q' \in G$. Hence from (2.11), applying Proposition 1.3(iii) and Lemma 2.14, it follows that

$$\begin{aligned}(a \vee c)\backslash c &= \big((p+q) \vee c\big)\backslash c = \big((p+q) \vee (q \vee c)\big)\backslash c \\ &= \big((p+q) \vee (d+q)\big)\backslash c = \big((p \vee d) + q\big)\backslash c \\ &= \big((p'+d) + q\big)\backslash c = \big(p' + (d+q)\big)\backslash c \\ &= \big(p' + (q \vee c)\big)\backslash c = p' + (q \vee c)\backslash c \\ &= p' + q' \in F + G \subseteq H.\end{aligned}$$

Now we prove that $\mathcal{H}$ satisfies (F3). Given $H \in \mathcal{H}$, let $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F + G \subseteq H$. Take $F' \in \mathcal{F}$ satisfying (F3), and $G' \in \mathcal{G}$ satisfying (F3) with G in place of F. Let $H' = F + G'$; clearly $H' \in \mathcal{H}$.

If $b\backslash a \in H'$, where $a, b \in L$ and $a \leq b$, then we may find $p \in F$ and $q \in G$ such that $b\backslash a = p + q$. Let

$$p' = a/(p+a), \quad q' = a/(q+a).$$

As $(p+a)\backslash a = p \in F'$, we have $p' \in F$; similarly $q' \in G$. Hence

$$b = p + q + a = p + a + q' = a + p' + q',$$

and therefore $a/b = p' + q' \in F + G \subseteq H$.

Similarly we see that $\mathcal{H}$ satisfies (F4), and we conclude that $\mathcal{H}$ is a D-filter, as claimed.

It follows from (2.9) that both $\mathcal{F}$ and $\mathcal{G}$ are finer than $\mathcal{H}$. To complete the proof, consider any D-filter $\mathcal{H}'$ such that both $\mathcal{F}$ and $\mathcal{G}$ are finer than $\mathcal{H}'$. We show that $\mathcal{H}' \subseteq \mathcal{H}$.

Let $H \in \mathcal{H}$. By (F1), there exists $H' \in \mathcal{H}'$ such that $H' + H' \subseteq H$. Since $H' \in \mathcal{F} \cap \mathcal{G}$ we get $H' + H' \in \mathcal{H}$ and hence $H \in \mathcal{H}$, too.

(b) Let $\mathcal{F}$ be the set of all intersections of finite subsets of $\bigcup \Gamma$. We first show that $\mathcal{F}$ is a filter.

Let $F_1, F_2 \in \mathcal{F}$. One has $F_1 = \bigcap \mathcal{F}_1$ and $F_1 = \bigcap \mathcal{F}_1$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are finite subsets of $\bigcup \Gamma$. If $G = F_1 \cap F_2$, then $G \in \mathcal{F}$ because it is the intersection of $\mathcal{F}_1 \cup \mathcal{F}_2$, which is again a finite subset of $\bigcup \Gamma$. Now let $F \in \mathcal{F}$. Then $F = \bigcap\limits_{i=1}^{n} F_i$, where $F_i \in \mathcal{G}_i$ and $\mathcal{G}_i \in \Gamma$ for each $i \in \{1, 2, \ldots, n\}$. If $G \supset F$, let $A = G \setminus F$. For each i, one has $G_i = A \cup F_i \in \mathcal{G}_i$, and

$$\bigcap_{i=1}^{n} G_i = \bigcap_{i=1}^{n} (A \cup F_i) = A \cup \bigcap_{i=1}^{n} F_i = A \cup F = G.$$

Hence $G \in \mathcal{F}$.

Let $F \in \mathcal{F}$. We may write $F = \bigcap\limits_{i=1}^{n} F_i$, with $F_i \in \mathcal{G}_i \in \Gamma$. To see that property (F1) is verified, for each i, take F_i' in $\mathcal{G}_i$ satisfying (F1) with F_i in place of F, and let $F' = \bigcap_{i=1}^{n} F'_i$. Clearly $F' \in \mathcal{F}$. If $a, b \in F$, with $a \perp b$, then for each $i \in \{1, 2, \ldots, n\}$ we have $a, b \in F'_i$ and hence $a + b \in F_i$. Therefore $a + b \in F$.

Now we check property (F2). Let $F \in \mathcal{F}$. As above, $F = \bigcap_{i=1}^{n} F_i$, with $F_i \in \mathcal{G}_i \in \Gamma$. For each i, take $G_i \in \mathcal{G}_i$ satisfying (F2) with F_i in place of F.

Let $G = \bigcap_{i=1}^{n} G_i$; clearly $G \in \mathcal{F}$. If $a \in G$ and $c \in L$, then for each $i \in \{1, 2, \ldots, n\}$ we have $a \in G_i$ and hence $(a \vee c)\backslash c \in F_i$. Therefore $(a \vee c)\backslash c \in F$.

Finally we show that $\mathcal{F}$ satisfies (F3) and (F4). Consider $F \in \mathcal{F}$; we have $F = \bigcap_{i=1}^{n} F_i$, with $F_i \in \mathcal{G}_i \in \Gamma$. For each i, take $G_i, H_i \in \mathcal{G}_i$ satisfying (F3) (respectively (F4)) with F_i in place of F. Let $G = \bigcap_{i=1}^{n} G_i$ and $H = \bigcap_{i=1}^{n} H_i$; clearly $G, H \in \mathcal{F}$. Given $a, b \in L$ with $a \leq b$, suppose that $b\backslash a \in G$ (or $a/b \in H$). Then for each $i \in \{1, 2, \ldots, n\}$ we have $b\backslash a \in G_i$ (respectively $a/b \in H_i$) and hence $a/b \in F_i$ (respectively $b\backslash a \in F_i$); we conclude that $a/b \in F$ (respectively $b\backslash a \in F$).

Since it is clear that each $\mathcal{G} \in \Gamma$ is contained in $\mathcal{F}$ (indeed every G in $\mathcal{G}$ is the intersection of $\{G\}$, which a finite subset of $\bigcup \Gamma$), it remains to prove that any D-filter which is finer than all filters in Γ is finer than $\mathcal{F}$, too.

So let $\mathcal{G}' \in \mathcal{FND}(L)$ such that $\mathcal{G} \subset \mathcal{G}'$ for every $\mathcal{G} \in \Gamma$. Given $F \in \mathcal{F}$, one has $F = \bigcap_{i=1}^{n} F_i$ where $F_i \in \mathcal{G}_i \in \Gamma$, hence $F_i \in \mathcal{G}'$, for each $i \in \{1, 2, \ldots, n\}$. As $\mathcal{G}'$ is a filter, we have $F \in \mathcal{G}'$. We conclude that $\mathcal{F} \subset \mathcal{G}'$.

(c) Given $F \in \mathcal{F}$ and $G \in \mathcal{G}$, since

$$F \cap G = \{a \wedge a : \ a \in F \cap G\} \subseteq \{a \wedge b : \ a \in F, \ b \in G\} = F \wedge G,$$

it remains to prove that there exist $F' \in \mathcal{F}$ and $G' \in \mathcal{G}$ such that $F' \wedge G' \subseteq F \cap G$.

Take $F' \in \mathcal{F}$ satisfying Proposition 2.3(1) and $G' \in \mathcal{G}$ also satisfying Proposition 2.3(1), with G in place of F. If $a \in F'$ and $b \in G'$, then $a \wedge b \leq b$ hence $a \wedge b \in F$; similarly $a \wedge b \in G$. We conclude that $a \wedge b \in F \cap G$. □

The final result of this section establishes that the lattice of all D-uniformities on pseudo-D-lattice is distributive, which generalizes the analogous fact for the lattice of FN-topologies on a Boolean algebra.

This is of interest because, on the contrary, the lattice of all topologies (or uniformities) on a set with more than two elements is far from being distributive, and it is not even modular.

Corollary 2.17. *Both $\mathcal{FND}(L)$ and $\mathcal{DU}(L)$ are distrbutive (complete) lattices.*

Proof. In view of Corollary 2.10, we just prove the assertion for $\mathcal{FND}(L)$. It suffices to consider $\mathcal{F}, \mathcal{G}_1, \mathcal{G}_2 \in \mathcal{FND}(L)$ and show that $(\mathcal{F} \vee \mathcal{G}_1) \wedge (\mathcal{F} \vee G_2) \subseteq \mathcal{F} \vee (\mathcal{G}_1 \wedge G_2)$.

Given $H \in (\mathcal{F} \vee \mathcal{G}_1) \wedge (\mathcal{F} \vee \mathcal{G}_2)$, take $F_1, F_2 \in \mathcal{F}$, $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}_2$ with $(F_1 \cap G_1) + (F_2 \cap G_2) \subseteq H$; let $F = F_1 \cap F_2$ and choose $F' \in \mathcal{F}$ satisfying Proposition 2.3(1). We complete the proof by showing that $F' \cap (G_1 + G_2) \subseteq (F_1 \cap G_1) + (F_2 \cap G_2)$.

Let $a \in F' \cap (G_1 + G_2)$; choose $a_1 \in G_1$ and $a_2 \in G_2$ such that $a = a_1 + a_2$. Since $a_1 \leq a$ and $a \in F'$, one has $a_1 \in F \subset F_1$ and hence $a_1 \in F_1 \cap G_1$; similarly one sees that $a_2 \in F_2 \cap G_2$. Therefore $a = a_1 + a_2 \in (F_1 \cap G_1) + (F_2 \cap G_2)$. □

3. k-submeasures

In the classical context of Boolean algebras, submeasures are simple generalizations of real-valued measures, and it is well-known [13] that any FN-topology is generated by a family of submeasures. More generally, it has been shown in [10: Theor. 2.5] that a any lattice uniformity on an MV-algebra is generated by a family of submeasures.

On the other hand, in order to generate lattice uniformities on orthomodular lattices [1], and hence also D-uniformities on D-lattices [6], one has to resort to suitable modifications of submeasures, which have been called k-submeasures.

The definition of k-submeasure given here is a convenient generalization of the one given in [6: Def. 3.1]. It is formulated in such a way that the results below turn out to work.

Definition 3.1. Let $k \geq 1$. A function $\eta\colon L \to [0, +\infty]$ is called *k-submeasure* if it satisifies the following conditions for every $a, b \in L$:

(S1) $\eta(0) = 0$;

(S2) If $a \leq b$ then $\eta(a) \leq \eta(b)$;

(S3) If $a \perp b$ then $\eta(a + b) \leq k\eta(a) + \eta(b)$;

(S4) $\eta\big((a \vee b)\backslash b\big) \leq k\eta(a)$;

(S5) If $a \leq b$ then $\eta(a/b) \leq k^2\eta(b\backslash a)$ and $\eta(b\backslash a) \leq k^2\eta(a/b)$.

A 1-submeasure will simply called *submaesure*; in this case condition (S5) clearly becomes: If $a \leq b$ then $\eta(a/b) = \eta(b\backslash a)$.

For every $\varepsilon > 0$, let

$$S_\varepsilon = \big\{(x, y) \in [0, +\infty[\times [0, +\infty[:\ |x - y| < \varepsilon\big\} \cup \{(+\infty, +\infty)\}.$$

Then $\{S_\varepsilon : \ \varepsilon > 0\}$ is a base for a uniformity $\mathcal{S}$ on $[0, +\infty]$ which coincides on $[0, +\infty[$ with the usual uniformity and for which $+\infty$ is an isolated point. In the sequel we will always endow $[0, +\infty]$ with this uniformity.

Notation 3.2. If η is a fixed k-subsmeasure on L, and ε is a positive real number, we set

$$F_\varepsilon = \{a \in L : \ \eta(a) < \varepsilon\}.$$

We begin by showing that a single k-submeasure generates a D-uniformity.

Proposition 3.3. *For every k-subsmeasure η on L, the collection*

$$\{^\triangle F_\varepsilon : \ \varepsilon > 0\}$$

(where F_ε is defined according to Notation 3.2 *above) is a base for a D-uniformity $\mathcal{U}(\eta)$, which is the coarsest D-uniformity making η uniformly continuous.*

Proof. Since clearly $F_{\varepsilon_1} \cap F_{\varepsilon_2} = F_{\min\{\varepsilon_1, \varepsilon_2\}}$, the collection $\{F_\varepsilon : \ \varepsilon > 0\}$ is a base for a filter $\mathcal{F}$. Let us show that $\mathcal{F}$ is a D-filter.

Indeed, given $F \in \mathcal{F}$, take $\varepsilon > 0$ with $F_\varepsilon \subseteq F$. It is easily seen that (F1) is verified if we let $G = F_{\frac{\varepsilon}{k+1}}$, that (F2) is verified if we let $G = F_{\frac{\varepsilon}{k}}$, and that both (F3) and (F4), are verified if we let $G = F_{\frac{\varepsilon}{k^2}}$.

Now let $\mathcal{U}(\eta)$ be the D-uniformity generated by $^\triangle\mathcal{B} = \{^\triangle F : \ F \in \mathcal{F}\}$ as in Theorem 2.8. We show that η is uniformly continuous with respect to $\mathcal{U}(\eta)$.

Fix $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{k}$. Given $a, b \in L$, as $a \vee b = (a \,{}^*\!\triangle\, b) + (a \wedge b)$, if $(a, b) \in {}^\triangle F_\delta$ then

$$\begin{aligned} \eta(a \vee b) &= \eta\big((a \,{}^*\!\triangle\, b) + (a \wedge b)\big) \\ &\leq k\eta(a \,{}^*\!\triangle\, b) + \eta(a \wedge b) < \varepsilon + \eta(a \wedge b). \end{aligned} \tag{3.1}$$

Thus, if $\eta(a \vee b) = +\infty$, also $\eta(a \wedge b) = +\infty$, hence by (S2) we have $\eta(a) = \eta(b) = +\infty$, too. On the other hand, if $\eta(a \vee b) < +\infty$, by (S2) both $\eta(a)$ and $\eta(b)$ are finite, and belong to the interval $[\eta(a \wedge b), \eta(a \vee b)]$; it follows from (3.1) that

$$|\eta(a) - \eta(b)| \leq \eta(a \vee b) - \eta(a \wedge b) < \varepsilon.$$

Therefore, in any case, we get $\big(\eta(a), \eta(b)\big) \in S_\varepsilon$.

Finally, let $\mathcal{V}$ be any D-uniformity making η uniformly continuous. It remains to prove that $\mathcal{U}(\eta) \subseteq \mathcal{V}$ and, to this end, it suffices to show that $\mathcal{F} \subseteq \mathcal{G}$, where $\mathcal{G}$ denotes the filter of neighbourhoods of 0 with respect to $\mathcal{V}$.

Given $F \in \mathcal{F}$, take $\varepsilon > 0$ with $F_\varepsilon \subseteq F$. As η is continuous at 0 with respect to $\mathcal{V}$, and $\eta(0) = 0$, we can find $G \in \mathcal{G}$ such that for every $a \in G$ we have $\eta(a) < \varepsilon$, i.e. $a \in F_\varepsilon$. It follows that $G \subseteq F_\varepsilon \subseteq F$, whence $F \in \mathcal{G}$. □

The reader should note that if μ is a $[0,+\infty[$-valued modular measure then it is a submeasure; moreover, in this case, the D-uniformity $\mathcal{U}(\mu)$ above coincides with the one given by Theorem 1.8.

Now our aim is to prove a sort of converse of the previous result, namely that every D-uniformity is generated by a family of k-submeasures. But first we need some preliminary facts.

Lemma 3.4. *For every $a, b \in L$ we have*

$$a \,{}^*\!\triangle\, b = \big((a \vee b)\backslash a\big) + \big(a\backslash(a \wedge b)\big) \tag{3.2}$$

and

$$a \triangle^* b = \big((a \wedge b)/a\big) + \big(a/(a \vee b)\big). \tag{3.3}$$

P r o o f. Indeed

$$\begin{aligned}
&\big((a \vee b)\backslash a\big) + \big(a\backslash(a \wedge b)\big)\\
&= \Big(\big(((a \vee b)\backslash a) + (a\backslash(a \wedge b))\big) + (a \wedge b)\Big)\backslash(a \wedge b)\\
&= \Big(\big((a \vee b)\backslash a\big) + \big((a\backslash(a \wedge b)) + (a \wedge b)\big)\Big)\backslash(a \wedge b)\\
&= \big(((a \vee b)\backslash a) + a\big)\backslash(a \wedge b) = (a \vee b)\backslash(a \wedge b) = a \,{}^*\!\triangle\, b.
\end{aligned}$$

The proof of (3.3) is similar. □

As a tool for the main result of this section, we need the concept of k-pseudometric.

Definition 3.5. Let $k \geq 1$. A pseudometric ϱ on L will be called *k-pseudometric* if it satisifies the following conditions for every $a, b, c, d \in L$:

(P1) $\varrho(a \wedge c, b \wedge c) \leq \varrho(a, b)$;

(P2) If $a \perp c$ and $b \perp c$ then $\varrho(a + c, b + c) \leq k\varrho(a, b)$;

(P3) If $c \perp a$ and $c \perp b$ then $\varrho(c + a, c + b) \leq k\varrho(a, b)$;

(P4) $\varrho\big((a \vee c)\backslash c, (b \vee c)\backslash c\big) \leq k\varrho(a, b)$;

(P5) $\varrho\big(c/(a \vee c), c/(b \vee c)\big) \leq k\varrho(a, b)$.

From a k-pseudometric ϱ, we get a k-submeasure $\tilde{\eta}$ in a natural way. Furthermore, the uniformity generated by $\tilde{\eta}$ coincides with the one induced by ϱ.

Proposition 3.6. *Given $k \geq 1$, let ϱ be a k-pseudometric on L. Then the mapping $\tilde{\eta}\colon a \mapsto \varrho(a, 0)$ is a k-submeasure, and $\mathcal{U}(\tilde{\eta})$ coincides with the uniformity induced by ϱ.*

P r o o f. It is clear that $\tilde{\eta}(0) = 0$. If $a \le b$, it follows from (P1) that

$$\tilde{\eta}(a) = \varrho(b \wedge a, 0 \wedge a) \le \varrho(b, 0) = \tilde{\eta}(b).$$

If $a \perp b$, by (P3) and the triangular inequality we have

$$\tilde{\eta}(a + b) \le \varrho(a + b, b) + \varrho(b, 0) \le k\varrho(a, 0) + \varrho(b, 0) = k\tilde{\eta}(a) + \tilde{\eta}(b).$$

For every $a, b \in L$, it follows from (P4) that

$$\tilde{\eta}\big((a \vee b)\backslash b\big) = \varrho\big((a \vee b)\backslash b, (0 \vee b)\backslash b\big) \le k\big(\varrho(a, 0) + \varrho(b, b)\big) = k\tilde{\eta}(a).$$

For every $a, b \in L$, with $a \le b$, it follows from (P5) and (P2) that

$$\begin{aligned} \tilde{\eta}(a/b) &= \varrho(a/b, 0) = \varrho\big((a/b), (a/a)\big) \le k\varrho(b, a) \\ &= k\varrho\big((b\backslash a) + a, 0 + a\big) \le k^2\varrho(b\backslash a, 0) = k^2\tilde{\eta}(b\backslash a); \end{aligned}$$

similarly one shows that $\tilde{\eta}(b\backslash a) \le k^2\tilde{\eta}(a/b)$. Therefore $\tilde{\eta}$ is a k-submeasure.

Now denote by $\mathcal{V}$ the uniformity induced by ϱ. We prove that $\mathcal{V} = \mathcal{U}(\tilde{\eta})$. To this end, since the collection of all sets of the form

$$V_\varepsilon = \big\{(a, b) \in L \times L : \ \varrho(a, b) < \varepsilon\big\}$$

(where $\varepsilon > 0$) is a base for $\mathcal{V}$, in view of Theorem 2.8 and Proposition 3.3, it suffices to show that, for every $\varepsilon > 0$ there exists $\delta > 0$ such that ${}^{\triangle}F_\delta \subseteq V_\varepsilon$ and $V_\delta \subseteq {}^{\triangle}F_\varepsilon$ (where F_ε and F_δ are defined according to Notation 3.2).

So, fix $\varepsilon > 0$ and let $\delta = \frac{\varepsilon}{2k^2}$. If $(a, b) \in {}^{\triangle}F_\delta$, applying (P1) and (P2) we have

$$\begin{aligned} \varrho(a, b) &\le \varrho(a, a \wedge b) + \varrho(a \wedge b, b) \\ &= \varrho\big((a \vee b) \wedge a, (a \wedge b) \wedge a\big) + \varrho\big((a \vee b) \wedge b, (a \wedge b) \wedge b\big) \\ &\le 2\varrho(a \vee b, a \wedge b) = 2\varrho\big((a \mathbin{^*\!\triangle} b) + (a \wedge b), 0 + (a \wedge b)\big) \\ &\le 2k\varrho(a \mathbin{^*\!\triangle} b, 0) = 2k\tilde{\eta}(a \mathbin{^*\!\triangle} b) < 2k\delta = \frac{2k\varepsilon}{2k^2} \le \varepsilon, \end{aligned}$$

hence $(a, b) \in V_\varepsilon$.

On the other hand, if $(a, b) \in V_\delta$, we have:

$$\begin{aligned} &\tilde{\eta}(a \mathbin{^*\!\triangle} b) \le \varrho\big(a \mathbin{^*\!\triangle} b, (a \vee b)\backslash a\big) + \varrho\big((a \vee b)\backslash a, 0\big) \\ &= \varrho\Big(\big((a \vee b)\backslash a\big) + \big(a\backslash(a \wedge b)\big), (a \vee b)\backslash a\Big) \\ &\quad + \varrho\big((b \vee a)\backslash a, (a \vee a)\backslash a\big) && \text{by (3.2)} \\ &\le k\varrho\big(a\backslash(a \wedge b), 0\big) + k\varrho(b, a) && \text{by (P3), (P4)} \end{aligned}$$

$$
\begin{aligned}
&= k\varrho\big(\big(a\backslash(a\wedge b)\big)\wedge\big(a\backslash(a\wedge b)\big),\big(b\backslash(a\wedge b)\big)\wedge\big(a\backslash(a\wedge b)\big)\big)+k\varrho(a,b) && \text{by 1.3(vii)}\\
&\le k\varrho\big(a\backslash(a\wedge b), b\backslash(a\wedge b)\big)+k\varrho(a,b) && \text{by (P1)}\\
&= k\varrho\big(\big(a\vee(a\wedge b)\big)\backslash(a\wedge b),\big(b\vee(a\wedge b)\big)\backslash(a\wedge b)\big)+k\varrho(a,b)\\
&\le k^2\varrho(a,b)+k\varrho(a,b) && \text{by (P4)}\\
&= (k+1)k\varrho(a,b) < (k+1)k\delta \le \varepsilon.
\end{aligned}
$$

Therefore $(a\,{}^*\!\triangle\, b)\in {}^{\triangle}F_\varepsilon$. □

Our final result establishes that a D-uniformity $\mathcal{U}$ on a pseudo-D-lattice L is always generated by a family of k-submeasures on L, with $k>1$; for a pseudometrizable D-uniformity, a single k-submeasure suffice. Moreover one can choose $k=1$ (that is a family of submeasures) when $\mathcal{U}$ is generated by a group-valued modular measure.

Theorem 3.7. *Let $\mathcal{U}\in\mathcal{DU}(L)$. The following hold:*

(a) *For every $k>1$ there exists a family $(\tilde\eta_\lambda)_{\lambda\in\Lambda}$ of k-submeasures such that $\mathcal{U}=\sup\limits_{\lambda\in\Lambda}\mathcal{U}(\tilde\eta_\lambda)$. Moreover, if $\mathcal{U}$ has a countable base, we can choose $|\Lambda|=1$.*

(b) *If $\mathcal{U}$ is generated by a modular measure $\mu\colon L\to G$ (where G is a topological Abelian group) then there exists a family $(\tilde\eta_\lambda)_{\lambda\in\Lambda}$ of submeasures such that $\mathcal{U}=\sup\limits_{\lambda\in\Lambda}\mathcal{U}(\tilde\eta_\lambda)$.*

Proof.

(a) Let $k>1$. For every $a,b\in L$, define

$$
\begin{aligned}
f(a,b)&=a\wedge b,\\
g(a,b)&=(a\wedge{}^{\perp}b)+b=\big(a\wedge(1\backslash b)\big)+b,\\
h_1(a,b)&=(a\vee b)\backslash b,\\
h_2(a,b)&=b/(a\vee b).
\end{aligned}
\tag{3.4}
$$

By [28: Prop. 1.1(b)], $\mathcal{U}$ has a base consisiting of sets $U\subseteq L\times L$ such that, for every $(a,a')\in U$ and every $b\in L$,

$$\big(f(a,b),\,f(a',b)\big)=\big(f(b,a),\,f(b,a')\big)\in U.$$

Moreover, since g, h_1 and h_2 are $\mathcal{U}$-uniformly continuous, from [28: Prop. 1.2] it follows that $\mathcal{U}$ is generated by a family $(\varrho_\lambda)_{\lambda\in\Lambda}$ of pseudometrics (where we can choose $|\Lambda|=1$ if $\mathcal{U}$ has a countable base) such that for every $\lambda\in\Lambda$ and every $a,a',b,b'\in L$

$$
\begin{aligned}
\varrho_\lambda\big(f(a,b),\,f(a',b')\big)&\le \varrho_\lambda(a,a')+\varrho_\lambda(b,b'),\\
\varrho_\lambda\big(g(a,b),\,g(a',b')\big)&\le k\big(\varrho_\lambda(a,a')+\varrho_\lambda(b,b')\big),\\
\varrho_\lambda\big(h_1(a,b),\,h_1(a',b')\big)&\le k\big(\varrho_\lambda(a,a')+\varrho_\lambda(b,b')\big),\\
\varrho_\lambda\big(h_2(a,b),\,h_2(a',b')\big)&\le k\big(\varrho_\lambda(a,a')+\varrho_\lambda(b,b')\big).
\end{aligned}
$$

Clearly each ϱ_λ satisfies (P1), (P2) and (P3), as well as (P4) and (P5), so that it is a k-pseudometric. Therefore, applying Proposition 3.6, the conclusion follows.

(b) Let $(p_\lambda)_{\lambda\in\Lambda}$ be a family of group seminorms generating the topology of G. By [20: Theor. 3], $\mathcal{U}$ is generated by the family of pseudometrics $(\varrho_\lambda)_{\lambda\in\Lambda}$ where, for every $\lambda \in \Lambda$,

$$\varrho_\lambda(a,b) = \sup\big\{p_\lambda\big(\mu(s)-\mu(r)\big) :\ a\wedge b \le r \le s \le a\vee b\big\};$$

moreover, each ϱ_λ satisfies (P1) and the following:

$$\forall a,b,c\in L \quad \varrho_\lambda(a\vee c, b\vee c) \le \varrho_\lambda(a,b). \tag{3.5}$$

In view of Proposition 3.6, we complete the proof by showing that each ϱ_λ is a 1-pseudometric (i.e. a k-pseudometric with $k=1$).

Fix $\lambda\in\Lambda$. Given $a,b\in L$, we claim that

$$\begin{aligned}\varrho_\lambda(a,b) &= \sup\big\{p_\lambda\big(\mu(t)\big) :\ t \le a \,{}^*\!\triangle\, b\big\}\\ &= \sup\big\{p_\lambda\big(\mu(u)\big) :\ u \le a \,\triangle^{\!*}\, b\big\}.\end{aligned} \tag{3.6}$$

To prove the first equality, consider $t \le a \,{}^*\!\triangle\, b$. Since $t \perp a\wedge b$, we can define $r = a\wedge b$ and $s = t + r$. Thus

$$a\wedge b = r \le s \le (a \,{}^*\!\triangle\, b) + (a\wedge b) = a\vee b$$

and

$$\mu(s)-\mu(r) = \mu(s\backslash r) = \mu\big((t+r)\backslash r\big) = \mu(t).$$

Hence

$$p_\lambda\big(\mu(t)\big) = p_\lambda\big(\mu(s)-\mu(r)\big) \le \varrho_\lambda(a,b).$$

It follows that $\sup\big\{p_\lambda\big(\mu(t)\big) :\ t\le a \,{}^*\!\triangle\, b\big\} \le \varrho_\lambda(a,b)$.

Conversely, consider $r,s\in[a\wedge b, a\vee b]$ with $r\le s$, and let $t = s\backslash r$. By Proposition 1.2(vi), we have

$$t = s\backslash r = \big(s\backslash(a\wedge b)\big)\backslash\big(r\backslash(a\wedge b)\big) \le s\backslash(a\wedge b) \le (a\vee b)\backslash(a\wedge b) = a \,{}^*\!\triangle\, b.$$

Hence

$$p_\lambda\big(\mu(s)-\mu(r)\big) = p_\lambda\big(\mu(s\backslash r)\big) = p_\lambda\big(\mu(t)\big) \le \sup\big\{p_\lambda\big(\mu(t)\big) :\ t\le a \,{}^*\!\triangle\, b\big\}.$$

Therefore $\varrho_\lambda(a,b) \le \sup\big\{p_\lambda\big(\mu(t)\big) :\ t \le a{}^*\!\triangle b\big\}$, whence the first equality in (3.6) follows.

In a similar way one can verify the second equality in (3.6), and this completes the proof of the claim.

Now let $a,b,c\in L$. By Lemma 2.4(a), we have

$$\big((a\vee c)\backslash c\big) \,{}^*\!\triangle\, \big((b\vee c)\backslash c\big) = (a\vee c) \,{}^*\!\triangle\, (b\vee c).$$

Applying (3.5) and (3.6), we get

$$\varrho_\lambda\big((a\vee c)\backslash c, (b\vee c)\backslash c\big) = \varrho_\lambda\big((a\vee c),(b\vee c)\big) \le \varrho_\lambda(a,b)$$

which gives (P4) with $k=1$. The proof of (P5) with $k=1$ is analogous.

If $a \perp c$ and $b \perp c$ then, by Lemma 2.4(e), we have $(a + c) \,{}^*\!\triangle\, (b + c) = a \,{}^*\!\triangle\, b$. Applying (3.6), we get

$$\varrho_\lambda(a + c, b + c) = \varrho_\lambda(a, b)$$

which gives (P2) with $k = 1$. Similarly one shows (P3) with $k = 1$, and the proof is complete. □

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Dipartimento di Matematica e Informatica
Università della Basilicata
Viale dell'Ateneo Lucano 10
85100 Potenza
ITALY

E-mail: anna.avallone@unibas.it
paolo.vitolo@unibas.it