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ASSOCIATIVITY OF OPERATIONS ON ORTHOMODULAR LATTICES

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Dedicated to Professor David Foulis

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ABSTRACT. It is known that orthomodular lattices admit 96 binary operations, out of which 16 are commutative. We clarify which of them are associative.

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1. Motivation

The history of orthomodular lattices (OMLs) goes back to Birkhoff and von Neumann [2] who introduced them as a mathematical structure allowing to describe events in a quantum mechanical system [7]. Their non-distributivity causes problems with evaluation of complex expressions which cannot be transformed to a unique normal form. E.g., the solvability of the word problem in orthomodular lattices seems to be a difficult question [8,13]. As a substitute for distributivity, several tools were designed, e.g., Foulis-Holland theorem [4,9], the *focusing technique* [5,6], or computation in the free OML with 2 free generators (or larger finite OMLs derived from it) proposed in [18]. The latter has been implemented in computer programs, see [10, 11, 15]. All these approaches fail whenever we have a variable non-commuting with at least two other variables. Thus their use is limited to formulas with up to 2 variables or satisfying compatibility restrictions. This does not allow, e.g., to test associativity. Particular results have been obtained in [3, 15] in the case when one variable commutes with the other two. Here we clarify the question of which operations on OMLs are associative.

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2. Basic tools

An *orthomodular lattice* is an algebra $(L, \wedge, \vee, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, $'$ (*orthocomplementation*) is its anti-isomorphism and the following identities hold for all $a, b \in L$: $a \wedge a' = 0$, $a \vee b = a \vee (a' \wedge (a \vee b))$ (the latter identity is called the *orthomodular law*; see [1, 12] for its equivalent formulations). Orthomodular lattices form an algebraic basis for the description of event structures of quantum mechanical systems. For basics about orthomodular lattices, we refer to [1, 7, 12]. Throughout this paper, L denotes an OML (we use this abbreviated notation instead of $(L, \wedge, \vee, ', 0, 1)$).

We distinguish a *sublattice* of an OML, which is a subset closed under the join and meet, and a *sub-orthomodular lattice* (*sub-OML*) which is closed under the join, meet, and orthocomplementation. A sub-OML which is a Boolean algebra (with the operations inherited from the OML) is called a *Boolean subalgebra* of an OML. Two elements a, b of an OML *commute* (are *compatible*) if they are contained in a Boolean subalgebra. There are many equivalent formulations of this property, e.g., the equalities $a = (a \wedge b) \vee (a \wedge b')$ or $a \,\underline{\mathrm{com}}\, b = 1$, where $a \,\underline{\mathrm{com}}\, b = (a \wedge b) \vee (a \wedge b') \vee (a' \wedge b) \vee (a' \wedge b')$ is the *lower commutator* of a, b. An element of an OML is called *central* if it commutes with all other elements. The set of all central elements of an OML L is called the *center* of L and denoted by $C(L)$. For elements $a, b \in L$ such that $a \leq b$ we define the *interval* $[a, b] = \{x \in L \mid a \leq x \leq b\}$.

The modular ortholattice MO2 is the OML $\{0, 1, x, y, x', y'\}$ whose elements satisfy $x^* \wedge y^* = 0$, $x^* \vee y^* = 1$ for all $x^* \in \{x, x'\}$, $y^* \in \{y, y'\}$. Its center is $\{0, 1\}$.

The approach of [18] is based on the use of the free orthomodular lattice with 2 free generators a, b, denoted by $\mathrm{F}(a, b)$.

Proposition 2.1. ([1, 12]) *The free OML $\mathrm{F}(a, b)$ with 2 free generators a, b is isomorphic to the product of the Boolean algebra $2^4 \cong [0, c]$ (with atoms $a \wedge b$, $a \wedge b'$, $a' \wedge b$, $a' \wedge b'$) and the modular ortholattice MO2 $\cong [0, c']$ (with atoms $a \wedge c'$, $a' \wedge c'$, $b \wedge c'$, $b' \wedge c'$), where $c = a \,\underline{\mathrm{com}}\, b$ is the lower commutator of a, b; the corresponding isomorphism is $h\colon x \mapsto (x \wedge c, x \wedge c')$. The OML $\mathrm{F}(a, b)$ has $2^4 \times 6 = 96$ elements, including 8 atoms (listed above). Its center is the Boolean algebra with 5 atoms $a \wedge b$, $a \wedge b'$, $a' \wedge b$, $a' \wedge b'$, c'. An element $x \in \mathrm{F}(a, b)$ is central iff $x \wedge c' \in \{0, c'\}$.*

The complete list of the elements of $\mathrm{F}(a, b)$ is presented in [1] together with their unique codes (called *Beran's numbers* in [15] and subsequent papers).[1] Any formula composed of the elements of $\mathrm{F}(a, b)$ corresponds to a binary operation

[1]The table of Beran's numbers is presented at http://cmp.felk.cvut.cz/~navara/FOML/ (with the permission of the author)

on OMLs and to a unique element of $\mathrm{F}(a,b)$. (Some of them are independent of some variable and thus can be understood as operations with a smaller arity.) As this OML is finite, the whole computation can be made automatically, e.g., by the use of a computer program. In [18], a graphical representation of the elements of $\mathrm{F}(a,b)$ was suggested; it allows to memorize these elements and also simplifies the operations, performing them independently on the Boolean factor and on the MO2 factor. Later on, Megill implemented this idea in a computer program[2] which, given a formula in two variables, returns the Beran's number of the corresponding element of $\mathrm{F}(a,b)$. This program was presented in [15] and extensively used in [16, 17] and other papers. Such results could be hardly obtained without computer support.

An implementation of the method of [18] has been done in [10] and its principles described in [11]. It returns the Beran's number, as well as the graphical representation according to [18] and the corresponding TEX macro for its typesetting.[3] Beside simplification of formulas and testing equalities, it allows to answer questions with more complicated logical structure. It admits to introduce further variables commuting with all other variables.

Free orthomodular lattices with more than two *free* generators are *infinite* [12], thus this technique cannot be extended to them. That causes problems when studying properties described using more than two variables, among them distributivity and associativity.

3. Preceding results

Let us start by summarizing the situation in Boolean algebras.

Proposition 3.1. *There are $2^4 = 16$ binary Boolean operations. Among them, the following 8 are associative: The disjunction $\vee$, the conjunction $\wedge$, the equivalence $\leftrightarrow$ and its complement $\not\leftrightarrow$ (XOR),*

$$a \leftrightarrow b = \begin{cases} 1 & \textit{if } a = b\,, \\ 0 & \textit{otherwise}\,, \end{cases}$$

$$a \not\leftrightarrow b = \begin{cases} 0 & \textit{if } a = b\,, \\ 1 & \textit{otherwise}\,, \end{cases}$$

and 4 operations which are essentially of a smaller arity — two constants 0, 1, and two projections $\lhd$, $\rhd$,

$$\begin{aligned} a \lhd b &= a\,, \\ a \rhd b &= b\,. \end{aligned} \tag{1}$$

[2] http://us.metamath.org/downloads/quantum-logic.tar.gz

[3] The style file can be downloaded from ftp://math.feld.cvut.cz/pub/navara/foml2.sty

Each Boolean operation has 6 corresponding OML operations. (In the free OML with 2 free generators, the corresponding elements form an interval isomorphic to MO2.) E.g., the Boolean implication gives rise to 6 (quantum) implications $\to_i$, $i = 0, \dots, 5$. They are defined as the following OML polynomials (the enumeration is taken from [15]):

symbol definition	Beran's number, name
$a \to_0 b = a' \vee b$	94, classical arrow
$a \to_1 b = a' \vee (a \wedge b)$	78, Sasaki arrow
$a \to_2 b = b \vee (a' \wedge b')$	46, Dishkant arrow
$a \to_3 b = (a' \wedge b) \vee (a' \wedge b') \vee (a \wedge (a' \vee b))$	30, Kalmbach arrow
$a \to_4 b = (a \wedge b) \vee (a' \wedge b) \vee (b' \wedge (a' \vee b))$	62, non-tolens arrow
$a \to_5 b = (a \wedge b) \vee (a' \wedge b) \vee (a' \wedge b')$	14, relevance arrow

In Boolean algebras, all right-hand-sides of these 6 equations coincide, and the same happens in OMLs if a, b commute. The implications $\to_i$, $i = 1, \dots, 5$, can be characterized as those binary OML operations which satisfy the Birkhoff-von Neumann requirement [2]

$$a \to_i b = 1 \quad \text{iff} \quad a \leq b \tag{2}$$

(see [12]).

The 6 implications $\to_i$, $i = 0, \dots, 5$, give rise to the corresponding (quantum) conjunctions $\wedge_i$ and disjunctions $\vee_i$ defined as follows [15]:

$$a \vee_i b = a' \to_i b\,,$$
$$a \wedge_i b = (a \to_i b')'\,.$$

Remark 3.1. In general, we have to distinguish a Boolean operation from its 6 corresponding OML operations. We use the same notation in special cases when one particular OML operation has an established natural meaning. E.g., the projections $\lhd, \rhd$ are defined by (1) also in OMLs. Table 1 summarizes these cases.

TABLE 1. Six (associative) OML operations having natural Boolean counterparts

symbol	Boolean operation	OML operation	Beran's number
0	least element	least element	1
1	greatest element	greatest element	96
$\lhd$	left projection	left projection	22
$\rhd$	right projection	right projection	39
$\vee$	disjunction	join	92
$\wedge$	conjunction	meet	2

Proposition 3.2. *The* 6 *OML operations in* Table 1 *are associative.*

Proof. The lattice operations $\vee$, $\wedge$ are known to be associative by definition, the verification for the remaining 4 (which are essentially of a smaller arity) is trivial. □

The associativity equation

$$(x * y) * z = x * (y * z) \tag{3}$$

was proved to hold for some other OML operations for special arguments:

Proposition 3.3. ([3, 15]) *Let x, y, z be elements of an OML such that one of them commutes with the other two. If $* \in \{\wedge_i, \vee_i \mid i = 0, \dots, 5\}$, then associativity equation* (3) *holds.*

Proof. See [3] for $i = 1, 2, 5$ and [15] for $i = 3, 4$. The case $i = 0$ leads to the lattice operations $\wedge_0 = \wedge$ and $\vee_0 = \vee$ which are known to be associative. □

Kröger [14] showed that the presence of commuting elements is a sufficient condition for associativity equation (3) in Boolean skew lattices. In an arbitrary OML, commuting elements are sufficient for (3), but it is not a necessary condition.

4. Which operations need to be tested for associativity

We pose the question whether there are other associative operations in OMLs than those specified in Proposition 3.2 (among all 96 OML binary operations). The following trivial observation (formulated as a proposition for further reference) reduces the options to one half:

Proposition 4.1. *In the special case of a Boolean algebra, each OML binary operation $*$ reduces to a unique Boolean operation, $\square$. If $*$ is associative (in OMLs), so is $\square$ (in Boolean algebras).*

Thus we have to search among the 48 OML operations which extend the 8 associative Boolean operations from Proposition 3.1. They are listed in Table 2.[4] As we shall see, most of these 48 operations are non-associative.

[4]The period 16 is due to the system of Beran's numbers.

Table 2. The OML operations extending the 8 associative Boolean operations

Boolean operation	Beran's numbers of corresponding OML operations
constant 0	$1, 17, 33, 49, 65, 81$
constant 1	$16, 32, 48, 64, 80, 96$
left projection $\lhd$	$6, 22, 38, 54, 70, 86$
right projection $\rhd$	$7, 23, 39, 55, 71, 87$
disjunction $\vee$	$12, 28, 44, 60, 76, 92$
conjunction $\wedge$	$2, 18, 34, 50, 66, 82$
equivalence $\leftrightarrow$	$8, 24, 40, 56, 72, 88$
non-equivalence $\not\leftrightarrow$	$9, 25, 41, 57, 73, 89$

5. Results

We first tested the operations for special cases of associativity equation (3) where only two variables are present:

$$(x * x) * y = x * (x * y)\,, \tag{4}$$

$$(x * x') * y = x * (x' * y)\,, \tag{5}$$

$$(x * y) * y = x * (y * y)\,, \tag{6}$$

$$(x * y') * y = x * (y' * y)\,, \tag{7}$$

$$(x * y) * x = x * (y * x)\,, \tag{8}$$

$$(x * y) * x' = x * (y * x')\,. \tag{9}$$

An example of such approach follows:

Example 5.1. Let x, y be non-compatible atoms of MO2. Let $*$ be one of the OML binary operations which, restricted to x, y, acts as the left projection,

$$x * y = x\,.$$

This occurs iff the Beran's number of $*$ is in $\{17, 18, \dots, 32\}$. Then the right-hand side of (8) is

$$x * (y * x) = x * y = x$$

and (8) reduces to

$$x * x = x\,.$$

This means that $*$, acting on the sub-OML generated by x, must be idempotent. This sub-OML is $\{0, 1, x, x'\}$, i.e., the free Boolean algebra with 1 generator x. We see that the Boolean counterpart of $*$ must be idempotent.

Among associative operations on Boolean algebras, idempotence holds for $\wedge$, $\vee$, $\lhd$, $\rhd$ and does not hold for $0, 1, \leftrightarrow, \not\leftrightarrow$. For the corresponding OML operations (denoted by Beran's numbers), this means that $18, 22, 23, 28$ may (and do) satisfy (8), while $17, 24, 25, 32$ violate (8) and are non-associative. Other operations from $\{17, 18, \ldots, 32\}$ were already excluded because they are not associative in Boolean algebras.

In view of Example 5.1, among operations with Beran's numbers in $\{17, 18, \ldots, 32\}$, only four, $18, 22, 23, 28$, remain candidates which have to be tested more. For this, we may use (4)–(9). All this requires only computations in two variables which can be performed by a program [10].

Example 5.2. The use of the computational tool of M. Hyčko is illustrated as follows: The Birkhoff-von Neumann condition (2) for implication $\to_1$ can be verified by the input `a i1 b = 1 <=> a <= b` whose interpretation is obvious. It returns `True`. To prove that the binary operation with Beran's number 24 does not satisfy equation (8), we insert the following input: `B3(24,B3(24,x,y),x) = B3(24,x,B3(24,y,x))`, where `B` stands for "Beran", `3` for the arity of the operator, `24` for the Beran's number of the required operation, and the remaining two items are its arguments. The program returns `False`.

Only those operations which pass these tests need to be checked by more general associativity criteria. (Among them 22, the left projection $\lhd$, is known to be associative.)

Proposition 5.1. *The following operations violate at least one of equations* (4)–(9)*:*

(1) *The operations $*$ with Beran's number in the set* $\{6, 12, 24, 25, 50, 54, 72\}$ *violate at least equation* (4)*.*

(2) *The operations $*$ with Beran's number in the set* $\{55, 60, 66, 70, 76, 89\}$ *violate at least equation* (5)*.*

(3) *The operations $*$ with Beran's number in the set* $\{7, 32, 65, 71, 80, 82\}$ *violate at least equation* (6)*.*

(4) *The operations $*$ with Beran's number in the set* $\{8, 9, 40, 41, 73, 87, 88\}$ *violate at least equation* (7)*.*

(5) *The operations $*$ with Beran's number in the set* $\{17, 33, 48, 49, 56, 57, 64\}$ *violate at least equation* (8)*.*

(6) *The operations $*$ with Beran's number in the set* $\{18, 23, 28, 34, 38, 44, 86\}$ *violate at least equation* (9)*.*

Proof. The proof was made using the computational tool[5] [10] and verified "by hand" using procedures analogous to those of Exs. 5.1 and 5.2. We omit the technical details here. □

[5] http://www.mat.savba.sk/~hycko/oml

The approach of Prop. 5.1 disproved associativity of 40 OML operations, but it is not applicable to the lower commutator

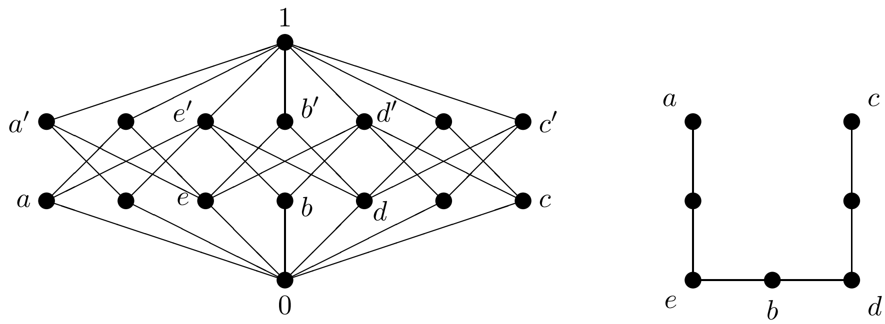
$$a \,\underline{\mathrm{com}}\, b = (a \wedge b) \vee (a \wedge b') \vee (a' \wedge b) \vee (a' \wedge b')$$

and the upper commutator[6]

$$a \,\overline{\mathrm{com}}\, b = (a \vee b) \wedge (a \vee b') \wedge (a' \vee b) \wedge (a' \vee b')\,.$$

These satisfy all equations (4)–(9) and another approach is needed.

Proposition 5.2. *The lower, resp. upper, commutator,* $\underline{\mathrm{com}}$, *resp.* $\overline{\mathrm{com}}$, *is not associative.*



a b

Figure 1. Hasse and Greechie diagram of the OML from Prop. 5.2

Proof. Let us take the OML depicted by the Hasse diagram in Fig. 1a and the Greechie diagram in Fig. 1b. Then

$$(a \,\underline{\mathrm{com}}\, b) \,\underline{\mathrm{com}}\, c = e \,\underline{\mathrm{com}}\, c = d\,,$$
$$a \,\underline{\mathrm{com}}\, (b \,\underline{\mathrm{com}}\, c) = a \,\underline{\mathrm{com}}\, d = e \neq (a \,\underline{\mathrm{com}}\, b) \,\underline{\mathrm{com}}\, c\,.$$

Similar argument works for the upper commutator:

$$(a \,\overline{\mathrm{com}}\, b) \,\overline{\mathrm{com}}\, c = e' \,\overline{\mathrm{com}}\, c = d'\,,$$
$$a \,\overline{\mathrm{com}}\, (b \,\overline{\mathrm{com}}\, c) = a \,\overline{\mathrm{com}}\, d' = e' \neq (a \,\overline{\mathrm{com}}\, b) \,\overline{\mathrm{com}}\, c\,.$$

□

Now we are ready to formulate the main result:

[6]The terminology "lower/upper commutator" is taken from [1]. In general, they are incomparable. In the special case of Boolean algebras, they satisfy the opposite ordering, $\overline{\mathrm{com}} = 0 \leq \underline{\mathrm{com}} = 1$.

Theorem 5.1. *The only associative operations in OMLs are the* 6 *operations in Table 1.*

Proof. This is just a combination of the previous propositions which cover all 96 possible cases. Based on Prop. 3.1, Prop. 4.1 excludes 48 operations. Prop. 5.1 rejects further 40 possibilities and Prop. 5.2 disproves associativity of 2 commutators (not covered by the preceding propositions). What remains are the 6 associative operations from Prop. 3.2. □

6. Conclusions

We have found all binary associative operations on orthomodular lattices. Among the 48 operations whose Boolean counterparts are associative, there are only 6 operations which are associative in OMLs. Other OML operations satisfy the associativity equation only under additional conditions on their arguments, e.g., that one of the variables commutes with the remaining two.

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