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A NOTE ON IDEALS IN SYNAPTIC ALGEBRAS

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Dedicated to Professor David J. Foulis

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ABSTRACT. The notion of a synaptic algebra was introduced by David Foulis. Synaptic algebras unite the notions of an order-unit normed space, a special Jordan algebra, a convex effect algebra and an orthomodular lattice. In this note we study quadratic ideals in synaptic algebras which reflect its Jordan algebra structure. We show that projections contained in a quadratic ideal from a p-ideal in the orthomodular lattice of projections in the synaptic algebra and we find a characterization of those quadratic ideals which are generated by their projections.

1. Introduction

The notion of a synaptic algebra was introduced by David Foulis in [8], where also some of the basic theory was developed. A synaptic algebra is an abstract version of the partially ordered Jordan algebra of all bounded Hermitian operators on a Hilbert space. As the word 'synaptic' (derived from the Greek word 'sunaptein' meaning to join together) suggests, synaptic algebras unite the notions of an order-unit normed space [1], a special Jordan algebra [21], a convex effect algebra [15], and an orthomodular lattice [4, 18]. Examples of synaptic algebras are selfadjoint part of a von Neumann algebra [24], an AW*-algebra

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[19], or a Rickart C*-algebra [16], and also JW-algebras [23], AJW-algebras [23: §20], JB-algebras [2, 3], and the ordered special Jordan algebras studied in [22]. Unlike these examples a synaptic algebra need not be a Banach space under the order-unit norm. A special case of synaptic algebras are generalized Hermitian (GH) algebras introduced and studied in [12, 13]. Projections in a GH algebra form a σ-complete orthomodular lattice (a σ-OML), while in a synaptic algebra projections also form an orthomodular lattice (an OML), but this lattice need not be σ-complete. Properties of projections in synaptic algebras were studied in [10]. Synaptic algebras of finite rank which include all positive-definite spin factors [11], are GH-algebras.

In this article we study ideals in synaptic algebras which reflect its Jordan algebra structure. In accordance with [23], we call these ideals quadratic ideals. We show that projections contained in a quadratic ideal from a p-ideal (i.e., an ideal closed under perspectivity) in the orthomodular lattice of projections in the synaptic algebra. It is well known that a quotient of an OML with respect to a p-ideal is again an OML [4,18]. Under a natural additional condition on the p-ideals, we find a characterization of those quadratic ideals which are generated by their projections. This characterization is similar to the characterization for ideals in weakly Rickart *-rings [5: Ch. 5].

We review the definition and basic properties of a synaptic algebra A in Section 2 below. In Section 3 we summarize important properties of the orthomodular lattice P of projections in A and of the relations between projections and symmetries in A. In Section 4 we introduce the notion of a quadratic ideal and study the relations between quadratic ideals in A and p-ideals in P.

2. Basic properties of a synaptic algebra

In the sequel, *R is a linear associative algebra over $\mathbb{R}$ with unity element* 1*, A is a vector subspace of R, $1 \in A$, and A is a partially ordered real vector space with positive cone* $A^+ = \{a \in A : \ 0 \leq a\}$. If $a, b \in A$, then the product ab, which may or may not belong to A, is understood to be the product of a and b in R. If $a, b \in A$, we write aCb iff *a commutes with b*, i.e., $ab = ba$, and we define $C(a) := \{b \in A : \ aCb\}$. Also, if $B \subseteq A$, we define the *commutant* $C(B) := \bigcap_{b\in B} C(b)$, and if $b \in CC(a) := C(C(a))$, we say that *b double commutes with a*. The commutant $C(A)$ is called the *center* of A.

2.1. Definition. The vector subspace A of R is a *synaptic algebra* with *enveloping algebra* R [8: Definition 1.1] iff the following conditions are satisfied:

(i) With 1 as the order unit, A is an order-unit normed space with order-unit norm $\|\cdot\|$.

(ii) If $a \in A$ then $a^2 \in A^+$.

(iii) If $a, b \in A^+$, then $aba \in A^+$; moreover, if aCb, then $ab \in A^+$.

(iv) If $a \in A$ and $b \in A^+$, then $aba = 0 \implies ab = ba = 0$.

(v) If $a \in A^+$, there exists $a^{1/2} \in A^+ \cap CC(a)$ such that $(a^{1/2})^2 = a$.

(vi) If $a \in A$, there exists $a^{\circ} \in A$ such that $(a^{\circ})^2 = a^{\circ}$ and, for all $b \in A$, $ab = 0 \iff a^{\circ}b = 0$.

(vii) If $1 \leq a \in A$, there exists $a^{-1} \in A$ such that $aa^{-1} = a^{-1}a = 1$.

(viii) If $a, b \in A$, $a_1 \leq a_2 \leq a_3 \leq \ldots$ is an ascending sequence of pairwise commuting elements of $C(b)$ and $\lim_{n\to\infty} \|a - a_n\| = 0$, then $a \in C(b)$.

In the sequel, A is a synaptic algebra with unit 1 and with enveloping algebra R. Also, $P := \{p \in A : \ p = p^2\}$ is the set of all idempotents in A and $E := \{e \in A : \ 0 \leq e \leq 1\}$ is the "unit interval" in A. We refer to elements $p \in P$ as "projections" and to elements $e \in E$ as "effects," and we understand that both P and E are partially ordered by the restrictions of the partial order $\leq$ on A. To avoid trivialities, we assume that A is "non-degenerate", i.e., $0 \neq 1$. Since A is nondegenerate, we can (and shall) follow the standard convention of identifying each real number $\lambda \in \mathbb{R}$ with the element $\lambda 1 \in A$, so that $\mathbb{N} \subseteq \mathbb{R} \subseteq A \subseteq R$.

As a consequence of Definition 2.1(ii), A is a *special Jordan algebra* [21] under the Jordan product

$$a \circ b := \frac{1}{2}((a+b)^2 - a^2 - b^2) = \frac{1}{2}(ab + ba).$$

Let $a, b \in A$. If aCb, then $a \circ b = ab = ba \in A$. Also, $a \circ 1 = a$, so A is a *unital* Jordan algebra, and if $1 < n \in \mathbb{N}$, then by induction, $a^n = a \circ a^{n-1} \in A$, so A is closed under the formation of real polynomials in its elements.

As A is an order unit space [1: p. 69], it is required that A is *archimedean*, i.e., for all $a, b \in A$,

$$[na \leq b \ \text{ for all } \ n \in \mathbb{N}] \implies a \leq 0.$$

That 1 is an *order unit* (sometimes called a *strong order unit*) in A means that

$$\text{for each } \ a \in A \ \text{ there exists } \ 0 < \lambda \in \mathbb{R} \ \text{ such that } \ a \leq \lambda.$$

(Recall that we are identifying λ with $\lambda 1$.) The *order-unit norm* $\|\cdot\|$, which is defined on A by

$$\|a\| := \inf\bigl\{0 < \lambda \in \mathbb{R} : \ -\lambda \le a \le \lambda\bigr\},$$

is a norm on the real linear space A. The positive cone A^+ is norm closed in A ([8: Theorem 4.6 (iii)]), as is $C(a)$ for every $a \in A$ ([8: Theorem 8.11]).[1]

By [8: Lemma 1.6], the order-unit norm has the following properties. For all $a, b \in A$ and all $0 < \lambda \in \mathbb{R}$:

(i) $-\|a\| \le a \le \|a\|$.

(ii) $-b \le a \le b \implies \|a\| \le \|b\|$.

(iii) $\|a^2\| = \|a\|^2$.

(iv) $\|a \circ b\| \le \|a\|\|b\|$.

(v) $aCb \implies \|ab\| \le \|a\|\|b\|$.

If $a \in A^+$, then the *square root* $a^{1/2}$ of a in Definition 2.1(v) is uniquely determined and $a^{1/2} \in CC(a)$ ([8: Theorem 2.2]). If $a \in A$, then $a^2 \in A^+$ (Definition 2.1(ii)), whence a has an *absolute value*

$$|a| := (a^2)^{1/2} \in CC(a^2) \subseteq CC(a),$$

which is uniquely determined by the properties $|a| \in A^+$ and $|a|^2 = a^2$.

If $a \in A$, then by [8: Theorem 3.3],

$$a^+ := \frac{1}{2}(|a| + a) \in A^+ \cap CC(a) \quad \text{and} \quad a^- := \frac{1}{2}(|a| - a) \in A^+ \cap CC(a);$$

moreover, $a = a^+ - a^-$, $|a| = a^+ + a^-$, $a^+a^- = a^-a^+ = 0$.

3. Projections and symmetries

By [8: Theorem 5.6], under the partial order inherited from A, the set P of projections forms an orthomodular lattice (OML) with $p \mapsto p^\perp := 1 - p$ as the orthocomplementation [4, 18]. Thus, for all $p, q \in P$:

(i) $0 \le p \le 1$.

(ii) p and q have a *supremum* (least upper bound) $p \vee q$ and an *infimum* (greatest lower bound) $p \wedge q$ in P.

(iii) $p \vee p^\perp = 1$ and $p \wedge p^\perp = 0$.

[1] In the presence of conditions (i)–(vii) of Definition 2.1, condition (viii) is equivalent to the requirement that $C(a)$ is norm closed for all $a \in A$.

(iv) $p \le q \implies q^\perp \le p^\perp$.

(v) $(p^\perp)^\perp = p$.

(vi) $p \le q \implies q = p \vee (q \wedge p^\perp)$. By the *De Morgan laws*,

(vii) $(p \vee q)^\perp = p^\perp \wedge q^\perp$ and $(p \wedge q)^\perp = p^\perp \vee q^\perp$.
Also, as $\|p\|^2 = \|p^2\| = \|p\|$, we have

(viii) $p \neq 0 \iff \|p\| = 1$.

Let $p, q \in P$. The projections p and q are *orthogonal*, in symbols, $p \perp q$, iff $p \le q^\perp$. The relation $\perp$ is symmetric, and if $p \perp q$, then $p \wedge q = 0$ and $p + q \in P$ with $p + q = p \vee q$ ([8: Lemma 5.3]). If $p \le q$, then $q - p \in P$, $p \perp (q - p)$, and $q - p = q \wedge p^\perp$.

The projections p and q are said to be (Mackey) *compatible* iff there are pairwise orthogonal projections $p_1, q_1, d \in P$ such that $p = p_1 \vee d$ and $q = q_1 \vee d$. Compatibility of p and q is equivalent to the condition that $p = (p \wedge q) \vee (p \wedge q^\perp)$. If either $p \le q$ or $p \perp q$, it is clear that p and q are compatible. It is not difficult to show that p and q are compatible iff pCq (for instance, see [9: Lemma 3.2]); moreover,

$$pCq \implies pq = qp \in P \text{ with } pq = p \wedge q \text{ and } p \vee q = p + q - pq.$$

The following fact (*Foulis-Holland theorem*) will be used routinely in what follows: If at least two of the relations pCq, pCr, qCr hold, then $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ and $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$.

Notice that the unit interval E of A is a convex effect algebra in the sense of [15]. We clearly have $P \subseteq E$. By [8: Th. 2.4], for $e \in E, p \in P$ we have $e \le p$ iff $e = ep$ iff $e = pe$ iff $e = pep$ iff $e = ep = pe$. Also, projections are exactly the extreme points in the convex set E. Moreover, $p \in E$ is a projection iff whenever $e, f, e + f \in E$, then $e, f \le p$ implies $e + f \le p$.

If $a \in A$, then the projection $a^{\mathrm{o}} \in P$ in Definition 2.1(vi) is called the *carrier projection* of a. By [8: Theorem 2.7], a^{o} is the unique projection such that, for all $b \in A$, $ab = 0 \iff a^{\mathrm{o}}b = 0$; moreover, by [8: Theorem 2.10], $a^{\mathrm{o}} \in CC(a)$, $a^{\mathrm{o}} = |a|^{\mathrm{o}}$, and for all $b \in A$,

$$ab = 0 \iff a^{\mathrm{o}}b^{\mathrm{o}} = 0 \iff b^{\mathrm{o}}a^{\mathrm{o}} = 0 \iff ba = 0.$$

If $p \in P$, then by [8: Theorem 2.10(iii)], $a^{\mathrm{o}} \le p \iff a = ap$, and by [8: Lemma 2.9(ii)] $a = ap \iff a = pa$. Also, $n \in \mathbb{N} \implies (a^n)^{\mathrm{o}} = a^{\mathrm{o}}$, hence there are no nonzero nilpotents in A.

For every $a \in A$, $a^{\mathrm{o}} = (a^+)^{\mathrm{o}} + (a^-)^{\mathrm{o}}$. The *signum of* a, defined by

$$\mathrm{sgn}(a) := (a^+)^{\mathrm{o}} - (a^-)^{\mathrm{o}},$$

satisfies $\operatorname{sgn}(a) \in CC(a)$, $(\operatorname{sgn}(a))^2 = a^{\mathrm{o}} = (\operatorname{sgn}(a))^{\mathrm{o}}$, $|a| = \operatorname{sgn}(a)a$, and $a = |a| \operatorname{sgn}(a)$ ([8: Theorem 3.6]). The formula $a = |a| \operatorname{sgn}(a)$ is called the *polar decomposition* of a.

3.1. Lemma.

(i) *For any two* $a, b \in A$, $(a+b)^{\mathrm{o}} \leq a^{\mathrm{o}} \vee b^{\mathrm{o}}$.

(ii) *If* $0 \leq a, b$, *then* $(a+b)^{\mathrm{o}} = a^{\mathrm{o}} \vee b^{\mathrm{o}}$.

Proof.

(i) By [8: Th. 2.7], if $x \in A, p \in P$, $x^{\mathrm{o}} \leq p \iff x = xp$. Since $a^{\mathrm{o}}, b^{\mathrm{o}} \leq a^{\mathrm{o}} \vee b^{\mathrm{o}}$, we have $(a+b)(a^{\mathrm{o}} \vee b^{\mathrm{o}}) = a(a^{\mathrm{o}} \vee b^{\mathrm{o}}) + b(a^{\mathrm{o}} \vee b^{\mathrm{o}}) = a+b$, whence $(a+b)^{\mathrm{o}} \leq a^{\mathrm{o}} \vee b^{\mathrm{o}}$.

(ii) On the other hand, $0 \leq a, b \leq a+b$ imply $a^{\mathrm{o}}, b^{\mathrm{o}} \leq (a+b)^{\mathrm{o}}$ by [8: Theorem 2.10(viii)], hence $a^{\mathrm{o}} \vee b^{\mathrm{o}} \leq (a+b)^{\mathrm{o}}$. □

In what follows, we need the following theorem.

3.2. Theorem. ([10: Th. 2.4] *Let* $p \in P$ *and define*

$$pAp := \{pap : \ a \in A\} = \{b \in A : \ b = bp = pb\}.$$

Then pAp *is norm-closed in* A *and, with the partial order inherited from* A, pAp *is a synaptic algebra with unit* p *and enveloping algebra* pRp. *Moreover, the order-unit norm on* pAp *is the restriction to* pAp *of the order-unit norm on* A, *and for all* $a, b \in pAp$, *we have:* $a \circ b$, a^{o}, $|a|$, a^+, a^-, $\operatorname{sgn}(a) \in pAp$, $0 \leq a \implies a^{1/2} \in pAp$, *and* a^{o} *is the carrier of* a *as calculated in* pAp.

An element $s \in A$ such that $s^2 = 1$ is called a *symmetry*, and it is obvious that there is a bijective correspondence $p \leftrightarrow s$ between projections $p \in P$ and symmetries $s \in A$ given by $s = 2p - 1$ and $p = \frac{1}{2}(s+1)$. If s is a symmetry, then so is $-s$, and if $p \leftrightarrow s$, then $p^{\perp} \leftrightarrow -s$. Evidently, $1 \leftrightarrow 1$, $0 \leftrightarrow -1$, and ± 1 are the only symmetries in $\mathbb{R} \subseteq A$. If s is a symmetry, then $\|s\|^2 = \|s^2\| = \|1\| = 1$, and as $0 \leq \frac{1}{2}(s+1) \leq 1$, it follows that $-1 \leq s \leq 1$.

An element $t \in A$ is called a *partial symmetry* iff $t^2 \in P$. If $t^2 = p \in P$, then $t^{\mathrm{o}} = (t^2)^{\mathrm{o}} = p$, whence t is a symmetry in the synaptic algebra pAp (Theorem 3.2). Moreover, $tp^{\perp} = p^{\perp}t = 0$, whence $s := t + p^{\perp}$ is a symmetry, and we shall refer to s as the *canonical extension* of the partial symmetry t to a symmetry s.

If $p \in P$, then $s := 2p - 1 = p - p^{\perp}$ is a difference of two orthogonal projections. More generally, if p and q are orthogonal projections, it is clear that $t = p - q$ is a partial symmetry with $t^2 = p + q = p \vee q$. Furthermore, every partial symmetry t

can be written as a difference of two orthogonal projections; in fact, $(t^+)^{\rm o} \perp (t^-)^{\rm o}$ with $t = (t^+)^{\rm o} - (t^-)^{\rm o} = \operatorname{sgn}(t)$, and $|t| = t^2 = (t^+)^{\rm o} + (t^-)^{\rm o} = (t^+)^{\rm o} \vee (t^-)^{\rm o} = t^{\rm o}$.

If $a \in A$, then $t := \operatorname{sgn}(a) = (a^+)^{\rm o} - (a^-)^{\rm o} \in CC(a)$ is a partial symmetry with $t^2 = (a^+)^{\rm o} + (a^-)^{\rm o} = a^{\rm o} = |a|^{\rm o} = t^{\rm o} \in CC(a)$. If $s := t + (t^{\rm o})^\perp$ is the canonical extension of t to a symmetry, then since $a(a^{\rm o})^\perp = |a|(a^{\rm o})^\perp = 0$, we have $s \in CC(a)$ with $|a| = sa$ and $a = s|a|$.

If $a, b \in A$, then $aba = 2a \circ (a \circ b) - a^2 \circ b \in A$; hence the *quadratic mapping* [8: Definition 4.1] $J_a : A \to A$ given by

$$J_a(b) := aba \qquad \text{for all} \quad b \in A$$

is well-defined. By [8: Theorem 4.2 and Lemma 4.3], J_a is an order preserving, linear and norm-continuous mapping on A. If $p \in P$, then J_p is called the *compression on A with focus p*. The OML of projections in the synaptic algebra pAp (Theorem 3.2) is the interval $P[0,p] := \{q \in P : \ 0 \le q \le p\}$, partially ordered by the restriction of the partial order on P, and with $q \mapsto q^{\perp_p} := p \wedge q^\perp$ as the orthocomplementation.

If $s \in A$ is a symmetry, then J_s is called the *symmetry mapping* on A. Obviously, J_s is a synaptic automorphism of period two on A.

If $p, q \in P$, then by [8: Theorem 5.6]

$$(J_p(q))^{\rm o} = (pqp)^{\rm o} = p \wedge (q \vee p^\perp) = \phi_p(q),$$

i.e., the restriction to P of the mapping $b \mapsto (J_p(b))^{\rm o}$ is the Sasaki projection ϕ_p on the OML P. The Sasaki mapping has the following generalization: if $a \in A$, the mapping $\phi_a \colon P \to P$ defined by

$$\phi_a(q) := (J_a(q))^{\rm o} = (aqa)^{\rm o} \qquad \text{for all} \quad q \in P$$

is called the *Sasaki mapping* on P determined by a. If s is a symmetry in A and $p \in P$, then $\phi_s(p) = J_s(p)$ [10: Lemma 6.3], in particular for every $b \in A$, $\phi_s(b^{\rm o}) = J_s(b^{\rm o}) = (sbs)^{\rm o}$.

3.3. Definition. A symmetry s is said to *exchange projections* e, f if $ses = f$, i.e., $J_s(e) = f$.

Clearly, if e, f are projections with $f = ses$, then $e = sfs$.

3.4. Theorem. *If $t \in A$ is a partial symmetry such that $tet = f$ and $tft = e$ for a pair of projections e, f then t can be extended to a symmetry $s \in A$ exchanging e and f.*

Proof. Assume the hypotheses and let $p := t^2$. We have $e = tft = t^2et^2 = pep$, so $e = pe = ep$. Similarly, $f = pf = fp$, and hence $p \geq e, f$. Set $s = t + (1 - p)$, then s is a symmetry, the canonical extension of t. From $(1 - p)e = 0$ we obtain that $tet = ses = f$. □

By [10: Corollary 6.6], the following version of the so-called *parallelogram rule* ([20: §11]) holds in P: If $p, q \in P$, there is a symmetry that exchanges $(p \vee q) - p$ and $q - (p \wedge q)$.

In the remaining part of this section, we study relations between symmetry exchange and perspectivity. First, following Dixmier [6], we introduce the following important notions, that make sense also in synaptic algebras.

3.5. Definition.

(i) We say that two projections e and f are in *position p′* if $e \wedge (1 - f) = 0 = (1 - e) \wedge f$.

(ii) Projections e and f are said to be in *position p″* if $e \wedge f = 0 = (1 - e) \wedge (1 - f)$.

3.6. Lemma.

(i) *Projections e and f are in position p″ iff they are complementary; e and f are in position p′ iff e and $1 - f$ are complements, or equivalently, $1 - e$ and f are complements.*

(ii) *For any two projections e and f, the projections $(e \vee f) - f$ and $e - (e \wedge f)$ are in position p′, as are the projections $e - (e \wedge (1 - f))$ and $f - (f \wedge (1 - e))$. The latter are nonzero iff $ef \neq 0$.*

Proof.

(i) is straightforward.

(ii) Write $1 - e = e^{\perp}, e \in P$. We have $((e \vee f) - f) \wedge (1 - (e - (e \wedge f)) = (e \vee f) \wedge f^{\perp} \wedge (e^{\perp} \vee (e \wedge f))$. Observe that $e \wedge f$ commutes with $e^{\perp}$ and with $(e \vee f) \wedge f^{\perp}$, therefore by the Foulis-Holland theorem we obtain $(e \vee f) \wedge f^{\perp} \wedge (e^{\perp} \vee (e \wedge f)) = (e \vee f) \wedge ((f^{\perp} \wedge e^{\perp}) \vee f^{\perp} \wedge (e \wedge f)) = 0$. The other relation in position p′ is obtained by symmetry. The second statement in (ii) is obtained by replacing f by $1 - f$. □

3.7. Lemma. *Let e and f be projections in position p″. If s is a symmetry exchanging e and f and we set $g = (1 + s)/2$, then e and g are in position p′, as are f and g.*

P r o o f. Let $0 \leq x \leq e \wedge (1-g)$. Then $ex = x = xe$ and $gx = 0$. Also $sx = 2gx - x = -x = xs$. Now since $fx = sesx = -sex = -sx = x = xf$, we have $x \leq e \wedge f = 0$. Hence $e \wedge (1-g) = 0$. Next let $0 \leq x \leq (1-e) \wedge g$. Then $gx = x = xg$ and $ex = 0$.Now $sx = sgx = (2g-1)gx = gx = x$ so $fx = fsx = sex = 0$ and $x \leq (1-e) \wedge (1-f) = 0$. Thus $(1-e) \wedge g = 0$. The proof that f and g are in position p$'$ is similar. □

3.8. Lemma. *For projections e and f the following are equivalent:*

(i) *e and f are in position p$'$.*

(ii) *e and f are the carrier projections of efe and fef, respectively.*

(iii) *For suitably chosen projections $g, h \in P$, $e = (g \vee h) - h$ and $f = g - (g \wedge h)$.*

(iv) *For any $h \in G$, $(efe)h = 0$ iff $eh = 0$ and $(fef)h = 0$ iff $fh = 0$.*

P r o o f. Follows immediately from the definitions of position p$'$ and carrier projection, and relations between ϕ_s and J_s. □

3.9. Corollary. *If e and f are projections in position p$'$, then there is a symmetry $s \in A$ that exchanges them.*

P r o o f. Follows from Lemma 3.8(iii) and the parallelogram rule. □

We recall that two elements in an OML are said to be *perspective* if they have a common complement. The following statements hold in any OML ([18]).

3.10. Lemma. *If $e, f \leq g$ have a common relative complement h in g, then $h \vee g^{\perp}$ is a common complement for e and f.*

3.11. Lemma. *Let h be a common complement for e and f. If g is a projection which is compatible with h and is orthogonal to e and f, then $e \vee g$ and $f \vee g$ are perspective with $h \wedge g^{\perp}$ as a common complement.*

P r o o f. Set $k := h \wedge g^{\perp}$. Then owing to compatibility of g and h, $g \vee h = g \vee k$, hence $(e \vee g) \vee k = e \vee (g \vee h) = 1$. Also $(e \vee g) \wedge k = (e \vee g) \wedge g^{\perp} \wedge h = e \wedge h = 0$. Similarly, k is a complement for $f \vee g$. □

3.12. Theorem. *Let $e, f \in P$ be exchanged by a symmetry $s \in A$. Then e and f are perspective in P.*

P r o o f. As $e \wedge f \leq e, f$, we have $s(e \wedge f)s \leq ses, sfs$, hence $s(e \wedge f)s \leq e \wedge f$. On the other hand, from $s^2 = 1$ it follows that $e \wedge f \leq s(e \wedge f)s$, so $s(e \wedge f)s = e \wedge f$. Then we also have $s(e - (e \wedge f))s = f - (e \wedge f)$. Let $p := (e - (e \wedge f)) \vee (f - (e \wedge f))$. Then $e - (e \wedge f)$ and $f - (e \wedge f)$ are in position p$'$ in the OML $[0, p]$. Put $s_0 := psp$.

It is easy to check that $e-(e\wedge f)$ and $f-(e\wedge f)$ are exchanged by s_0. Put $g:=(p+s_0)/2$. Then g is a projection in $[0,p]$. Similarly as in Lemma 3.7 we show that $e-(e\wedge f)$ and g are in position p′ in pAp, as are $f-(e\wedge f)$ and g. By Lemma 3.6 $e-(e\wedge f)$ and $f-(e\wedge f)$ have a common complement $p\wedge g^{\perp}=p-g$ in $[0,p]$. Thus by Lemma 3.10, $e-(e\wedge f)$ and $f-(e\wedge f)$ have $p\wedge g^{\perp}\vee p^{\perp}=g^{\perp}=1-g$ as a common complement. But $e\wedge f$ is orthogonal to $e-(e\wedge f)$ and $f-(e\wedge f)$, and therefore commutes with p , and also with $1-g$. By Lemma 3.11, e and f are perspective with $(1-g)\wedge(1-(e\wedge f))$ as a common complement. □

The converse of Theorem 3.12 need not hold in general, but we have the following.

3.13. Theorem. *If e and f are perspective projections then there are symmetries $s,t\in A$ with $tsest=f$.*

P r o o f. If g is a common complement of e and f, then e and $1-g$ are in position p′, as are f and $1-g$ (Lemma 3.6). Thus there are symmetries $s,t\in G$ with $ses=1-g$ and $tft=1-g$, by Corollary 3.9. Hence $f=tsest$. □

The next theorem shows that in the special case when e and f are orthogonal, the converse of Theorem 3.12 holds.

3.14. Theorem. *If e and f are orthogonal perspective projections then there is a symmetry v exchanging e and f.*

P r o o f. By Theorem 3.13 there are symmetries s,t such that $tsest=f$. Now let $x:=tse, x^*:=est$.[2] Then $x^*x=e$ and $xx^*=f$. Using the relations $xe=x=fx$ and $x^2=xefx=0$, $ex^*=x^*$, $x^*f=x^*$, $(x^*)^2=0$ and orthogonality of e and f, we see that $v:=x+x^*+(1-e-f)$ is a symmetry in A exchanging e and f. □

4. Ideals of the synaptic algebra

In this section, we study the interplay between ideals and projections of a synaptic algebra A. The following definition is analogous to that in [24].

[2]Here x,x^* belong to R, not necessarily to A, but $x+x^*=est+tse=(e+t)s(e+t))-(ese+tst)\in A$.

4.1. Definition. A linear subspace I of A is called a *quadratic ideal* iff $ada \in I$ whenever $a \in A$, $d \in I$.

4.2. Lemma. *A subset I of A is a quadratic ideal iff I is a Jordan ideal in A.*

Proof. ([7]) The proof follows from the following two equalities:

$$ada = \frac{1}{2}[(a(da + ad) + (da + ad)a) - (a^2 d + da^2)], \tag{1}$$

$$ad + da = (a + 1)d(a + 1) - ada - d. \tag{2}$$

□

If I is a quadratic ideal, write $\tilde{I} := I \cap P$ for the set of all projections in I.

4.3. Theorem. *If I is a quadratic ideal of A, the set $\tilde{I}$ of projections in I has the following properties:*

(i) *if $p \in \tilde{I}$ and $q \le p$, then $q \in \tilde{I}$;*

(ii) *if $p \in \tilde{I}$ and q and p are perspective, then $q \in \tilde{I}$;*

(iii) *if $p, r \in \tilde{I}$, then $p \vee r \in \tilde{I}$.*

Proof.

(i) follows from $q = qpq$.

(ii) If p, q are perspective projections, then by Theorem 3.13, there are symmetries $s, t \in A$ with $q = (t(sps)t)$, whence if $p \in \tilde{I}$, then $q \in \tilde{I}$.

(iii) Suppose $p, r \in \tilde{I}$. By (i), $p - (p \wedge r) \in \tilde{I}$. But $p - (p \wedge r)$ and $(p \vee r) - r$ are exchanged by a symmetry by the parallelogram rule, hence by (ii), $(p \vee r) - r \in \tilde{I}$, and whence $p \vee r = ((p \vee r) - r) + r \in \tilde{I}$. □

Recall that a nonempty subset J of any OML L is called a *p-ideal* of L iff it satisfies (i) - (iii) of Theorem 4.3. Alternatively, a subset J of L is a p-ideal iff (1) $p, q \in J \implies p \vee q \in J$ and (2) $p \in J, q \in L \implies (p \vee q^{\perp}) \wedge q = \phi_q(p) \in J$ [4, 18]. Theorem 4.3 asserts that projections in any quadratic ideal I of A form a p-ideal in the OML P.

Let $V := \{a_1 a_2 \ldots a_n : \ a_i \in A, \ i = 1, 2, \ldots, n\} \subseteq R$ be the set of all finite products of elements of A. If $v \in V, v = a_1 a_2 \ldots a_n$, put $v^* := a_n a_{n-1} \ldots a_1$. If $b \in A$, then $vbv^* = a_1 a_2 \ldots a_n b a_n a_{n-1} \ldots a_1 = J_{a_1} \circ J_{a_2} \circ \cdots \circ J_{a_n}(b) \in A$.

If K is any nonempty subset of A, it is easily seen that the set J of all finite sums $\sum_{i=1}^{n} \alpha_i v_i k_i v_i^*$, where $\alpha_i \in \mathbb{R}$, $v_i \in V$, $k_i \in K$, $n \in \mathbb{N}$, is the smallest quadratic ideal containing K, that is, the quadratic ideal generated by K. In accordance with [5], we call a quadratic ideal J a *restricted quadratic ideal* iff J coincides with the ideal generated by $\tilde{J}$.

4.4. Theorem. *Let J be a p-ideal in P satisfying the following additional property:*

$$p \in J, a \in A \implies \phi_a(p) = (apa)^{\mathrm{o}} \in J. \tag{3}$$

Then the set

$$\widehat{J} := \{x \in A : \ x^{\mathrm{o}} \in J\}$$

is the quadratic ideal in A generated by J.

P r o o f. Assume that $x, y \in \widehat{J}$, then $x^{\mathrm{o}}, y^{\mathrm{o}} \in J$, and since $(x+y)^{\mathrm{o}} \leq x^{\mathrm{o}} \vee y^{\mathrm{o}}$, we have $x + y \in \widehat{J}$. If $x \in \widehat{J}$, $\alpha \in \mathbb{R}$, then $(\alpha x)^{\mathrm{o}} = x^{\mathrm{o}}$, so $\alpha x \in \widehat{J}$. This proves linearity of $\widehat{J}$.

By [8: Cor. 3.7(ii)], $x^{\mathrm{o}} = |\ x\ |^{\mathrm{o}}$, and by [8: Theor. 3.3(viii)], $x^{\mathrm{o}} = (x^+)^{\mathrm{o}} + (x^-)^{\mathrm{o}} = (x^+)^{\mathrm{o}} \vee (x^-)^{\mathrm{o}}$. Consequently, $x \in \widehat{J}$ iff $|\ x\ | \in \widehat{J}$ iff $x^+, x^- \in \widehat{J}$. Moreover, by [8: Theorem 4.9(v)], if $0 \leq x$, then $(axa)^{\mathrm{o}} = (ax^{\mathrm{o}}a)^{\mathrm{o}}$.

Assume that $x \in \widehat{J}$, $a \in A$. Then $axa = ax^+a - ax^-a$, and $(x^+)^{\mathrm{o}}, (x^-)^{\mathrm{o}} \in J$ implies $(ax^+a)^{\mathrm{o}} = (a(x^+)^{\mathrm{o}}a) \in J$, $(ax^-a)^{\mathrm{o}} = (a(x^-)^{\mathrm{o}}a)^{\mathrm{o}} \in J$, and since $(axa)^{\mathrm{o}} \leq (ax^+a)^{\mathrm{o}} \vee (ax^-a)^{\mathrm{o}} \in J$, we have $(axa)^{\mathrm{o}} \in J$, hence $axa \in \widehat{J}$. This proves that $\widehat{J}$ is a quadratic ideal in A.

It remains to prove that $\widehat{J}$ is generated by J. Let $x \in \widehat{J}$, then $x^{\mathrm{o}} \in J$, and owing to $(x^+)^{\mathrm{o}}, (x^-)^{\mathrm{o}} \leq x^{\mathrm{o}}$, we have $(x^+)^{\frac{1}{2}} x^{\mathrm{o}} (x^+)^{\frac{1}{2}} = x^+$, $(x^-)^{\frac{1}{2}} x^{\mathrm{o}} (x^-)^{\frac{1}{2}} = x^-$, so that x^+ and x^- belong to the ideal generated by J, , and hence also $x = x^+ - x^-$ belongs to the ideal generated by J. □

We have the following characterization of restricted ideals.

4.5. Theorem. *The following conditions on a quadratic ideal I of A are equivalent:*

(a) *I is restricted and $\tilde{I}$ is a p-ideal satisfying (3).*

(b) *$x \in I$ implies $x^{\mathrm{o}} \in I$.*

P r o o f.

(b) $\implies$ (a): If (b) holds, then $x^{\mathrm{o}} \in \tilde{I}$ whenever $x \in I$, and we show as in the proof of Theorem 4.4, that x belongs to the ideal generated by $\tilde{I}$. If $p \in \tilde{I} \subseteq I$, then for any $a \in A$, $apa \in I$ yields $(apa)^{\mathrm{o}} \in I$, which shows that $\tilde{I}$ satisfies condition (3).

(a) $\implies$ (b): We have $\tilde{I} \subseteq I$. By Theorem 4.4, $\widehat{\tilde{I}}$ is the smallest quadratic ideal containing $\tilde{I}$, and since by (a) I is restricted, we have $\widehat{\tilde{I}} = I$. □

4.6. Remark. By [14: Lemma 1.11, Proposition 1.12], [17: 1.4.4], the quotient of A with respect to an ideal I has the following properties. If I is a quadratic ideal of A, then the quotient A/I, endowed with the operations $[x]+[y] = [x+y]$, $\alpha[x] = [\alpha x]$, $[x] \circ [y] = [x \circ y]$, $x, y \in A, \alpha \in \mathbb{R}$, becomes a real Jordan algebra. With quotient norm $|| \, [x] \, ||_0 = \inf_{y \in [x]} || \, y \, ||$, $(A/I, || \, . \, ||_0)$ becomes a normed space which is norm-complete iff $(A, || \, . \, ||)$ is norm-complete. The set $(A^+ + I)/I$ defines a pre-order on A/I, and we have $[x] \leq [y]$ iff there is $a \in I$ such that $x \leq y + a$. This pre-order is a partial order if and only if $(A^+ + I)/I$ is a (strict) cone, which happens if and only if I is order-convex, that is, if $x, z \in I$ and $y \in A$ with $x \leq y \leq z$, then $y \in I$. Moreover, [1] is an order unit in A/I if and only if for every $x \in A$, there are $a, b \in I$ and $\alpha, \beta \in \mathbb{R}$ such that $\alpha + a \leq x \leq \beta + b$.

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