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PROPERTY (A) OF THIRD-ORDER ADVANCED DIFFERENTIAL EQUATIONS

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ABSTRACT. In the paper we offer criteria for property (A) of the third-order nonlinear functional differential equation with advanced argument

$$\big(a(t)(x'(t))^{\gamma}\big)'' + p(t)f\big(x(\sigma(t))\big) = 0,$$

where $\int^{\infty} a^{-1/\gamma}(s)\,\mathrm{d}s = \infty$. We establish new comparison theorems for deducing property (A) of advanced differential equations from that of ordinary differential equations without deviating argument. The presented comparison principle fill the gap in the oscillation theory.

1. Introduction

We present criteria for property (A) of the third-order advanced functional differential equations

$$\left(a(t)\left(x'(t)\right)^{\gamma}\right)'' + p(t)f(x(\sigma(t))) = 0, \tag{E}$$

where $a, p, \sigma, \in C([t_0, \infty))$, $f \in C((-\infty, \infty))$. Throughout the paper, we will always assume that

(H_1) γ is the ratio of two positive odd integers,

(H_2) $a(t), p(t)$ are positive,

(H_3) $\sigma(t) \ge t$, $\sigma(t)$ nondecreasing,

(H_4) $xf(x) > 0$, $f'(x) \ge 0$ for $x \ne 0$, $-f(-xy) \ge f(xy) \ge f(x)f(y)$ for $xy > 0$,

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We assume that (E) is in a canonical form, that is,

$$R(t) = \int_{t_0}^{t} a^{-1/\gamma}(s)\,\mathrm{d}s \to \infty \qquad \text{as} \quad t \to \infty. \tag{1.1}$$

By a solution of Eq. (E) we mean a function $x(t) \in C^1([T_x, \infty))$, $T_x \geq t_0$, which has the property $a(t)(x'(t))^\gamma \in C^2([T_x, \infty))$ and satisfies Eq. (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : \ t \geq T\} > 0$ for all $T \geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise, it is called to be nonoscillatory.

Remark 1. All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

We start with the classification of the possible nonoscillatory solutions of (E).

Lemma 1. *Let $x(t)$ be a nonoscillatory solution of* (E). *Then $x(t)$ satisfies one of the following conditions*

(C$_0$) $x(t)x'(t) < 0, \quad x(t)\left[a(t)\left[x'(t)\right]^\gamma\right]' > 0, \quad x(t)\left[a(t)\left[x'(t)\right]^\gamma\right]'' < 0,$

(C$_2$) $x(t)x'(t) > 0, \quad x(t)\left[a(t)\left[x'(t)\right]^\gamma\right]' > 0, \quad x(t)\left[a(t)\left[x'(t)\right]^\gamma\right]'' < 0,$

eventually.

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (E), say $x(t) > 0$ for $t \geq t_0$. It follows from (E) that $\left[a(t)\left[x'(t)\right]^\gamma\right]'' < 0$, eventually. Thus, $\left[a(t)\left[x'(t)\right]^\gamma\right]'$ is decreasing and of fixed sign eventually.

If $\left[a(t)\left[x'(t)\right]^\gamma\right]' < 0$, then it follows from (1.1) that $a(t)\left[x'(t)\right]^\gamma < 0$, which implies $x(t) < 0$. This is a contradiction and we conclude that $\left[a(t)\left[x'(t)\right]^\gamma\right]' > 0$, eventually. Consequently, $a(t)\left[x'(t)\right]^\gamma$ is of fixed sign for all t large enough. Therefore, either Case (C$_0$) or Case (C$_2$) holds. The proof is complete. □

There are considerably less results concerning differential equations with advanced arguments than those for delay differential equations. We attempt to fill this gap and present new comparison theorems in which the compared and the comparing equations are of the same order.

Following Foster and Grimmer [10] and Kusano and Naito [14], we say that $x(t)$ is a solution of degree 0 if it satisfies (C$_0$), while $x(t)$ satisfying (C$_2$) is said to be of degree 2. If we denote by $\mathscr{N}_\ell$ the set of all nonoscillatory solution of degree ℓ, then the set $\mathscr{N}$ of all nonoscillatory solutions of (E) has the following decomposition

$$\mathscr{N} = \mathscr{N}_0 \cup \mathscr{N}_2.$$

For the particular case of (E), namely for differential equation

$$x'''(t) + p(t)x(t) = 0$$

the set $\mathscr{N}_0 \neq \emptyset$, i.e. there always exists (see e.g. [11]) a nonoscillatory solution satisfying the Case (C_0) of Lemma 1. This fact led to the following definition. We say that (E) has property (A) if

$$\mathscr{N} = \mathscr{N}_0.$$

Since

$$\left\{\left(a(t)\left(x'(t)\right)^{\gamma}\right)'' + p(t)f(x(\sigma(t)))\right\} \operatorname{sgn} x(t) \leq 0 \tag{1.2}$$

has the same structure of nonoscillatory solutions as (E) does, we can use the above-mentioned definition also for property (A) of (1.2).

Property (A) of various types of differential equations has been studied by many authors (see [1]–[17]).

When studying properties of differential equations with deviating arguments, many times we are not able to apply technique presented for ordinary equations without deviating argument. In such cases, comparison theorems for deducing properties of equations with deviating argument from those of ordinary equations are very useful. Mahfoud in [16] offers a comparison theorem that permit to study properties of delay differential equations

$$x^{(n)}(t) + p(t)f(x(\tau(t))) = 0,$$

from that of ordinary differential equations

$$x^{(n)}(t) + \frac{p(\tau^{-1}(t))}{\tau'(\tau^{-1}(t))} f(x(t)) = 0.$$

This comparison theorem has been generalized by Kusano and Naito [14] and Dzurina [7] to differential equations with quasi-derivatives. But the corresponding comparison principle for advanced differential equations is still missing. In this paper, we try to fill this gap in the oscillation theory and we present desirable comparison result for the third order advanced differential equations. The results obtained, permit immediately to transfer known/incoming criteria for property (A) from ordinary equation to advanced one.

2. Main results

It is convenient to prove our main result by means of a series of lemmas, as follows. The following result is crucial and essential for the intended comparison theorem.

Lemma 2. *Assume $z(t) > 0$, $z'(t) > 0$, $\left(a(t)\left[z'(t)\right]^{\gamma}\right)' > 0$, eventually. Then for arbitrary $k \in (0,1)$*

$$\frac{z(\sigma(t))}{z(t)} \geq k\frac{R(\sigma(t))}{R(t)}, \tag{2.1}$$

eventually.

P r o o f. It follows from the monotonicity of $w(t) = a(t)\,[z'(t)]^\gamma$ that

$$z(\sigma(t)) - z(t) = \int_t^{\sigma(t)} z'(s)\,\mathrm{d}s = \int_t^{\sigma(t)} w^{1/\gamma}(s) a^{-1/\gamma}(s)\,\mathrm{d}s$$

$$\geq w^{1/\gamma}(t) \int_t^{\sigma(t)} a^{-1/\gamma}(s)\,\mathrm{d}s = w^{1/\gamma}(t)\big[R(\sigma(t)) - R(t)\big].$$

That is,

$$\frac{z(\sigma(t))}{z(t)} \geq 1 + \frac{w^{1/\gamma}(t)}{z(t)}\big[R(\sigma(t)) - R(t)\big]. \tag{2.2}$$

On the other hand, since $z(t) \to \infty$ as $t \to \infty$, then for any $k \in (0,1)$ there exists a t_1 large enough, such that

$$kz(t) \leq z(t) - z(t_1) = \int_{t_1}^{t} w^{1/\gamma}(s) a^{-1/\gamma}(s)\,\mathrm{d}s$$

$$\leq w^{1/\gamma}(t) \int_{t_1}^{t} a^{-1/\gamma}(s)\,\mathrm{d}s \leq w^{1/\gamma}(t) R(t)$$

or equivalently

$$\frac{w^{1/\gamma}(t)}{z(t)} \geq \frac{k}{R(t)}. \tag{2.3}$$

Using (2.3) in (2.2), we get

$$\frac{z(\sigma(t))}{z(t)} \geq 1 + \frac{k}{R(t)}\big[R(\sigma(t)) - R(t)\big] \geq k\frac{R(\sigma(t))}{R(t)}.$$

This completes the proof. □

Now, we are prepared to offer our main result.

Theorem 1. *Assume that for some $k \in (0,1)$ the third order ordinary differential inequality*

$$\left\{ \left(a(t)\,(x'(t))^\gamma\right)'' + p(t) f\left(k\frac{R(\sigma(t))}{R(t)}\right) f(x(t)) \right\} \operatorname{sgn} x(t) \leq 0, \tag{E_1}$$

has property (A), *then so does Eq.* (E).

P r o o f. Assume the contrary, let $x(t)$ be a nonoscillatory solution of Eq. (E), such that $x(t) \in \mathscr{N}_2$. We may assume that $x(t) > 0$, then

$$x'(t) > 0, \quad \big[a(t)\,[x'(t)]^\gamma\big]' > 0, \qquad \big[a(t)\,[x'(t)]^\gamma\big]'' < 0.$$

Integrating (E) from t to ∞, one gets

$$\begin{aligned}\left(a(t)\left[x'(t)\right]^{\gamma}\right)' &\geq \int_t^{\infty} p(s)f(x\left[\sigma(s)\right])\,\mathrm{d}s\\ &\geq \int_t^{\infty} p(s)f\left(k\frac{R(\sigma(s))}{R(s)}\right) f(x(s))\,\mathrm{d}s,\end{aligned}$$

where we have used (2.1) and (H_4). An integration from t_1 to t, yields

$$a(t)\left[x'(t)\right]^{\gamma} \geq \int_{t_1}^{t}\int_v^{\infty} p(s)f\left(k\frac{R(\sigma(s))}{R(s)}\right) f(x(s))\,\mathrm{d}s\,\mathrm{d}v,$$

or equivalently

$$x'(t) \geq a^{-1/\gamma}(t)\left[\int_{t_1}^{t}\int_v^{\infty} p(s)f\left(k\frac{R(\sigma(s))}{R(s)}\right) f(x(s))\,\mathrm{d}s\,\mathrm{d}v\right]^{1/\gamma}.$$

Integrating once more from t_1 to t, we see that

$$x(t) \geq \int_{t_1}^{t} a^{-1/\gamma}(u)\left[\int_{t_1}^{u}\int_v^{\infty} p(s)f\left(k\frac{R(\sigma(s))}{R(s)}\right) f(x(s))\,\mathrm{d}s\,\mathrm{d}v\right]^{1/\gamma}\mathrm{d}u.$$

Let us denote the right hand side of this inequality by $z(t)$. Then $x(t) > z(t) > 0$, moreover, $z(t)$ is of degree 2 and

$$\begin{aligned}0 &= \left[a(t)\left[z'(t)\right]^{\gamma}\right]'' + p(t)f\left(k\frac{R(\sigma(t))}{R(t)}\right) f(x(t))\\ &\geq \left[a(t)\left[z'(t)\right]^{\gamma}\right]'' + p(t)f\left(k\frac{R(\sigma(t))}{R(t)}\right) f(z(t)).\end{aligned}$$

This means that $z(t)$ is a solution of the class $\mathscr{N}_2$ of (E_1), which contradicts property (A) of (E_1). We conclude that (E) has property (A). □

Corollary 1. *Assume that for some* $k \in (0,1)$ *the third order ordinary differential equation*

$$x'''(t) + p(t)f\left(k\frac{R(\sigma(t))}{R(t)}\right) f(x(t)) = 0, \tag{E_2}$$

has property (A), *then so does the advanced differential equation*

$$x'''(t) + p(t)f(x\left[\sigma(t)\right]) = 0, \tag{E_3}$$

P r o o f. By [14: Corollary 1] or [7: Theorem 2], (E_2) has property (A) if and only if the corresponding differential inequality

$$\left\{x'''(t) + p(t)f\left(k\frac{R(\sigma(t))}{R(t)}\right)f(x(t))\right\}\operatorname{sgn} x(t) \le 0,$$

enjoys this property. The assertion of the corollary now, follows from Theorem 1. □

Now, we recall two criteria for property (A) of the third order ordinary equations (see e.g. [7: Theorem 5], [12: Theorems 1.7, 1.7']) .

Theorem A. *Assume that*

$$\liminf_{t\to\infty} t^2 \int_t^\infty p(s)\,\mathrm{d}s > \frac{1}{3\sqrt{3}}$$

or

$$\liminf_{t\to\infty} t \int_t^\infty s\,p(s)\,\mathrm{d}s > \frac{2}{3\sqrt{3}},$$

then the third order ordinary equation

$$x'''(t) + p(t)x(t) = 0 \tag{E_4}$$

has property (A).

Applying Corollary 1, we can immediately extend the conclusions of Theorem A to advanced differential equations.

Theorem 2. *Assume that*

$$\liminf_{t\to\infty} t^2 \int_t^\infty \frac{\sigma(s)}{s}\,p(s)\,\mathrm{d}s > \frac{1}{3\sqrt{3}} \tag{2.4}$$

or

$$\liminf_{t\to\infty} t \int_t^\infty \sigma(s)\,p(s)\,\mathrm{d}s > \frac{2}{3\sqrt{3}}, \tag{2.5}$$

then the third order advanced equation

$$x'''(t) + p(t)x(\sigma(t)) = 0 \tag{E_5}$$

has property (A).

We illustrate all our results in the following example.

Example 1. Consider the third order advanced differential equation

$$x'''(t) + \frac{a}{t^4}\, x(t^2) = 0, \qquad t \geq 1, \tag{E$_6$}$$

with $a > 0$. Both conditions (2.4) and (2.5) reduce to just one condition

$$a > \frac{2}{3\sqrt{3}},$$

which by Theorem 2 guarantees property (A) of (E_6). Note that according to [14: Theorem 1], advanced equation (E_5) has property (A) if so does equation (E_4). But this comparison theorem fails for (E_6) since the corresponding differential equation

$$x'''(t) + \frac{a}{t^4}\, x(t) = 0 \tag{E$_7$}$$

has not property (A). This fact follows from the following observations. It is easy to see that differential inequality

$$\left\{x''' + \frac{a}{t^4}\, x(t)\right\} \operatorname{sgn} x(t) \leq 0$$

has a solution $x(t) = (8at^{3/2})/3$ which is of degree 2. But [14: Theorem 2] ensures that equation (E_7) has also a solution of degree 2, i.e. (E_7) does not enjoy property (A).

3. Summary

In this paper, we have presented new comparison theorems for deducing property (A) of the advanced third order equation from that of the suitable ordinary third order differential inequality.

The presented comparison method enable us to immediately extend criteria for property (A) of ordinary equation to advanced equations so that our results support backward the research on ordinary third order equations. It remains an open problem, how to extend presented results to cover also the higher order differential equations. This problem seems to be far from easy, since our comparison theorem is based on the crucial estimate (2.1) of Lemma 2, which essentially utilizes

$$z(t) > 0, \quad z'(t) > 0, \quad \left(a(t)\,[z'(t)]^{\gamma}\right)' > 0,$$

eventually and cannot be applied for the case which appears at higher order equations, namely

$$z(t) > 0, \quad z'(t) > 0, \quad \left(a(t)\,[z'(t)]^{\gamma}\right)' < 0.$$

Really, if we consider $z(t) = t^{1/2}$, $\sigma(t) = t^2$, $a(t) = 1$, $\gamma = 1$, then for any $k \in (0, 1)$ we have

$$\frac{z(\sigma(t))}{z(t)} = t^{1/2} \leq kt = k\frac{R(\sigma(t))}{R(t)},$$

which is opposite to (2.1).

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