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ON COEFFICIENTS OF KAPTEYN-TYPE SERIES

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ABSTRACT. Quite recently Jankov and Pogány [JANKOV, D.—POGÁNY, T. K.: *Integral representation of Schlömilch series*, J. Classical Anal. **1** (2012) 75–84] derived a double integral representation of the Kapteyn-type series of Bessel functions. Here we completely describe the class of functions $\Lambda = \{\alpha\}$, which generate the mentioned integral representation in the sense that the restrictions $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$ is the sequence of coefficients of the input Kapteyn-type series.

1. Introduction and motivation

The series of Bessel functions of the first kind of the type

$$\mathfrak{K}^{\mu}_{\nu,\beta}(z) := \sum_{n\geq 1} \alpha_n\, J_{\nu+\beta n}\left((\mu+n)z\right), \qquad z\in\mathbb{C}, \tag{1}$$

where ν, μ, α_n and $\beta > 0$ are constants, is called a *Kapteyn-type series*. This series were considered recently by Jankov and Pogány [8: Theorem 1, Corollary 1] in deriving integral expression for $\mathfrak{K}^{\mu}_{\nu,\beta}(z)$. Letting $\beta = 1$, $\mu = \nu$ in (1) we arrive at the *Kapteyn series of Bessel functions* named after Willem Kapteyn [10], who introduced the series $\mathfrak{K}^{\nu}_{\nu,1}(z) \equiv \mathfrak{K}_{\nu}(z)$ in 1893, (see also [2]). The importance of the latter series one extends from pulsar physics [14] through radiation from rings of discrete charges [15, 23], electromagnetic radiation [20], quantum modulated systems [3, 16], traffic queueing problems [5] and to plasma physics problems in ambient magnetic fields [13, 21]. For further applications see also the paper [22] and the cited references therein. It is also important to mention that the solution of the famous *Kepler's equation* [7, 17, 19]

$$E - \varepsilon \sin E = M,$$

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where $M \in (0,\pi)$, $\varepsilon \in (0,1]$, can be expressed *via* a Kapteyn series

$$E = M + 2\sum_{n\ge 1} \frac{\sin(nM)}{n}\, J_n(n\varepsilon)\,.$$

The problem of computing the coefficients α_n of Kapteyn series has been considered for example by Kapteyn [11] who concluded that it is possible to expand an arbitrary function f, which is analytic throughout the region

$$D_a = \Big\{ z \in \mathbb{C}:\ \Omega(z) = \Big| \frac{z\exp\{\sqrt{1-z^2}\}}{1+\sqrt{1-z^2}} \Big| \le a \le 1 \Big\},$$

into a series of Bessel functions of the first kind (1) (see, for example [4,11,24]):

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathfrak{L}} \Theta_0(\zeta) f(\zeta)\,\mathrm{d}\zeta + \frac{1}{\pi \mathrm{i}} \sum_{n\ge 1} \oint_{\mathfrak{L}} \Theta_n(\zeta) f(\zeta)\,\mathrm{d}\zeta \cdot J_n(nz), \qquad z \in D_a,$$

where the integration contour $\mathfrak{L}$ is the curve on which $\Omega(z) = a$. Here the function Θ_n is the so-called *Kapteyn polynomial* defined by

$$\Theta_0(z) = \frac{1}{z}, \qquad \Theta_n(z) = \frac{1}{4}\sum_{k=0}^{[\frac{n}{2}]} \frac{(n-2k)^2(n-k-1)!}{k!} \Big(\frac{nz}{2}\Big)^{2k-n}, \quad n \in \mathbb{N}\,.$$

The series $\mathfrak{K}_\nu(z)$ is convergent and represents an analytic function (see [24: p. 559]) throughout the domain $\Omega(z) < \liminf\limits_{n\to\infty} |\alpha_n|^{-1/(\nu+n)}$. But, when $z = x \in \mathbb{R}$, the convergence region depends on the nature of the sequence $(\alpha_n)_{n\in\mathbb{N}}$. Nielsen [18: p. 45] showed that the power function x^ν can be expanded in the form

$$x^\nu = \nu^2\, 2^\nu \sum_{n\ge 0} \frac{\Gamma(\nu+n)\, n!}{(\nu+2n)^{\nu+1}}\, J_{\nu+2n}\left((\nu+2n)x\right), \qquad |x| < \Omega(1)\,.$$

Nielsen [18: p. 48] concluded the Kapteyn-series result

$$\sum_{n\ge 0} a_n \Big(\frac{x}{2}\Big)^{n+\nu} = \sum_{n\ge 0}\sum_{p=0}^{[\frac{n}{2}]} \frac{(\nu+n-2p)^2\,\Gamma(\nu+n-p)}{p!\,(\nu+n)^{n-2p}}\, a_{n-2p} \cdot \frac{J_{\nu+n}\left((\nu+n)x\right)}{(\nu+n)^{\nu+1}}\,,$$

valid for all $|x| < \min\{\Omega(\rho), \Omega(1)\}$, where $\Omega(\rho)$ denotes the positive solution of the equation $x\,\mathrm{e}^{1+x^2/4} = 2\rho$.

Now, we recall the integral expression [8] for $\mathfrak{K}^{\mu}_{\nu,\beta}(z)$ established by Jankov and Pogány.

Theorem A. ([8: Theorem 1]) *Let* $\alpha \in \mathrm{C}^1(\mathbb{R}_+)$, $\alpha|_{\mathbb{N}} = (\alpha_n)_{n\in\mathbb{N}}$ *and assume that the series* $\sum\limits_{n\ge1} n^{-1/3}\alpha_n$ *absolutely converges. Then, for all* $\beta > 0$, $2(\nu+\beta)+1 > 0$ *and*

$$x \in \left(0, 2\min\left(1, \beta\,\mathrm{e}^{-1}\left(\limsup_{n\to\infty} |\alpha_n|^{1/n}\right)^{-1/\beta}\right)\right) =: \mathcal{I}_{\alpha,\beta}$$

we have the integral representation

$$\mathfrak{K}^{\mu}_{\nu,\beta}(x) = -\int_1^\infty \int_0^{[u]} \frac{\partial}{\partial u}\left(\frac{\Gamma(\beta u+\nu+1/2)}{(\mu+u)^{\beta u+\nu}}\, J_{\beta u+\nu}\big((\mu+u)\,x\big)\right) \times \mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) \mathrm{d}u\,\mathrm{d}s, \qquad (2)$$

where $\mathfrak{d}_x := 1+\{x\}\frac{\mathrm{d}}{\mathrm{d}x}$ *and* $[a]$ *and* $\{a\} = a-[a]$ *denote the integer and fractional part of* a*, respectively.*

2. Construction of the function class $\Lambda = \{\alpha\}$

The aim of this section is to describe the class of functions $\Lambda = \{\alpha\}$ which generate an integral representation like (2) for the corresponding Kapteyn-type series, in the sense that the restriction $\alpha\big|_{\mathbb{N}} = \big(\alpha_n\big)_{n\in\mathbb{N}}$ forms the coefficient array of the series (1). Knowing only the set of nodes $\big\{(n,\alpha_n)\big\}_{n\in\mathbb{N}}$, we will derive the class of functions α which depends on a suitable, integrable (on $\mathbb{R}_+$), scaling–function h, say. It is important to note that Jankov *et al.* [9] used a similar way of concluding the coefficient–function class for Neumann series of Bessel functions.

Theorem. *Let* $\beta > 0, 1+\min(\mu,\nu/\beta) > 0$ *and assume that* Theorem A *holds for a given Kapteyn-type series of Bessel functions. Suppose that the integrand in* (2)*, is such that*

$$\frac{\partial}{\partial u}\left(\frac{\Gamma(\beta u+\nu+1/2)}{(\mu+u)^{\beta u+\nu}}\, J_{\beta u+\nu}\big((\mu+u)\,x\big)\right) \int_0^{[u]} \mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) \mathrm{d}s,$$

is $\mathrm{L}^1(\mathbb{R}_+)$*-integrable and let*

$$h(u) := \frac{\partial}{\partial u}\left(\frac{\Gamma(\beta u+\nu+1/2)}{(\mu+u)^{\beta u+\nu}}\, J_{\beta u+\nu}\big((\mu+u)\,x\big)\right) \int_0^{u} \mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) \mathrm{d}s\,.$$

Then the following formula holds

$$\alpha(u) = \begin{cases} \dfrac{\Gamma(\nu+\beta k+1/2)}{(\mu+k)^{\nu+\beta k}} \, \dfrac{\mathrm{d}}{\mathrm{d}u} \dfrac{h(u)}{\mathcal{H}(u)}\Big|_{u=k+}, & u=k, \ \ k\in\mathbb{N}, \\ \dfrac{\Gamma(\nu+\beta u+1/2)}{\{u\}\,(\mu+u)^{\nu+\beta u}} \Big(\dfrac{h(u)}{\mathcal{H}(u)} - \dfrac{h(k+)}{\mathcal{H}(k)}\Big), & 1<u\neq k, \ \ k\in\mathbb{N}, \end{cases} \tag{3}$$

where

$$\mathcal{H}(u) := \frac{\partial}{\partial u}\left(\frac{\Gamma(\beta u+\nu+1/2)}{(\mu+u)^{\beta u+\nu}}\, J_{\beta u+\nu}\big((\mu+u)\,x\big)\right).$$

P r o o f. Assume that the integral representation (2) holds for some class of functions $\Lambda = \{\alpha\}$ such that $\alpha|_{\mathbb{N}}$ represents the coefficient array appearing in $\mathfrak{K}^{\mu}_{\nu,\beta}(x)$. Suppose that the function $\widetilde{h} \in \mathrm{L}^1(\mathbb{R}_+)$ is defined by

$$\widetilde{h}(u) := \mathcal{H}(u) \int_0^{[u]} \mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) \mathrm{d}s\,. \tag{4}$$

Because $u \sim [u]$ for large u, using (4) we conclude that

$$\int_0^{u} \mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) \mathrm{d}s = \frac{h(u)}{\mathcal{H}(u)}, \tag{5}$$

where

$$h(u) := \frac{\widetilde{h}(u) \displaystyle\int_0^{u} \mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) \mathrm{d}s}{\displaystyle\int_0^{[u]} \mathfrak{d}_s\left(\frac{\alpha(s)(\mu+s)^{\nu+\beta s}}{\Gamma(\nu+\beta s+1/2)}\right) \mathrm{d}s} \sim \widetilde{h}(u), \qquad u\to\infty.$$

If we differentiate (5) with respect to u, we get

$$\begin{aligned} \{u\}\alpha'(u) + \left(1+\{u\}\left(\ln(\mu+u)^{\beta} + \frac{\nu+\beta u}{\mu+u} - \beta\,\psi(\nu+\beta u+1/2)\right)\right)\alpha(u) \\ = \frac{\Gamma(\nu+\beta u+1/2)}{(\mu+u)^{\nu+\beta u}} \cdot \frac{\partial}{\partial u}\frac{h(u)}{\mathcal{H}(u)}\,; \end{aligned} \tag{6}$$

here $\psi(\cdot) = \big(\ln\Gamma(\cdot)\big)'$ stands for the *digamma function*, [1: p. 13]. For $u \equiv k \in \mathbb{N}$, we have the coefficient–set (α_k). When $u \in (k, k+1)$, where k is a fixed positive integer, from (6) we deduce

$$\alpha'(u) + \Big(\frac{1}{u-k} + \ln(\mu+u)^\beta + \frac{\nu+\beta u}{\mu+u} - \beta\,\psi(\nu+\beta u+1/2)\Big)\alpha(u) = \frac{\Gamma(\nu+\beta u+1/2)}{(u-k)\,(\mu+u)^{\nu+\beta u}} \cdot \frac{\partial}{\partial u}\frac{h(u)}{\mathcal{H}(u)}\,.$$

Now it is easy to find the solution of the previous linear ODE in the form

$$\alpha(u) = \frac{\Gamma(\nu+\beta u+1/2)}{\{u\}\,(\mu+u)^{\nu+\beta u}}\Big(C_k + \frac{h(u)}{\mathcal{H}(u)}\Big),$$

where C_k denotes the integration constant. Thus we conclude that for $u \geq 1$ it holds $\alpha(u) = \alpha_k$ for $u = k$, $k \in \mathbb{N}$ and

$$\alpha(u) = \frac{\Gamma(\nu+\beta u+1/2)}{\{u\}\,(\mu+u)^{\nu+\beta u}}\left(C_k + \frac{h(u)}{\mathcal{H}(u)}\right), \qquad 1 < u \neq k, \quad k \in \mathbb{N}.$$

Using Landau's bounds [12] for the first kind Bessel function

$$|J_\nu(x)| \leq b_L\,\nu^{-1/3}, \qquad b_L = \sqrt[3]{2}\,\sup_{x\in\mathbb{R}_+} \mathrm{Ai}(x),$$
$$|J_\nu(x)| \leq c_L\,|x|^{-1/3}, \qquad c_L = \sup_{x\in\mathbb{R}_+} x^{1/3}J_0(x)\,,$$

where $\mathrm{Ai}(\cdot)$ stands for the familiar Airy function

$$\mathrm{Ai}(x) := \frac{\pi}{3}\sqrt{\frac{x}{3}}\,\Big[J_{-1/3}\big(2(x/3)^{3/2}\big) + J_{1/3}\big(2(x/3)^{3/2}\big)\Big]\,,$$

we have the estimate

$$\big|\mathfrak{K}^\mu_{\nu,\beta}(x)\big| \leq \max\left(\frac{b_L}{\sqrt[3]{\beta}}, \frac{c_L}{\sqrt[3]{|x|}}\right)\sum_{n\geq 1}\frac{|\alpha_n|}{\big(n+\min(\mu,\nu/\beta)\big)^{1/3}} \sim \sum_{n\geq 1}\frac{|\alpha_n|}{n^{1/3}}\,,$$

which converges by assumption. So, it is sufficient to take $\alpha(u) \to 0$, as $k \to \infty$.

Let us find the constant C_k. Because

$$\alpha_k = \lim_{u\to k+}\alpha(u) = \lim_{u\to k+}\Gamma(\nu+\beta u+1/2)\,\lim_{u\to k+}\frac{C_k + \dfrac{h(u)}{\mathcal{H}(u)}}{(u-k)\,(\mu+u)^{\nu+\beta u}}$$

becomes an indeterminate form for

$$C_k = -\frac{h(k+)}{\mathcal{H}(k)},$$

by L'Hospital's rule we conclude

$$\alpha_k = \frac{\Gamma(\nu+\beta k+1/2)}{(\mu+k)^{\nu+\beta k}} \cdot \frac{\mathrm{d}}{\mathrm{d}u}\frac{h(u)}{\mathcal{H}(u)}\bigg|_{u=k+}\,.$$

Finally, we get the desired formula

$$\alpha(u) = \begin{cases} \dfrac{\Gamma(\nu+\beta k+1/2)}{(\mu+k)^{\nu+\beta k}} \, \dfrac{\mathrm{d}}{\mathrm{d}u} \dfrac{h(u)}{\mathcal{H}(u)}\Big|_{u=k+}, & u=k, \ k\in\mathbb{N}, \\ \dfrac{\Gamma(\nu+\beta u+1/2)}{\{u\}\,(\mu+u)^{\nu+\beta u}} \left(\dfrac{h(u)}{\mathcal{H}(u)} - \dfrac{h(k+)}{\mathcal{H}(k)} \right), & 1<u\neq k, \ k\in\mathbb{N}, \end{cases}$$

such that finishes the proof of the Theorem. □

Remark. Specifying $\beta = 1$, $\mu = \nu$ in Theorem, we deduce the coefficient function class Λ for the Kapteyn-series $\mathfrak{K}_\nu(x)$ associated with the integral representation result by Baricz et al. [2: Theorem 1].

3. Examples

In the Introduction we pointed out a wide range of applications of Kapteyn-series, while in this section, we present two illustrative examples for function $\widetilde{h} \in \mathrm{L}^1(\mathbb{R}_+)$, which describes the convergence rate to zero of the integrand in (4) at infinity. Simultaneously, the associated function $h(u) \sim \widetilde{h}(u)$, $u \to \infty$ and finally related coefficient–functions α are obtained by (3). We remark that in both examples $\mathcal{H}(u)$ remains the same as in Theorem.

Example 1. Let $\widetilde{h}(u) = [u]^s(\,\mathrm{e}^{[u]}+1)^{-1}$, $s>0$; then

$$\int_0^\infty \widetilde{h}(u)\,\mathrm{d}u = \sum_{n\geq 1} \frac{n^s}{\mathrm{e}^n+1}\,,$$

which is a convergent series, so $\widetilde{h} \in \mathrm{L}^1(\mathbb{R}_+)$. Next, we have

$$[u]^s\big(\,\mathrm{e}^{[u]}+1\big)^{-1} \sim u^s\big(\,\mathrm{e}^u+1\big)^{-1} = h(u) \qquad u\to\infty\,.$$

Othersides

$$\int_0^\infty h(u)\,\mathrm{d}u = (1-2^{-s})\Gamma(1+s)\zeta(1+s),$$

where ζ stands for the Riemann Zeta function. For $s>0$, from (3) it follows that the coefficent-function is of the form

$$\alpha(u) = \begin{cases} \dfrac{\Gamma(\nu+\beta k+1/2)}{(\mu+k)^{\nu+\beta k}} \, \dfrac{\mathrm{d}}{\mathrm{d}u} \dfrac{u^s}{(\,\mathrm{e}^u+1)\mathcal{H}(u)}\Big|_{u=k+}, & u=k\in\mathbb{N}, \\ \dfrac{\Gamma(\nu+\beta u+1/2)}{\{u\}(\mu+u)^{\nu+\beta u}} \left(\dfrac{u^s}{\mathcal{H}(u)\,(\,\mathrm{e}^u+1)} \right. & \\ \qquad\qquad \left. -\dfrac{k^s}{\mathcal{H}(k)\,(\,\mathrm{e}^k+1)} \right), & 1<u\neq k\in\mathbb{N}. \end{cases}$$

Example 2. Take $\widetilde{h}(u) = \mathrm{e}^{-s[u]}J_0(a\,[u])$, where $s \geq 0$, $a \in \mathbb{R}$. Since

$$\int\limits_0^\infty \mathrm{e}^{-s[u]}J_0(a\,[u])\,\mathrm{d}u = \sum_{n\geq 0}^\infty \mathrm{e}^{-sn}J_0(an),$$

the auxiliary function $\widetilde{h}$ belongs to $\mathrm{L}^1(\mathbb{R}_+)$. Further

$$\mathrm{e}^{-s[u]}J_0(a\,[u]) \sim \mathrm{e}^{-su}J_0(au) = h(u) \qquad u \to \infty\,,$$

and

$$\int\limits_0^\infty h(u)\,\mathrm{d}u = (s^2+a^2)^{-1/2}\,,$$

being the integral the Laplace transform of $J_0(au)$. Thus, by (3) we conclude

$$\alpha(u) = \begin{cases} \dfrac{\Gamma(\nu+\beta k+1/2)}{(\mu+k)^{\nu+\beta k}}\,\dfrac{\mathrm{d}}{\mathrm{d}u}\dfrac{\mathrm{e}^{-su}J_0(au)}{\mathcal{H}(u)}\Bigg|_{u=k+}, & u = k \in \mathbb{N},\\ \dfrac{\Gamma(\nu+\beta u+1/2)}{\{u\}(\mu+u)^{\nu+\beta u}}\left(\dfrac{\mathrm{e}^{-su}J_0(au)}{\mathcal{H}(u)}\right. & \\ \qquad\qquad\left. -\dfrac{\mathrm{e}^{-sk}J_0(ak)}{\mathcal{H}(k)}\right), & 1 < u \neq k \in \mathbb{N}. \end{cases}.$$

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