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BM-ALGEBRAS AND RELATED TOPICS

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ABSTRACT. Some connections between BM-algebras and its related topics are studied. It is proved that the class of medial BH-algebras coincides with the class of BM-algebras. Moreover, the congruence lattice of a BM-algebra is investigated.

1. Introduction

Y. Imai and K. Iséki introduced in [4] the concept of BCK-algebras. It is well known that BCK-algebras are inspired by some implicational logic. There exist several generalizations of BCK-algebras such as BCI-algebras ([5]), BCH-algebras ([3]), BCC-algebras ([10]), BZ-algebras ([14]), and BH-algebras ([6]). J. Neggers and H. S. Kim [13] introduced the notion of a B-algebra. Ch. B. Kim and H. S. Kim defined BG-algebras ([7]) and BM-algebras ([8]).

The interrelationships between some classes of algebras mentioned before are visualized in Figure 1. (An arrow indicates proper inclusion, that is, if $\mathcal{X}$ and $\mathcal{Y}$ are classes of algebras, then $\mathcal{X} \to \mathcal{Y}$ means $\mathcal{X} \subset \mathcal{Y}$.)

BF-algebras ([15]) and BRK-algebras ([1]) are other generalizations of BM-algebras.

In this paper we discuss relations between BM-algebras and BH/BRK/BCH/BCI/BG/BF/B-algebras. We show that a BH-algebra is a BM-algebra if and only if it is medial. Moreover, it is proved that the variety of all BM-algebras is congruence permutable (hence the congruence lattice of a BM-algebra is modular).

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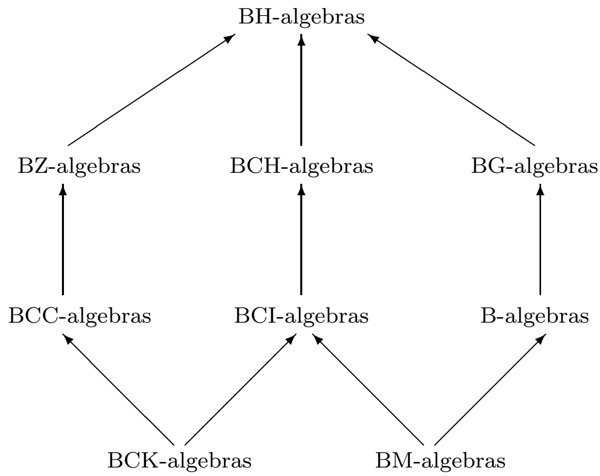

FIGURE 1.

2. Preliminaries

An algebra $(A; *, 0)$ of type $(2, 0)$ (i.e., a nonempty set A with a binary operation $*$ and a constant 0) is said to be a *BH-algebra* ([6]) if it satisfies the following axioms:
(B1) $x * x = 0$;
(B2) $x * 0 = x$;
(BH) $x * y = y * x = 0 \implies x = y$.

A BH-algebra with the condition

(BCH) $(x * y) * z = (x * z) * y$

is called a *BCH-algebra*. In [3], it is proved that $(A; *, 0)$ is a BCH-algebra if and only if it obeys (B1), (BH), and (BCH).

A BH-algebra satisfying the identity

(BCI) $((x * y) * (x * z)) * (z * y) = 0$

is called a *BCI-algebra*. Recall that according to the H. S. Li's axiom system ([11]), an algebra $(A; *, 0)$ of type $(2, 0)$ is a BCI-algebra if and only if it obeys (B2), (BH), and (BCI).

A *BCK-algebra* ([4]) is a BCI-algebra satisfying the following additional axiom:

(BCK) $0 * x = 0$.

Remark 2.1. We know that every BCK-algebra is a BCI-algebra and every BCI-algebra is a BCH-algebra and every BCH-algebra is a BH-algebra.

Let $(A;*,0)$ be an algebra of type $(2,0)$ satisfying identities (B1) and (B2). We say that A is a *B-algebra* (resp. *BF/BG-algebra*) if A obeys axiom (B) (resp. (BF)/(BG)), where:

(B) $(x*y)*z = x*[z*(0*y)]$;

(BF) $0*(x*y) = y*x$;

(BG) $x = (x*y)*(0*y)$.

An algebra $(A;*,0)$ of type $(2,0)$ is called a *BM-algebra* ([8]) if it satisfies (B2) and the following axiom:

(BM) $(x*y)*(x*z) = z*y$.

Remark 2.2. From [8: Theorem 2.6] it follows that every BM-algebra is a B-algebra. By [7: Theorem 2.2, Proposition 2.8], every B-algebra is a BG-algebra and every BG-algebra is a BH-algebra. Consequently, BM-algebras are BH-algebras. It is easy to see that (BM) implies (BCI). Therefore the class of BM-algebras is a subclass of the class of BCI-algebras.

Recently, R. K. Bandaru ([1]) introduced the notion of a BRK-algebra. An algebra $(A;*,0)$ of type $(2,0)$ satisfying (B2) and the following axiom:

(BRK) $(x*y)*x = 0*y$

is called a *BRK-algebra*.
We will denote by **BRK** (resp. **BH/BCH/BCI/BCK/BF/BG/B/BM**) the class of all BRK-algebras (resp. BH/BCH/BCI/BCK/BF/BG/B/BM-algebras). Let **BFH** denote the class of all algebras $(A;*,0)$ of type $(2,0)$ satisfying identities (B1) and (B2). By [1: Proposition 2.6], in any BRK-algebra identity (B1) holds. It is obvious that (BCH) implies (BRK). Consequently,

$$\mathbf{BCH} \subset \mathbf{BRK} \subset \mathbf{BFH}. \tag{1}$$

We get by Remark 2.1 that

$$\mathbf{BCK} \subset \mathbf{BCI} \subset \mathbf{BCH} \subset \mathbf{BH} \tag{2}$$

and by Remark 2.2 we have

$$\mathbf{BM} \subset \mathbf{B} \qquad \text{and} \qquad \mathbf{BM} \subset \mathbf{BCI}. \tag{3}$$

From (1)–(3) and [15: Figure 1] we obtain the interrelatioships between some of the concepts mentioned above which are depicted in Figure 2.

Proposition 2.3. ([8]) *Let $(A;*,0)$ be a BM-algebra. Then for all $x,y,z \in A$ we have:*

(a) $x*x = 0$;
(b) $0*(0*x) = x$;
(c) $0*(x*y) = y*x$;
(d) $(x*z)*(y*z) = x*y$;
(e) $(x*y)*z = (x*z)*y$.

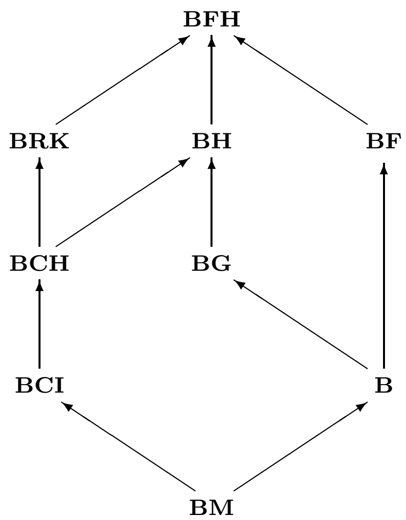

Figure 2.

A B-algebra $(A; *, 0)$ is said to be 0-*commutative* ([13]) if $x*(0*y) = y*(0*x)$ for any $x, y \in A$. In [8], it is proved that $(A; *, 0)$ is a BM-algebra if and only if it is a 0-commutative B-algebra.

H. S. Kim, Y. H. Kim, and J. Neggers ([9]) introduced the notion of a Coxeter algebra. An algebra $(A; *, 0) \in \mathbf{BFH}$ is called a *Coxeter algebra* if $(A; *)$ is a semigroup (that is, A obeys the associative law for $*$).

Proposition 2.4. ([8]) *Every Coxeter algebra is a BM-algebra.*

3. Results

Theorem 3.1. *Let $(A; *, 0)$ be a BM-algebra. Then for all $x, y, z, u \in A$ we have:*

(a) $x * (x * y) = y$;
(b) $x * y = 0 \implies x = y$;
(c) $x * (y * z) = z * (y * x)$;
(d) $(x * y) * (z * u) = (x * z) * (y * u)$;
(e) $(x * y) * (0 * y) = x$.

Proof.

(a) By the definition of a BM-algebra,

$$x * (x * y) = (x * 0) * (x * y) = y * 0 = y.$$

(b) Let $x * y = 0$. Applying (B2) and (a) we get

$$x = x * 0 = x * (x * y) = y.$$

(c) Using (a) and Proposition 2.3 (e) we obtain

$$\begin{aligned} x*(y*z) &= (y*(y*x))*(y*z) \\ &= (y*(y*z))*(y*x) = z*(y*x). \end{aligned}$$

(d) From (c) we have

$$(x*y)*(z*u) = u*(z*(x*y)) = u*(y*(x*z)) = (x*z)*(y*u).$$

(e) follows from (d), (B1) and (B2). □

J. Meng and X. L. Xin introduced the notion of an implicative BCI-algebra ([12]). A BCI-algebra $(A, *, 0)$ is called *implicative* if A satisfies

(iBCI) $x*(x*y) = (y*(y*x))*(x*y)$

for all $x, y \in A$.

Proposition 3.2. *Every BM-algebra is an implicative BCI-algebra.*

P r o o f. Let $(A; *, 0)$ be a BM-algebra. By Remark 2.2, A is a BCI-algebra. The condition (iBCI) follows from Theorem 3.1 (a). □

***Example* 3.3.** Let $A = \{0, 1\}$ and $*$ be defined by the following table:

$*$	0	1
0	0	0
1	1	0

Then $(A; *, 0)$ is an implicative BCI-algebra, which is not a BM-algebra.

A BM-algebra $(A; *, 0)$ is said to be *associative* if it satisfies the associative law, that is, $x*(y*z) = (x*y)*z$ for all $x, y, z \in A$.

Proposition 3.4. $(A; *, 0)$ *is an associative BM-algebra if and only if it is a Coxeter algebra.*

P r o o f. It is easy to see that every associative BM-algebra is a Coxeter algebra. Let $(A; *, 0)$ be a Coxeter algebra. By Proposition 2.4, A is a BM-algebra. We conclude from definition that A is associative. □

Theorem 3.5. *Let* $(A; *, 0)$ *be an algebra of type* $(2, 0)$ *satisfying* (B1) *and* (BCH). *Then the following statements are equivalent:*

(a) *A satisfies* (BG)*;*

(b) $0*(0*x) = x$ *for all* $x \in A$*;*

(c) $x*y = 0$ *implies* $x = y$ *for all* $x, y \in A$*;*

(d) *A is a BM-algebra.*

Proof.

(a) $\Longrightarrow$ (b): By (BG), $(x * x) * (0 * x) = x$. Hence $0 * (0 * x) = x$.

(b) $\Longrightarrow$ (c): Let $x * y = 0$. Applying (b), (BCH), and (B1) we get

$$x = 0 * (0 * x) = 0 * ((x * y) * x) = 0 * ((x * x) * y) = 0 * (0 * y) = y.$$

(c) $\Longrightarrow$ (d): First we prove that

$$z * (z * y) = y. \tag{4}$$

From (BCH) and (B1) we have $(z * (z * y)) * y = (z * y) * (z * y) = 0$. By (c), $z * (z * y) = y$, that is, (4) holds. Using (BCH) and (4) we obtain

$$(z * x) * (z * y) = (z * (z * y)) * x = y * x.$$

Thus (BM) is satisfied. Moreover, A also obeys (B2). Indeed, $x*0 = x*(x*x) = x$ by (4). Consequently, A is a BM-algebra.

(d) $\Longrightarrow$ (a): Let A be a BM-algebra. By Theorem 3.1 (e), A satisfies (BG). □

Theorem 3.6. *Let $(A; *, 0)$ be an algebra of type $(2, 0)$. Then the following statements are equivalent:*

(a) *A is a BM-algebra;*

(b) *A is a BF-algebra with condition* (BCH)*;*

(c) *A is a BG-algebra with condition* (BCH).

Proof.

(a) $\Longrightarrow$ (b): By Proposition 2.3 (a), (c), A is a BF-algebra. From Proposition 2.3 (e) we see that A obeys (BCH).

(b) $\Longrightarrow$ (a): Let A be a BF-algebra satisfying (BCH). By (BF) and (B2) we obtain $0 * (0 * x) = x * 0 = x$. Applying Theorem 3.5 we conclude that A is a BM-algebra.

(a) $\Longrightarrow$ (c): It follows from Proposition 2.3 (a), (e) and Theorem 3.1 (e).

(c) $\Longrightarrow$ (a): It follows from Theorem 3.5. □

Theorem 3.7. *Let $(A; *, 0)$ be an algebra of type $(2, 0)$. Then the following statements are equivalent:*

(a) *A is a BM-algebra;*

(b) *A is a BCH-algebra obeying the implication*

$$x * y = 0 \quad \textit{implies} \quad y * x = 0 \qquad \textit{for all} \quad x, y \in A.$$

Proof.

(a) $\Longrightarrow$ (b): It follows from Proposition 2.3 (a), (e) and Theorem 3.1 (b).

(b) $\Longrightarrow$ (a): Let $x, y \in A$. Suppose that $x * y = 0$. By assumption, $y * x = 0$. Using (BH) we have $x = y$. From Theorem 3.5 we deduce that A is a BM-algebra. □

Theorem 3.8. *Let $(A;*,0)$ be an algebra of type $(2,0)$ such that $0*(0*x)=x$ for all $x \in A$. Then the following statements are equivalent:*

(a) *A is a BM-algebra;*

(b) *A is a BCI-algebra;*

(c) *A is a BCH-algebra.*

P r o o f. Implications (a) $\Longrightarrow$ (b) and (b) $\Longrightarrow$ (c) are obvious.
(c) $\Longrightarrow$ (a) follows from Theorem 3.5. □

Proposition 3.9. *Let $(A;*,0)$ be a BM-algebra such that $0*x=0$ for all $x \in A$. Then $A=\{0\}$.*

P r o o f. Let $x \in A$. From Proposition 2.3 (b) we have $x=0*(0*x)=0*0=0$. Consequently, $A=\{0\}$. □

Corollary 3.10. *If $(A;*,0)$ is a BM-algebra and it is a BCK-algebra, then $A=\{0\}$.*

Let $(A;*,0) \in \mathbf{BFH}$. We say that A is *medial* if A obeys the condition

(M) $(x*y)*(z*u)=(x*z)*(y*u)$

for all $x,y,z,u \in A$. A BH/BRK/BCH/BCI/BG/BF/B-algebra satisfying (M) will be called *medial*.

Theorem 3.11. *An algebra $A \in \mathbf{BFH}$ is medial if and only if it is a BM-algebra.*

P r o o f. From Theorem 3.1 (d) it follows that every BM-algebra satisfies (M).
Conversely, let $(A;*,0) \in \mathbf{BFH}$. Suppose that A is medial. Observe that A obeys (BF). Indeed,

$$0*(x*y)=(y*y)*(x*y)=(y*x)*(y*y)=y*x.$$

Applying (BF) we have

$$(x*y)*(x*z)=(x*x)*(y*z)=0*(y*z)=z*y.$$

Therefore A satisfies (BM) and hence A is a BM-algebra. □

Corollary 3.12. *A BH/BRK/BCH/BCI/BG/BF/B-algebra is medial if and only if it is a BM-algebra.*

The class of all BM-algebras is a variety, which we denote by $\mathcal{BM}$. We recall that a variety $\mathcal{V}$ of algebras is said to be *congruence permutable* if all the algebras in $\mathcal{V}$ have permuting congruences.

Lemma 3.13. (see e.g. [2]) *Let $\mathcal{V}$ be a variety of algebras. The variety $\mathcal{V}$ is congruence permutable if and only if there is a 3-ary term p such that the identities $p(x,y,y)=x$ and $p(x,x,y)=y$ are valid in $\mathcal{V}$.*

Theorem 3.14. *The variety $\mathcal{BM}$ is congruence permutable.*

P r o o f. Let $(A; *, 0)$ be a BM-algebra and let $p(x, y, z) = x * (y * z)$. By (B1) and (B2),

$$p(x, y, y) = x * (y * y) = x * 0 = x.$$

From Theorem 3.1(a) we have

$$p(x, x, y) = x * (x * y) = y.$$

Applying Lemma 3.13 we conclude that the variety $\mathcal{BM}$ is congruence permutable. □

For an algebra A we denote by $\mathrm{Con}(A)$ the set of all congruences on A. With respect to the set inclusion, $\mathrm{Con}(A)$ forms a lattice. It is known (see for an example [2]) that if an algebra A has permuting congruences, then $\mathrm{Con}(A)$ is a modular lattice. From this we have:

THEOREM 3.15. *Let A be a BM-algebra. Then the lattice* $\mathrm{Con}(A)$ *is modular.*

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