VERSITA

Mathematica Slovaca

DOI: 10.2478/s12175-014-0261-3
Math. Slovaca **64** (2014), No. 5, 1093–1104

STATES ON BOUNDED COMMUTATIVE RESIDUATED LATTICES

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(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. We define states on bounded commutative residuated lattices and consider their property. We show that, for a bounded commutative residuated lattice X,

(1) If s is a state, then $X/\ker(s)$ is an MV-algebra.
(2) If s is a state-morphism, then $X/\ker(s)$ is a linearly ordered locally finite MV-algebra.

Moreover we show that for a state s on X, the following statements are equivalent:

(i) s is a state-morphism on X.
(ii) $\ker(s)$ is a maximal filter of X.
(iii) s is extremal on X.

1. Introduction

Since the notion of *state* is firstly defined on MV-algebras by Kôpka and Chovanec in [9], theory of states on algebras is applied to other algebras and now it is a hot research filed. For example, property of states on pseudo-MV algebras is considered in [2], on pseudo-BL algebras in [7], on non-commutative residuated Rℓ-monoids in [5,6]. In [7], it is proved that the notion of (Bosbach) state is the same as the notion of Riečan for good bounded Rℓ-monoids.

The algebras above all have the condition of divisibility (div): $x \wedge y = x \odot (x \to y)$, from which the algebras are distributive lattices. On the other hand there are no research about states on algebras without (div) so far. We

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2010 Mathematics Subject Classification: Primary 06B05, 03G10.
Keywords: state, state morphism, residuated lattice, MV-algebra.
This work was supported by JSPS KAKENHI Grant No. 24500024.

here define states on commutative residuated lattices and consider their property. In [2], it is proved that there exists a state on every bounded commutative Rℓ-monoid, but unfortunately we don't know whether there exist states on bounded commutative residuated lattices. In [4], a strong notion of *state-morphism* is defined and investigated its property on several algebras. It is also prove that if s is a state-morphism s for a bounded commutative Rℓ-monoid X, then the quotient algebra $X/\ker(s)$ is an MV-algebra.

We show a stronger result, namely, for a bounded commutative residuated lattice X,

(1) If s is a state, then $X/\ker(s)$ is an MV-algebra.

(2) If s is a state-morphism, then $X/\ker(s)$ is a linearly ordered locally finite MV-algebra.

Moreover we give a characterization theorem of extremal states on residuated lattices. Namely, for a state s on a bounded commutative residuated lattice X, the following statements are equivalent:

(i) s is a state-morphism on X.

(ii) $\ker(s)$ is a maximal filter of X.

(iii) s is extremal on X.

Our results are generalizations of those in [2, 4, 5] and [7].

2. Bounded commutative residuated lattice and state

We recall a definition of bounded commutative residuated lattices. An algebraic structure $(X, \wedge, \vee, \odot, \rightarrow, \mathbf{0}, \mathbf{1})$ is called a bounded commutative residuated lattice (simply called *residuated lattice*, CRL) if

(1) $(X, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice;

(2) $(X, \odot, \mathbf{1})$ is a commutative monoid with unit element $\mathbf{1}$;

(3) For all $x, y, z \in X$, $x \odot y \leq z$ if and only if $x \leq y \rightarrow z$.

For all $x \in X$, by x', we mean $x' = x \rightarrow 0$, which is a *negation* in a sense.

Some well-known algebras, MTL-algebras, BL-algebras, MV-algebras, Heyting algebras and so on, are considered as algebraic semantics for so-called fuzzy logics, monoidal t-norm logic, Basic logic, many valued logic, intuitionistic logic and so on, respectively. Moreover, any residuated lattice satisfying the *divisibility* condition (div) is called a *Rℓ-monoid* ([4]):

(div) $x \wedge y = x \odot (x \rightarrow y)$.

These algebras are axiomatic extensions of residuated lattices as follows:

$$\begin{aligned} \text{R}\ell\text{-monoid} &= \text{CRL} + \big\{x \wedge y = x \odot (x \to y)\big\} \\ \text{MTL} &= \text{CRL} + \{(x \to y) \vee (y \to x) = \mathbf{1}\} \\ \text{BL} &= \text{CRL} + \{x \wedge y = x \odot (x \to y)\} + \{(x \to y) \vee (y \to x) = \mathbf{1}\} \\ &= \text{MTL} + \{x \wedge y = x \odot (x \to y)\} \\ \text{MV} &= \text{BL} + \{x'' = x\} \end{aligned}$$

As to basic properties of residuated lattices, we have the following ([3,4,8,10]).

Proposition 2.1. *Let X be a residuated lattice. For all $x, y, z \in X$, we have*

(1) $\mathbf{0}' = \mathbf{1}$, $\mathbf{1}' = \mathbf{0}$

(2) $x \odot x' = \mathbf{0}$

(3) $x \leq y \iff x \to y = \mathbf{1}$

(4) $x \odot (x \to y) \leq y$

(5) $x \leq y \implies x \odot z \leq y \odot z$, $z \to x \leq z \to y$, $y \to z \leq x \to z$

(6) $\mathbf{1} \to x = x$

(7) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$

(8) $(x \vee y)' = x' \wedge y'$

According to [4], we define states on residuated lattices. Let X be a residuated lattice. A map $s\colon X \to [0,1]$ is called a *state* on X if it satisfies

(S1) $s(x) + s(x \to y) = s(y) + s(y \to x)$

(S2) $s(\mathbf{0}) = 0$ and $s(\mathbf{1}) = 1$

The condition (S1) above has other equivalent notions.

Proposition 2.2. *For a map $s\colon X \to [0,1]$ with meeting* (S2) *above, the following conditions are equivalent:*

(S1) $s(x) + s(x \to y) = s(y) + s(y \to x)$ *for all $x, y \in X$*

(S1)' $1 + s(x \wedge y) = s(x \vee y) + s(d(x,y))$ *for all $x, y \in X$, where* $d(x,y) = (x \to y) \wedge (y \to x)$

(S1)" $1 + s(x \wedge y) = s(x) + s(x \to y)$ *for all $x, y \in X$*

Proof.

(S1) $\implies$ (S1)': If we replace x and y by $x \vee y$ and $x \wedge y$ in (S1), respectively, then we have $s(x \vee y) + s((x \vee y) \to (x \wedge y)) = s(x \wedge y) + s((x \wedge y) \to (x \vee y))$.

Since

$$\begin{aligned} x \vee y \to x \wedge y &= (x \to x \wedge y) \wedge (y \to x \wedge y) \\ &= (x \to x) \wedge (x \to y) \wedge (y \to x) \wedge (y \to y) \\ &= \mathbf{1} \wedge (x \to y) \wedge (y \to x) \wedge \mathbf{1} \\ &= (x \to y) \wedge (y \to x) \\ &= d(x, y) \end{aligned}$$

and $s((x \wedge y) \to (x \vee y)) = s(\mathbf{1}) = 1$, we have $s(x \vee y) + s(d(x, y)) = s(x \wedge y) + 1$.

(S1)' $\Longrightarrow$ (S1)": We set y by $x \wedge y$ in (S1)'. Then we have $1 + s(x \wedge (x \wedge y)) = s(x \vee (x \wedge y)) + s((x \to x \wedge y) \wedge ((x \wedge y \to y))$. Since $x \to x \wedge y = (x \to x) \wedge (x \to y) = \mathbf{1} \wedge (x \to y) = x \to y$, it follows that $1 + s(x \wedge y) = s(x) + s(x \to y)$.

(S1)" $\Longrightarrow$ (S1): If we exchange x and y in the condition (S1)", then we have $1 + s(y \wedge x) = s(y) + s(y \to x)$. Since $s(x \wedge y) = s(y \wedge x)$, we get $s(x) + s(x \to y) = s(y) + s(y \to x)$. □

The following results are proved in [4] under the condition that the support algebras are Rℓ-monoids. We can show the same results under the only conditions of residuated lattices.

Proposition 2.3. *Let s be a state on a residuated lattice X. Then for any $x, y \in X$ we have,*

(S3) $s(x') = 1 - s(x)$

(S4) $s(x'') = s(x)$

(S5) $x \leq y \implies 1 + s(x) = s(y) + s(y \to x)$

(S6) $x \leq y \implies s(x) \leq s(y)$

(S7) $s(x \odot y) = 1 - s(x \to y')$

(S8) $s(x) + s(y) = s(x \odot y) + s(y' \to x)$

(S9) $s(x' \to y') = 1 + s(x) - s(x \vee y)$

(S10) $s(x' \vee y') = 1 - s(x) - s(y) + s(x \vee y)$

(S11) $s(x'' \vee y'') = s(x \vee y)$

(S12) $s(x) + s(y) = s(x \wedge y) + s(x \vee y)$

(S13) $s(d(x, y)) = s(d(x'', y''))$

P r o o f. We only show the cases of (S9)–(S13), because other cases can be proved similarly as in [4].

(S9): $s(x' \to y') = 1 + s(x) - s(x \vee y)$: Since $s(x') + s(x' \to y') = 1 + s(x' \wedge y') = 1 + s((x \vee y)')$, we have $s(x' \to y') = 1 - s(x') + s((x \vee y)') = 1 - 1 + s(x) + 1 - s(x \vee y) = 1 + s(x) - s(x \vee y)$.

(S10): $s(x' \vee y') = 1 - s(x) - s(y) + s(x \vee y)$: Since $s(x' \vee y') + s(x' \vee y' \to y') = s(y') + s(y' \to x' \vee y') = s(y') + 1$, it follows $s(x' \vee y') + s(x' \to y') = 1 + s(y')$ and thus

$$\begin{aligned} s(x' \vee y') &= 1 + s(y') - s(x' \to y') \\ &= 1 + 1 - s(y) - (1 + s(x) - s(x \vee y)) \\ &= 1 - s(x) - s(y) + s(x \vee y). \end{aligned}$$

(S11): $s(x'' \vee y'') = s(x \vee y)$: Since $s((x'' \vee y'')') = s(x''' \wedge y''') = s(x' \wedge y') = s((x \vee y)')$, we get that $s(x'' \vee y'') = 1 - s((x'' \vee y'')') = 1 - s((x \vee y)') = s(x \vee y)$.

(S12): $s(x) + s(y) = s(x \wedge y) + s(x \vee y)$: From $x \leq x \vee y$, we have $1 + s(x) = s(x \vee y) + s(x \vee y \to x) = s(x \vee y) + s(y \to x)$. This implies that $1 + s(x) + s(y) = s(x \vee y) + s(y) + s(y \to x) = s(x \vee y) + 1 + s(x \wedge y)$ and hence that $s(x) + s(y) = s(x \wedge y) + s(x \vee y)$. We note that the condition can be proved without divisibility nor pre-linearity condition $(a \to b) \vee (b \to a) = \mathbf{1}$.

(S13): $s(d(x,y)) = s(d(x'', y''))$: It is easy to prove that $s(d(x,y)) \leq s(d(x', y'))$ by $x \to y \leq y' \to x'$. Then we have

$$\begin{aligned} s(d(x'', y'')) &= 1 + s(x'' \wedge y'') - s(x'' \vee y'') \\ &= 1 + s((x' \vee y')') - s(x'' \vee y'') \\ &= 1 + 1 - s(x' \vee y') - s(x \vee y) \\ &= 1 + 1 - (1 - s(x) - s(y) + s(x \vee y)) - s(x \vee y) \\ &= 1 + s(x) + s(y) - s(x \vee y) - s(x \vee y) \\ &= 1 + s(x \wedge y) + s(x \vee y) - s(x \vee y) - s(x \vee y) \\ &= 1 + s(x \wedge y) - s(x \vee y) \\ &= s(d(x,y)). \end{aligned}$$

□

We note that especially (S12) and (S13) above are proved in several papers ([4–6]) under the condition of divisibility. But our proof says that the condition is not necessary to prove them.

It follows from the results without assumption of (div) that an important property of states on residuated lattices can be proved.

Lemma 2.1. *Let s be a state on a residuated lattice X. Then for all $x, y \in X$, we have*

(S14) $1 + s(d(x,y)) = s(x \to y) + s(y \to x)$

(S15) $s((x \to y) \vee (y \to x)) = 1$

(S16) $s(d(x,y)) = s(d(x \to y, y \to x))$.

Proof.

(S14): Since $s(x)+s(x \to y) = 1+s(x \wedge y)$ and $s(y)+s(y \to x) = 1+s(y \wedge x)$, we have $s(x) + s(y) + s(x \to y) + s(y \to x) = 2 + s(x \wedge y) + s(x \vee y)$. It follows from (S12) that

$$\begin{aligned} s(x \to y) + s(y \to x) &= 2 + s(x \wedge y) + s(x \wedge y) - (s(x) + s(y)) \\ &= 2 + s(x \wedge y) + s(x \wedge y) - s(x \wedge y) - s(x \vee y) \\ &= 1 + s(x \wedge y) - s(x \vee y) + 1 \\ &= s(d(x,y)) + 1. \end{aligned}$$

(S15): It follows from (S12) and (S14) that

$$\begin{aligned} 1 + s(d(x,y)) &= s(x \to y) + s(y \to x) \\ &= s((x \to y) \wedge (y \to x)) + s((x \to y) \vee (y \to x)) \\ &= s(d(x,y)) + s((x \to y) \vee (y \to x)) \end{aligned}$$

and thus $s((x \to y) \vee (y \to x)) = 1$.

(S16): Since $s((x \to y) \vee (y \to x)) = 1$ and (S1)', we have $1 + s(d(x,y)) = 1 + s((x \to y) \wedge (y \to x)) = s((x \to y) \vee (y \to x)) + s(d(x \to y, y \to x)) = 1 + s(d(x \to y, y \to x))$. Thus we get $s(d(x,y)) = s(d(x \to y, y \to x))$. □

3. Filter

We define filters of residuated lattices. Let X be a residuated lattice. A non-empty subset $F \subseteq X$ is called a *filter* of X if

(F1) If $x, y \in F$ then $x \odot y \in F$;

(F2) If $x \in F$ and $x \leq y$ then $y \in F$.

It is easy to prove that, for a non-empty subset F of X, F is a filter if and only if it satisfies the condition

(DS) If $x \in F$ and $x \to y \in F$ then $y \in F$.

For every filter F, we define a relation $\equiv_F$ on X as follows:

$$x \equiv_F y \iff x \to y,\, y \to x \in F.$$

We see that if F is a filter then $\equiv_F$ is a congruence. In this case, we consider a quotient algebra $X/F = \{x/F \mid x \in X\}$ and we consistently define operators, for x/F and $y/F \in X/F$

$$\begin{aligned} x/F \wedge y/F &= (x \wedge y)/F \\ x/F \vee y/F &= (x \vee y)/F \\ x/F \to y/F &= (x \to y)/F \\ x/F \odot y/F &= (x \odot y)/F \\ \mathbf{0} &= \mathbf{0}/F \\ \mathbf{1} &= \mathbf{1}/F. \end{aligned}$$

It is trivial that $X/F = (x/F, \wedge, \vee, \odot, \to, \mathbf{0}, \mathbf{1})$ is also a residuated lattice.

For a non-empty subset $S \subseteq X$, by $[S)$ we mean the filter generated by S. We have a concrete notation about $[S)$. Since it is easy to prove the following, we omit its proof.

Proposition 3.1. *If F is a filter and $a \in X$, then we have*

$$[F \cup \{a\}) = \{x \mid \exists u \in F,\ \exists n \in \mathbb{N}:\ u \odot a^n \leq x\}.$$

Proposition 3.2. *For all $x, y \in X$ and $m, n \in N$, we have $(x \vee y)^{m+n} \leq x^m \vee y^n$.*

A proper filter P (i.e., $P \neq X$) is called *prime* if it satisfies $x \in P$ or $y \in P$ provided $x \vee y \in P$ for all $x, y \in X$. A filter H is called *maximal* if there is no proper filter containing H properly. It is easy to prove that, for a filter F, F is a maximal filter if and only if there exists $n \geq 1$ such that $(x^n)' \in F$ for $x \notin F$.

For a state s on X, we define

$$\ker(s) = \big\{x \in X \mid s(x) = 1\big\},$$

the *kernel* of s. It is obvious that $\ker(s)$ is a proper filter of X. Since $\ker(s)$ is the filter, we can consider the quotient residuated lattice $X/\ker(s)$ by the filter $\ker(s)$.

Lemma 3.1. *If s is a state on X, then the following conditions are equivalent:*

(i) $x/\ker(s) = y/\ker(s)$

(ii) $s(x) = s(y) = s(x \wedge y)$

(iii) $s(x \wedge y) = s(x \vee y)$

Proof.

(i) $\Longrightarrow$ (ii): Suppose that $x/\ker(s) = y/\ker(s)$. This means that $x \to y$, $y \to x \in \ker(s)$ and hence $s(x \to y) = s(y \to x) = 1$. It follows from $s(x) + s(x \to y) = s(y) + s(y \to x) = 1 + s(x \wedge y)$ that $s(x) = s(y) = s(x \wedge y)$.

(ii) $\Longrightarrow$ (iii): We assume that $s(x) = s(y) = s(x \wedge y)$. Since $s(x) + s(y) = s(x \wedge y) + s(x \vee y)$, we have $s(x) = s(x \vee y)$ and thus $s(x \wedge y) = s(x \vee y)$.

(iii) $\Longrightarrow$ (i): Assume $s(x \wedge y) = s(x \vee y)$. Since $x \wedge y \leq x, y \leq x \vee y$, it follows from assumption that $s(x \wedge y) = s(x) = s(y) = s(x \vee y)$. The fact that $s(x) + s(x \to y) = 1 + s(x \wedge y) = s(y) + s(y \to x)$ yields to $s(x \to y) = 1 = s(y \to x)$. This means that $x \to y, y \to x \in \ker(s)$ and thus $x/\ker(s) = y/\ker(s)$. □

Lemma 3.2. *If s is a state on X, then $s(x \wedge y) = s(x \odot (x \to y))$.*

Proof. Since $s(x) + s(x \to y) = s(x \to y) + s(x) = s((x \to y) \odot x) + s(x' \to (x \to y)) = s(x \odot (x \to y)) + s(1) = s(x \odot (x \to y)) + 1$ by (S8), we have $s(x \odot (x \to y)) = s(x) + s(x \to y) - 1 = s(x \wedge y) + 1 - 1 = s(x \wedge y)$. □

If s is a state on X, denote by $\hat{X} = \{\hat{x} := x/\ker(s) \mid x \in X\}$ the corresponding quotient residuated lattice. Let $\hat{s}$ be the map on $\hat{X}$ defined by $\hat{s}(\hat{x}) = s(x)$ $(x \in X)$.

Theorem 3.3. *Let s be a state on X, then we have*

(i) *$\hat{s}$ is a state on $\hat{X}$.*
(ii) *$\hat{X} = X/\ker(s)$ is an MV-algebra.*

Proof. We only show the case of (ii), because (i) is proved easily (c.f. [4]).

For all $x, y \in X$, since $s((x \to y) \vee (y \to x)) = 1$, we have $(x \to y) \vee (y \to x) \in \ker(s)$. This implies $(\hat{x} \to \hat{y}) \vee (\hat{y} \to \hat{x}) = (x/\ker(s) \to y/\ker(s)) \vee (y/\ker(s) \to x/\ker(s)) = ((x \to y) \vee (y \to x))/\ker(s) = \mathbf{1}/\ker(s) = \hat{\mathbf{1}}$, that is, $(\hat{x} \to \hat{y}) \vee (\hat{y} \to \hat{x}) = \hat{\mathbf{1}}$. It follows from the lemma above that $s(x \odot (x \to y)) = s(x \wedge y) = s((x \odot (x \to y)) \wedge (x \wedge y))$ and hence that $x/\ker(s) \odot (x/\ker(s) \to y/\ker(s)) = x/\ker(s) \odot (x \to y)/\ker(s) = (x \odot (x \to y))/\ker(s) = (x \wedge y)/\ker(s) = x/\ker(s) \wedge y/\ker(s)$. This means that $\hat{x} \wedge \hat{y} = \hat{x} \odot (\hat{x} \to \hat{y})$, that is, $\hat{X} = X/\ker(s)$ satisfies the divisibility. Moreover, it follows from $s(x'') = s(x)$ and $x \leq x''$ that $s(x) = s(x \wedge x'') = s(x \vee x'')$ hence that $(\hat{x})'' = (x/\ker(s))'' = x''/\ker(s) = x/\ker(s) = \hat{x}$. Therefore $\hat{X} = X/\ker(s)$ is an MV-algebra. □

4. State-morphism

In this section, we define a state-morphism on a residuated lattice according to [4]. We consider the closed interval $[0,1]$ as the standard MV-algebra, where operations $\wedge_L$, $\vee_L$, $\odot_L$, $\to_L$ are defined by, for all $a,b \in [0,1]$

$$\begin{aligned} a \wedge_L b &= \min\{a,b\} \\ a \vee_L b &= \max\{a,b\} \\ a \odot_L b &= \max\{a+b-1,0\} \\ a \to_L b &= \min\{1-a+b,1\}. \end{aligned}$$

A map s from a residuated lattice X to the standard MV-algebra $[0,1]$ is said to be a *state-morphism* if, for all $x,y \in X$,

(sm1) $s(x \to y) = s(x) \to_L s(y)$

(sm2) $s(x \wedge y) = s(x) \wedge_L s(y)$

(sm3) $s(\mathbf{0}) = 0$ and $s(\mathbf{1}) = 1$.

A state s on X is said to be *extremal* if the equality $s = \lambda s_1 + (1-\lambda)s_2$ for some $\lambda \in (0,1)$ and states s_1, s_2 implies $s_1 = s_2$.

Proposition 4.1. *Every state-morphism s on a residuated lattice X is a state on X and $s(x \odot y) = s(x) \odot_L s(y)$.*

Proof. Let s be a state-morphism on X. We only show the case of (sm1). The other case (sm2) can be proved similarly. There are two cases to be considered

(i) $s(x) \le s(y)$ and

(ii) $s(x) > s(y)$.

In the first case (i) $s(x) \le s(y)$, since $s(x) \to_L s(y) = 1$ and $s(y) \to_L s(x) = 1 - s(y) + s(x)$, we have $s(x) + s(x \to y) = s(x) + 1$ and $s(y) + s(y \to x) = s(y) + 1 - s(y) + s(x) = 1 + s(x)$. Hence $s(x) + s(x \to y) = s(y) + s(y \to x)$. The same equation also holds in the case (ii). Thus, every state-morphism is the state.

Concerning to $s(x \odot y) = s(x) \odot_L s(y)$, we have

$$\begin{aligned} s(x \odot y) &= 1 - s(x \to y') = 1 - (s(x) \to_L s(y')) \\ &= 1 - \min\{1 - s(x) + s(y'), 1\} \\ &= \max\{1 - (1 - s(x) + s(y')), 0\} \\ &= \max\{s(x) - s(y'), 0\} = \max\{s(x) - 1 + s(y), 0\} \\ &= s(x) \odot_L s(y). \end{aligned}$$

□

Proposition 4.2. *A state s on X is a state-morphism if and only if* $\ker(s)$ *is a maximal filter.*

Proof. If $x \notin \ker(s)$, since $s(x) < 1$, then there exists $n \geq 1$ such that $(s(x))^n = 0$ by definition of $\odot$ in $[0, 1]$. It follows from the above that $s(x^n) = 0$. Since $s((x^n)') = 1 - s(x^n) = 1 - 0 = 1$, we have $(x^n)' \in \ker(s)$. This means that $\ker(s)$ is the maximal filter.

Conversely, suppose that $\ker(s)$ is a maximal filter. Since $\ker(s)$ is also a prime filter and $s((x \to y) \vee (y \to x)) = 1$, we have $x \to y \in \ker(s)$ or $y \to x \in \ker(s)$. In the case of $x \to y \in \ker(s)$, we note that $s(x) \leq s(y)$. Because, since $s(x) + s(x \to y) = s(y) + s(y \to x)$, we have $s(y) - s(x) = 1 - s(y \to x) \geq 0$. It follows from $s(x)+s(x \to y) = 1+s(x\wedge y)$ that $s(x) = s(x\wedge y)$ and $s(x\wedge y) \leq s(y)$, that is, $s(x \wedge y) = s(x) = \min\{s(x), s(y)\}$. The other case of $y \to x \in \ker(s)$ can be proved similarly. That is, $s(x \wedge y) = \min\{s(x), s(y)\}$.

It follows from this result that

$$\begin{aligned} s(x \to y) &= 1 + s(x \wedge y) - s(x) \\ &= 1 + \min\{s(x), s(y)\} - s(x) \\ &= \min\{1 + s(x) - s(x), 1 + s(y) - s(x)\} \\ &= \min\{1, 1 - s(x) + s(y)\} \\ &= s(x) \to_L s(y). \end{aligned}$$

These mean that s is the state-morphism on X. □

A residuated lattice X is called *locally finite* ([11]) if, for any $x \in X$ $(x \neq \mathbf{1})$ there exists $n \geq 1$ such that $x^n = \mathbf{0}$. By the same proof in [11], it is clear that:

Theorem 4.3. *For a filter H of a residuated lattice X, H is a maximal filter if and only if X/H is a locally finite residuated lattice.*

Further, if s is a state-morphism, since s is a state and $\ker(s)$ is the maximal filter, the quotient structure $X/\ker(s)$ is also a locally finite MV-algebra. Moreover, we have the following result.

Lemma 4.1. *If s is a state-morphism then $X/\ker(s)$ is linearly ordered.*

Proof. Since $s((x \to y) \vee (y \to x)) = 1$ for all $x, y \in X$, we have $x/\ker(s) \to y/\ker(s) = \mathbf{1}/\ker(s)$ or $y/\ker(s) \to x/\ker(s) = \mathbf{1}/\ker(s)$ and this means that $X/\ker(s)$ is linearly ordered. □

Lemma 4.2. *If s is a state-morphism, then $x/\ker(s) = y/\ker(s)$ if and only if $s(x) = s(y)$.*

Proof. Suppose that $x/\ker(s) = y/\ker(s)$. We have $s(x \to y) = s(y \to x) = 1$ from $x \to y$, $y \to x \in \ker(s)$. Since $s(x) + s(x \to y) = s(y) + s(y \to x)$, we get that $s(x) = s(y)$.

Conversely, if $s(x) = s(y)$, since s is the state-morphism, then we have $s(x \to y) = s(x) \to_L s(y) = 1 = s(y) \to_L s(x) = s(y \to x)$. This implies $x \to y, y \to x \in \ker(s)$ and thus $x/\ker(s) = y/\ker(s)$. □

It follows from the above we have:

Theorem 4.4. *If s is a state-morphism on a residuated lattice X, then $X/\ker(s)$ is a linearly ordered locally finite MV-algebra.*

It is proved in [1] that, for subalgebras X_1 and X_2 of the MV-algebra $[0,1]$, if X_1 and X_2 are isomorphic then $X_1 = X_2$. Moreover, it is also proved for MV-algebra X, X is simple if and only if it is a subalgebra of the standard MV-algebra $[0,1]$. It follows from the fact that a state on an MV-algebra is extremal if and only if it is a state-morphism ([5]), we can show the following result on bounded commutative residuated lattices with the similar proof in [4: Proposition 4.10], which can be proved for bounded commutative Rℓ-monoids.

Lemma 4.3. *Let S be a state on a bounded commutative residuated lattice X. The following statements are equivalent:*

(i) *s is an extremal state on X.*

(ii) *s is a state-morphism on X.*

It follows from the above that we have a characterization theorem about state-morphisms.

Theorem 4.5. *For a state s on a bounded commutative residuated lattice X, the following statements are equivalent:*

(i) *s is a state-morphism on X.*

(ii) $\ker(s)$ *is a maximal filter of X.*

(iii) *s is extremal on X.*

Acknowledgement. The author is indebted to the referee for his valuable suggestions which improved the result Theorem 4.5.

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Received 22. 5. 2012
Accepted 11. 7. 2012

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