SOME CLASS OF ANALYTIC FUNCTIONS RELATED TO CONIC DOMAINS

STANISŁAWA KANAS* — DORINA RĂDUCANU**

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ABSTRACT. For $q \in (0, 1)$ let the q -difference operator be defined as follows

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q - 1)} \quad (z \in \mathbb{U}),$$

where $\mathbb{U}$ denotes the open unit disk in a complex plane. Making use of the above operator the extended Ruscheweyh differential operator $R_q^\lambda f$ is defined. Applying $R_q^\lambda f$ a subfamily of analytic functions is defined. Several interesting properties of a defined family of functions are investigated.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions f on the open unit disk $\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1.1)$$

and let $\mathcal{S}$ be the class of univalent functions in $\mathcal{A}$.

Goodman [5] introduced the class $\mathcal{UCV}$ of uniformly convex functions. Here, a function $f \in \mathcal{A}$ is called *uniformly convex* if every (positively oriented) circular arc of the form $\{z \in \mathbb{U} : |z - \zeta| = r\}$ with $\zeta \in \mathbb{U}$ and $0 < r < |\zeta| + 1$, is mapped by f univalently onto a convex arc. In particular, $\mathcal{UCV}$ is a subset of univalent,

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convex functions. An analytic characterization for the members of $\mathcal{UCV}$ was obtained by Ma and Minda [13] and Rønning [19].

THEOREM 1.1. ([13], [19]) *Let $f \in \mathcal{A}$. Then $f \in \mathcal{UCV}$ if and only if*

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}). \quad (1.2)$$

Applying the well known Alexander relation Rønning in [18], [19] considered the class, denoted $\mathcal{S}_p$, consisting of functions $f \in \mathcal{A}$ satisfying the inequality

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{U}). \quad (1.3)$$

Discarding an assertion that ζ have to lie in the unit disk, and that a consideration concerns whole circular arcs with center at ζ , the first author and Wiśniowska in [7] introduced a family of k -uniformly convex functions, denoted $k\text{-}\mathcal{UCV}$. Let $0 \leq k < \infty$. A function $f \in \mathcal{A}$ is said to be k -uniformly convex in $\mathbb{U}$, if the image of every (positively oriented) circular arc of the form $\{z \in \mathbb{U} : |z - \zeta| = r\}$ with $\zeta \in \mathbb{C}$ and $|\zeta| \leq k$, is mapped by f univalently onto a convex arc. We note that $1\text{-}\mathcal{UCV} = \mathcal{UCV}$. An analytic characterization for the members of $k\text{-}\mathcal{UCV}$ was given in [7] (see also [8], [9], [6], [10] and [11] for related results).

THEOREM 1.2. ([7]) *Let $f \in \mathcal{A}$ and let $0 \leq k < \infty$. Then $f \in k\text{-}\mathcal{UCV}$ if and only if*

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > k \left| \frac{zf''(z)}{f'(z)} \right| \quad (z \in \mathbb{U}). \quad (1.4)$$

In [8] a class of k -starlike functions, denoted $k\text{-}\mathcal{ST}$, and related with $k\text{-}\mathcal{UCV}$ by the Alexander relation was considered. Such a class consists of functions $f \in \mathcal{A}$ which satisfy the inequality

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{U}). \quad (1.5)$$

We note that $1\text{-}\mathcal{ST} = \mathcal{ST}$. The classes $k\text{-}\mathcal{UCV}$ and $k\text{-}\mathcal{ST}$ have been generalized as follows (see [4], [16]).

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{ST}(k, \alpha)$ of k -starlike functions of order α , $0 \leq \alpha < 1$, if

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > k \left| \frac{zf'(z)}{f(z)} - 1 \right| + \alpha \quad (k \geq 0, \quad 0 \leq \alpha < 1). \quad (1.6)$$

Replacing f in (1.6) by $zf'(z)$, we obtain the inequality

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > k \left| \frac{zf''(z)}{f'(z)} \right| + \alpha. \quad (1.7)$$

Following the original definition the functions satisfying (1.7) are called k -uniformly convex of order α and denoted $\mathcal{UCV}(k, \alpha)$.

We note that $\mathcal{UCV}(k, 0) = k\text{-}\mathcal{UCV}$ and $\mathcal{ST}(k, 0) = k\text{-}\mathcal{ST}$. The classes $\mathcal{UCV}(1, \alpha)$ and $\mathcal{ST}(1, \alpha)$ were investigated in [1], [2] and [18].

The aim of this work is to adopt a notion of uniform convexity onto some classes defined by a difference operator. To do this, we first introduce a notion of q -difference operator related to the q -calculus (see, e.g., [3: Ch. 10]). Let $q > 0$. For any non-negative integer n , the q -integer number n $[n]$ is defined by:

$$[n] = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}, \quad [0] = 0. \quad (1.8)$$

The q -number shifted factorial is defined by $[0]! = 1$ and $[n]! = [1][2] \cdots [n]$. Clearly, $\lim_{q \rightarrow 1^-} [n] = n$ and $\lim_{q \rightarrow 1^-} [n]! = n!$. In general we will denote $[t] = \frac{1 - q^t}{1 - q}$ also for a non-integer number. Throughout this paper we will assume q to be a fixed number between 0 and 1.

DEFINITION 1.1. Let $f \in \mathcal{A}$, and let the q -derivative operator or q -difference operator be defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q - 1)} \quad (z \in \mathbb{U}). \quad (1.9)$$

It is easy to check that for $n \in \mathbb{N} := \{1, 2, \dots\}$ and $z \in \mathbb{U}$

$$\partial_q z^n = [n]z^{n-1}. \quad (1.10)$$

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Let the q -generalized Pochhammer symbol be defined as

$$[t]_n = [t][t+1][t+2] \cdots [t+n-1],$$

and for $t > 0$ let the q -gamma function be defined by

$$\Gamma_q(t+1) = [t]\Gamma_q(t) \quad \text{and} \quad \Gamma_q(1) = 1. \quad (1.11)$$

DEFINITION 1.2. For $f \in \mathcal{A}$ let the Ruscheweyh q -differential operator be defined as follows

$$R_q^\lambda f(z) = f(z) * F_{q, \lambda+1}(z) \quad (z \in \mathbb{U}, \lambda > -1) \quad (1.12)$$

where

$$F_{q, \lambda+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{n-1}}{[n-1]!} z^n \quad (z \in \mathbb{U}). \quad (1.13)$$

The symbol “ $*$ ” stands for Hadamard product (or convolution).

From (1.12) we obtain that

$$R_q^0 f(z) = f(z), \quad R_q^1 f(z) = z \partial_q f(z)$$

and

$$R_q^m f(z) = \frac{z \partial_q^m (z^{m-1} f(z))}{[m]!} \quad (m \in \mathbb{N}).$$

Making use of (1.12) and (1.13), the power series of $R_q^\lambda f(z)$ for f of the form (1.1) is given by

$$R_q^\lambda f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]! \Gamma_q(1+\lambda)} a_n z^n = z + \sum_{n=2}^{\infty} \frac{[\lambda+1]_{n-1}}{[n-1]!} a_n z^n \quad (z \in \mathbb{U}). \quad (1.14)$$

Note that

$$\lim_{q \rightarrow 1^-} F_{q, \lambda+1}(z) = \frac{z}{(1-z)^{\lambda+1}}$$

and

$$\lim_{q \rightarrow 1^-} R_q^\lambda f(z) = f(z) * \frac{z}{(1-z)^{\lambda+1}}.$$

Thus, we can say that Ruscheweyh q -differential operator reduces to the differential operator defined by Ruscheweyh [20] in the case when $q \rightarrow 1^-$.

It is easy to check that

$$z \partial_q (F_{q, \lambda+1}(z)) = \left(1 + \frac{[\lambda]}{q^\lambda}\right) F_{q, \lambda+2}(z) - \frac{[\lambda]}{q^\lambda} F_{q, \lambda+1}(z) \quad (z \in \mathbb{U}). \quad (1.15)$$

Making use of (1.12), (1.15) and the properties of Hadamard product, we obtain the following equality

$$z \partial_q (R_q^\lambda f(z)) = \left(1 + \frac{[\lambda]}{q^\lambda}\right) R_q^{\lambda+1} f(z) - \frac{[\lambda]}{q^\lambda} R_q^\lambda f(z) \quad (z \in \mathbb{U}). \quad (1.16)$$

If $q \rightarrow 1^-$, the equality (1.16) implies

$$z(R^\lambda f(z))' = (1+\lambda)R^{\lambda+1}f(z) - \lambda R^\lambda f(z) \quad (z \in \mathbb{U})$$

which is the well known recurrent formula for Ruscheweyh differential operator.

Using Ruscheweyh differential operator various new classes of convex and star-like functions have been defined. Therefore it seems natural to use Ruscheweyh q -differential operator to introduce the following class of functions.

DEFINITION 1.3. Let $0 \leq \alpha < 1$, $k \geq 0$ and $\lambda > -1$. A function $f \in \mathcal{A}$ is in the class $\mathcal{ST}(k, \alpha, \lambda, q)$ if it satisfies the condition

$$\operatorname{Re} \left\{ \frac{z \partial_q (R_q^\lambda f(z))}{R_q^\lambda f(z)} \right\} > k \left| \frac{z \partial_q (R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1 \right| + \alpha \quad (z \in \mathbb{U}). \quad (1.17)$$

Note that if $\lambda = 0$ and $q \rightarrow 1^-$ the class $\mathcal{ST}(k, \alpha, \lambda, q)$ reduces to the class $\mathcal{ST}(k, \alpha)$.

In a present work we study several properties of the family $\mathcal{ST}(k, \alpha, \lambda, q)$, e.g. necessary and sufficient conditions to be a member of $\mathcal{ST}(k, \alpha, \lambda, q)$, coefficients bounds and Fekete-Szegő problem.

2. Properties of the class $\mathcal{ST}(k, \alpha, \lambda, q)$

We begin this section with a sufficient condition for a function f to be in the class $\mathcal{ST}(k, \alpha, \lambda, q)$.

THEOREM 2.1. *Let $f \in \mathcal{A}$ be given by (1.1). If the inequality*

$$\sum_{n=2}^{\infty} ([n](k+1) - k - \alpha) \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} |a_n| \leq 1 - \alpha \quad (2.1)$$

holds true for some k ($0 \leq k < \infty$), $\lambda > -1$ and α ($0 \leq \alpha < 1$), then $f \in \mathcal{ST}(k, \alpha, \lambda, q)$. The result is sharp for the function

$$f_n(z) = z - \frac{(1-\alpha)[n-1]!\Gamma_q(1+\lambda)}{([n](k+1) - k - \alpha)\Gamma_q(n+\lambda)} z^n.$$

Proof. Making use of the Definition 1.3 it suffices to prove that

$$k \left| \frac{z \partial_q (R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1 \right| - \operatorname{Re} \left\{ \frac{z \partial_q (R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1 \right\} < 1 - \alpha.$$

Observe, that

$$\begin{aligned} & k \left| \frac{z \partial_q (R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1 \right| - \operatorname{Re} \left\{ \frac{z \partial_q (R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1 \right\} \\ & \leq (k+1) \left| \frac{z \partial_q (R_q^\lambda f(z))}{R_q^\lambda f(z)} - 1 \right| \\ & = (k+1) \left| \frac{\sum_{n=2}^{\infty} ([n]-1) \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} a_n z^{n-1}}{1 + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} a_n z^{n-1}} \right| \\ & < (k+1) \frac{\sum_{n=2}^{\infty} ([n]-1) \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} |a_n|}{1 - \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\lambda)}{[n-1]!\Gamma_q(1+\lambda)} |a_n|}. \end{aligned}$$

The last expression is bounded by $1 - \alpha$ if inequality (2.1) holds.

It is obvious that the function f_n satisfies inequality (2.1) thus the number $1 - \alpha$ can not be replaced by a larger number. Therefore, we only need to show that $f_n \in \mathcal{ST}(k, \alpha, \lambda, q)$. Since

$$\begin{aligned} k \left| \frac{z \partial_q (R_q^\lambda f_n(z))}{R_q^\lambda f_n(z)} - 1 \right| &= k \left| \frac{(1 - \alpha)(1 - [n])z^{n-1}}{([n](k + 1) - k - \alpha) - (1 - \alpha)z^{n-1}} \right| \\ &< \frac{k(1 - \alpha)}{k + 1}, \end{aligned}$$

and

$$\operatorname{Re} \left\{ \frac{z \partial_q (R_q^\lambda f_n(z))}{R_q^\lambda f_n(z)} \right\} = \operatorname{Re} \left\{ \frac{[n](k + 1) - k - \alpha - [n](1 - \alpha)z^{n-1}}{[n](k + 1) - k - \alpha - (1 - \alpha)z^{n-1}} \right\} > \frac{k + \alpha}{k + 1},$$

the condition (1.17) holds true for $f_n(z)$. Thus, $f_n \in \mathcal{ST}(k, \alpha, \lambda, q)$. $\square$

The next Corollary can be easily obtained from Theorem 2.1.

COROLLARY 2.1. *Let $f(z) = z + a_n z^n$. If*

$$|a_n| \leq \frac{(1 - \alpha)[n - 1]! \Gamma_q(1 + \lambda)}{([n](k + 1) - k - \alpha) \Gamma_q(n + \lambda)} \quad (n \geq 2),$$

then $f \in \mathcal{ST}(k, \alpha, \lambda, q)$.

Consider $p(z) = z \partial_q (R_q^\lambda f(z)) / R_q^\lambda f(z)$. We can rewrite the condition (1.17) into the form

$$\operatorname{Re} p(z) > k|p(z) - 1| + \alpha \quad (z \in \mathbb{U}). \quad (2.2)$$

It follows that the range of the expression $p(z)$ ($z \in \mathbb{U}$) is a conic domain

$$\Omega_{k, \alpha} = \{w \in \mathbb{C} : \operatorname{Re} w > k|w - 1| + \alpha\}, \quad (2.3)$$

or

$$\Omega_{k, \alpha} = \left\{ w = u + iv : u > k\sqrt{(u - 1)^2 + v^2} + \alpha \right\}, \quad (2.4)$$

where $0 \leq k < \infty$ and $0 \leq \alpha < 1$.

Note that $\Omega_{k, \alpha}$ is such that $1 \in \Omega_{k, \alpha}$ and $\partial\Omega_{k, \alpha}$ is a curve defined by

$$\partial\Omega_{k, \alpha} = \{w = u + iv : (u - \alpha)^2 = k^2(u - 1)^2 + k^2v^2\}. \quad (2.5)$$

Elementary computations show that $\partial\Omega_{k, \alpha}$ represents a conic section symmetric about the real axis. It follows that the domain $\Omega_{k, \alpha}$ is bounded by an ellipse for $k > 1$, by a parabola for $k = 1$ and by a hyperbola if $0 < k < 1$. Finally, for $k = 0$, $\Omega_{k, \alpha}$ is the right half plane $\operatorname{Re} w > \alpha$.

From (1.17) we obtain that $f \in \mathcal{ST}(k, \alpha, \lambda, q)$ if and only if

$$\frac{z \partial_q (R_q^\lambda f(z))}{R_q^\lambda f(z)} \in \Omega_{k, \alpha}. \quad (2.6)$$

Making use of the properties of the domain $\Omega_{k,\alpha}$ and (2.6) it follows that if $f \in \mathcal{ST}(k, \alpha, \lambda, q)$, then

$$\operatorname{Re} \frac{z \partial_q (R_q^\lambda f(z))}{R_q^\lambda f(z)} > \frac{k + \alpha}{k + 1} \quad (z \in \mathbb{U})$$

and

$$\left| \operatorname{Arg} \frac{z \partial_q (R_q^\lambda f(z))}{R_q^\lambda f(z)} \right| \leq \begin{cases} \arctan \frac{1-\alpha}{\sqrt{|k^2-\alpha^2|}} & \text{if } 0 \leq \alpha < 1, k > 0, \\ \frac{\pi}{2} & \text{if } k = 0. \end{cases}$$

Denote by $\mathcal{P}$ the class of analytic and normalized Carathéodory functions and by $p_{k,\alpha} \in \mathcal{P}$ the function such that $p_{k,\alpha}(\mathbb{U}) = \Omega_{k,\alpha}$. Following Ma and Minda notation [15] let $\mathcal{P}(p_{k,\alpha})$, where $0 \leq k < \infty$ and $0 \leq \alpha < 1$, denotes the following class of functions

$$\mathcal{P}(p_{k,\alpha}) = \{p \in \mathcal{P} : p(\mathbb{U}) \subset \Omega_{k,\alpha}\} = \{p \in \mathcal{P} : p \prec p_{k,\alpha} \text{ in } \mathbb{U}\}.$$

The functions which play the role of extremal functions for the class $\mathcal{P}(p_{k,\alpha})$, may be obtained by a simple modification of related functions described in [7] (see also [11]), and are defined by

$$p_{k,\alpha}(z) = \begin{cases} \frac{1+(1-2\alpha)z}{1-z} & \text{if } k = 0, \\ 1 + \frac{2(1-\alpha)}{\pi^2} \left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right)^2 & \text{if } k = 1, \\ \frac{1-\alpha}{1-k^2} \cos \left(A(k) i \log \frac{1+\sqrt{z}}{1-\sqrt{z}} \right) - \frac{k^2-\alpha}{1-k^2} & \text{if } 0 < k < 1, \\ \frac{1-\alpha}{k^2-1} \sin \left(\frac{\pi}{2K(t)} \int_0^{\frac{u(z)}{\sqrt{t}}} \frac{dx}{\sqrt{1-x^2} \sqrt{1-t^2 x^2}} \right) + \frac{k^2-\alpha}{k^2-1} & \text{if } k > 1. \end{cases} \quad (2.7)$$

with $A(k) = \frac{2}{\pi} \arccos k$,

$$u(z) = \frac{z - \sqrt{t}}{1 - \sqrt{t}z} \quad (0 < t < 1, z \in \mathbb{U}),$$

where t is chosen such that

$$k = \cosh \frac{\pi K'(t)}{4K(t)},$$

and $K(t)$ is Legendre's complete elliptic integral of the first kind and $K'(t)$ is complementary integral of $K(t)$.

Obviously, if $k = 0$

$$p_{0,\alpha}(z) = 1 + 2(1-\alpha)z + 2(1-\alpha)z^2 + \dots$$

For $k = 1$, we have (see [13] and also [19])

$$p_{1,\alpha}(z) = 1 + \frac{8}{\pi^2}(1-\alpha)z + \frac{16}{3\pi^2}(1-\alpha)z^2 + \dots \quad (2.8)$$

Using Taylor expansion in [7] and [8], for $0 < k < 1$, we have

$$p_{k,\alpha}(z) = 1 + \frac{1-\alpha}{1-k^2} \sum_{n=1}^{\infty} \left[\sum_{l=1}^{2n} 2^l \binom{A(k)}{l} \binom{2n-1}{2n-l} \right] z^n. \quad (2.9)$$

Finally, when $k > 1$

$$p_{k,\alpha}(z) = 1 + \frac{\pi^2(1-\alpha)}{4\sqrt{t}(k^2-1)K^2(t)(1+t)} \times \left\{ z + \frac{4K^2(t)(t^2+6t+1) - \pi^2}{24\sqrt{t}K^2(t)(1+t)} z^2 + \dots \right\}, \quad (2.10)$$

so that, denoting $p_{k,\alpha}(z) = 1 + P_1 z + P_2 z^2 + \dots$ ($P_j = P_j(k, \alpha)$, $j = 1, 2, \dots$), we get

$$P_1 = \begin{cases} \frac{8(1-\alpha)(\arccos k)^2}{\pi^2(1-k^2)} & \text{if } 0 \leq k < 1, \\ \frac{8(1-\alpha)}{\pi^2} & \text{if } k = 1, \\ \frac{\pi^2(1-\alpha)}{4\sqrt{t}(1+t)K^2(t)(k^2-1)} & \text{if } k > 1. \end{cases} \quad (2.11)$$

Let $f_{k,\alpha}(z) = z + A_2 z^2 + A_3 z^3 + \dots$ be the extremal function in the class $\mathcal{ST}(k, \alpha, \lambda, q)$. Then, the relation between the extremal functions in the classes $\mathcal{P}(p_{k,\alpha})$ and $\mathcal{ST}(k, \alpha, \lambda, q)$ is given by

$$p_{k,\alpha}(z) = \frac{z \partial_q (R_q^\lambda f_{k,\alpha}(z))}{R_q^\lambda f_{k,\alpha}(z)} \quad (z \in \mathbb{U}). \quad (2.12)$$

Making use of (1.14), (1.17) and (2.12) we obtain for $p_{k,\alpha}(z)$ the following coefficient relation

$$\frac{q\Gamma_q(n+\lambda)}{[n-2]!\Gamma_q(1+\lambda)} A_n = \sum_{m=1}^{n-1} \frac{\Gamma_q(m+\lambda)}{[m-1]!\Gamma_q(1+\lambda)} A_m P_{n-m}, \quad A_1 = 1. \quad (2.13)$$

In particular, by a straightforward computation, we get

$$A_2 = \frac{P_1}{q[1+\lambda]}, \quad (2.14)$$

$$A_3 = \frac{qP_2 + P_1^2}{q^2[1+\lambda][2+\lambda]}. \quad (2.15)$$

Since $\lambda > -1$, $q \in (0, 1)$ and P_n are nonnegative, it follows that A_n are nonnegative.

THEOREM 2.2. *Let $k \in [0, \infty)$, $\lambda > -1$, $q \in (0, 1)$ and $\alpha \in [0, 1)$. If f of the form (1.1) belongs to the class $\mathcal{ST}(k, \alpha, \lambda, q)$, then*

$$|a_2| \leq A_2 \quad \text{and} \quad |a_3| \leq A_3. \quad (2.16)$$

P r o o f. Let $p(z) = z\partial_q(R_q^\lambda f(z))/R_q^\lambda f(z)$. Using the relation (1.14) for $p(z) = 1 + p_1z + p_2z^2 + \dots$, we have

$$\frac{\Gamma_q(n+\lambda)}{[n-2]!\Gamma_q(1+\lambda)}qa_n = \sum_{m=1}^{n-1} \frac{\Gamma_q(m+\lambda)}{[m-1]!\Gamma_q(1+\lambda)}a_m p_{n-m}, \quad a_1 = 1. \quad (2.17)$$

Since $p_{k,\alpha}$ is univalent in $\mathbb{U}$, the function

$$q(z) = \frac{1 + p_{k,\alpha}^{-1}(p(z))}{1 - p_{k,\alpha}^{-1}(p(z))} = 1 + c_1z + c_2z^2 + \dots,$$

is analytic in $\mathbb{U}$ and $\operatorname{Re} q(z) > 0$. From

$$p(z) = p_{k,\alpha} \left(\frac{q(z) - 1}{q(z) + 1} \right) = 1 + \frac{1}{2}c_1P_1z + \left(\frac{1}{2}c_2P_1 + \frac{1}{4}c_1^2(P_2 - P_1) \right) z^2 + \dots,$$

we have

$$|a_2| = \frac{1}{q[1+\lambda]}|p_1| = \frac{1}{2q[1+\lambda]}|c_1P_1| \leq \frac{P_1}{q[1+\lambda]} = A_2, \quad (2.18)$$

where we used the inequality $|c_n| \leq 2$ and (2.13). In view of a relation $|p_1|^2 + |p_2| \leq P_1^2 + P_2$ (cf. [8]) and (2.14), we obtain

$$\begin{aligned} |a_3| &= \frac{|qp_2 + p_1^2|}{q^2[1+\lambda][2+\lambda]} \leq \frac{q(|p_2| + |p_1|^2) + (1-q)|p_1|^2}{q^2[1+\lambda][2+\lambda]} \\ &\leq \frac{q(P_2 + P_1^2) + (1-q)P_1^2}{q^2[1+\lambda][2+\lambda]} \leq \frac{qP_2 + P_1^2}{q^2[1+\lambda][2+\lambda]} = A_3. \end{aligned} \quad (2.19)$$

Thus, the proof of the theorem is completed. $\square$

THEOREM 2.3. *Let $0 \leq k < \infty$, $\lambda > -1$, $q \in (0, 1)$ and $0 \leq \alpha < 1$. If f of the form (1.1) is in the class $\mathcal{ST}(k, \alpha, \lambda, q)$, then*

$$|a_n| \leq \frac{P_1(P_1 + q)(P_1 + [2]q) \cdots (P_1 + [n-2]q)}{q^{n-1}[1+\lambda]_{n-1}}, \quad n \geq 2. \quad (2.20)$$

P r o o f. The result is clearly true for $n = 2$. Let n be an integer number with $n \geq 2$, and assume that the inequality is true for all $m \leq n-1$. Making use of (2.13), we have

$$\begin{aligned} |a_n| &= \left| \frac{[n-2]!\Gamma_q(1+\lambda)}{q\Gamma_q(n+\lambda)} \left(p_{n-1} + \sum_{m=2}^{n-1} \frac{\Gamma_q(m+\lambda)}{[m-1]!\Gamma_q(1+\lambda)}a_m p_{n-m} \right) \right| \\ &\leq \frac{[n-2]!\Gamma_q(1+\lambda)}{q\Gamma_q(n+\lambda)} \left\{ P_1 + \sum_{m=2}^{n-1} \frac{\Gamma_q(m+\lambda)}{[m-1]!\Gamma_q(1+\lambda)}|a_m|P_1 \right\} \end{aligned}$$

$$\leq \frac{[n-2]!\Gamma_q(1+\lambda)}{q\Gamma_q(n+\lambda)}P_1\left\{1+\sum_{m=2}^{n-1}\frac{\Gamma_q(m+\lambda)}{[m-1]!\Gamma_q(1+\lambda)}\right. \\ \left.\times\frac{P_1(P_1+q)\cdots(P_1+[m-2]q)}{q^{m-1}[1+\lambda]_{m-1}}\right\},$$

where we applied the induction hypothesis to $|a_m|$ and the Rogosinski result ([17]) $|p_n| \leq P_1$. Since

$$\frac{\Gamma_q(m+\lambda)}{\Gamma_q(1+\lambda)} = [1+\lambda]_{m-1}$$

we have

$$|a_n| \leq \frac{[n-2]!P_1}{q([1+\lambda])_{n-1}}\left\{1+\sum_{m=2}^{n-1}\frac{P_1(P_1+q)\cdots(P_1+[m-2]q)}{q^{m-1}[m-1]!}\right\}.$$

Applying again mathematical induction, we find that

$$1+\sum_{m=2}^{n-1}\frac{P_1(P_1+q)\cdots(P_1+[m-2]q)}{q^{m-1}[m-1]!} = \frac{(P_1+q)\cdots(P_1+[n-2]q)}{q^{n-2}[n-2]!}.$$

Consequently, the inequality (2.20) follows. $\square$

To obtain a solution of the Fekete-Szegő problem over the class $\mathcal{ST}(k, \alpha, \lambda, q)$, we need the following lemmas.

LEMMA 2.1. ([12], [15]) *If $q(z) = 1 + c_1z + c_2z^2 + \cdots$ is an analytic function with positive real part in $\mathbb{U}$, then*

$$|c_2 - vc_1^2| \leq 2 \max\{1; |2v-1|\}. \quad (2.21)$$

The result is sharp for the functions $q(z) = \frac{1+z^2}{1-z^2}$ or $q(z) = \frac{1+z}{1-z}$.

LEMMA 2.2. ([6]) *If $q(z) = 1 + c_1z + c_2z^2 + \cdots \in \mathcal{P}(p_k)$ is an analytic function in $\mathbb{U}$, then*

$$|c_2 - vc_1^2| \leq \begin{cases} P_1 - vP_1^2 & \text{if } v \leq 0, \\ P_1 & \text{if } 0 < v < 1, \\ P_1 + (v-1)P_1^2 & \text{if } v \geq 1. \end{cases} \quad (2.22)$$

The result is sharp for the functions $q(z) = \frac{1+z^2}{1-z^2}$ for $0 < v < 1$ and $q(z) = \frac{1+z}{1-z}$ otherwise.

THEOREM 2.4. *Let $0 \leq k < \infty$, $\lambda > -1$, $q \in (0, 1)$ and $0 \leq \alpha < 1$. Suppose that the function f given by (1.1) is in the class $\mathcal{ST}(k, \alpha, \lambda, q)$. Then, for a complex number μ*

$$|a_3 - \mu a_2^2| \leq \frac{2}{q[1+\lambda][2+\lambda]} \max\left\{1; \left|\frac{2\mu[2+\lambda]}{q[1+\lambda]} - 3\right|\right\}. \quad (2.23)$$

Moreover for a real parameter μ , we obtain more rigorous bounds, as follows.

$$|a_3 - \mu a_2^2| \leq \frac{1}{[1 + \lambda][2 + \lambda]} \begin{cases} P_1 - \frac{1}{q} \left(1 - \mu \frac{[2 + \lambda]}{[1 + \lambda]}\right) P_1^2 & \text{if } \mu \geq \frac{[1 + \lambda]}{[2 + \lambda]} \\ P_1 & \text{if } \mu \in \frac{[1 + \lambda]}{[2 + \lambda]} (1 - q, 1) \\ P_1 + \frac{1}{q} \left(1 - q - \mu \frac{[1 + \lambda]}{[2 + \lambda]}\right) P_1^2 & \text{if } \mu \leq (1 - q) \frac{[1 + \lambda]}{[2 + \lambda]}, \end{cases}$$

where P_1 given by (2.11). The results are sharp.

Proof. From (2.18) and (2.19), it follows that

$$a_2 = \frac{p_1}{q[1 + \lambda]} \quad (2.24)$$

and

$$a_3 = \frac{qp_2 + p_1^2}{q^2[1 + \lambda][2 + \lambda]}. \quad (2.25)$$

In view of (2.24) and (2.25), for a complex number μ , we have

$$|a_3 - \mu a_2^2| = \frac{1}{q[1 + \lambda][2 + \lambda]} \left| p_2 - \frac{p_1^2}{q} \left(\mu \frac{[2 + \lambda]}{[1 + \lambda]} - 1 \right) \right|.$$

Applying Lemma 2.1, we get

$$|a_3 - \mu a_2^2| \leq \frac{2}{q[1 + \lambda][2 + \lambda]} \max \left\{ 1, \left| \frac{2\mu[2 + \lambda]}{q[1 + \lambda]} - 3 \right| \right\},$$

which is the thesis. The sharpness of (2.23) follows from the sharpness of (2.22).

Assume now, that a parameter μ is real. Since

$$\begin{aligned} |a_3 - \mu a_2^2| &= \left| \frac{qp_2 + p_1^2}{q^2[1 + \lambda][2 + \lambda]} - \mu \frac{p_1^2}{q^2[1 + \lambda]^2} \right|, \\ &= \frac{1}{[1 + \lambda][2 + \lambda]} \left| p_2 + \left(\frac{1}{q} - \mu \frac{[2 + \lambda]}{q[1 + \lambda]} \right) p_1^2 \right| \end{aligned}$$

therefore, making use of (2.22) from Lemma 2.2 the thesis follows. $\square$

A necessary and sufficient condition for a function $f \in \mathcal{A}$ to be in the class $\mathcal{ST}(k, \alpha, \lambda, q)$ in terms of Hadamard product is given in the following theorem.

THEOREM 2.5. *Let $0 \leq k < \infty$, $\lambda > -1$, $q \in (0, 1)$ and $0 \leq \alpha < 1$. Then the function f belongs to the class $\mathcal{ST}(k, \alpha, \lambda, q)$ if and only if $(f * H_{q, \lambda})(z)/z \neq 0$ in $\mathbb{U}$, where*

$$H_{q, \lambda}(z) = F_{q, \lambda+2}(z) \left\{ 1 - \left(1 - \frac{F_{q, \lambda+1}(z)}{F_{q, \lambda+2}(z)} \right) \frac{w(t)q^\lambda + [\lambda]}{q^\lambda(w(t) - 1)} \right\} \quad (2.26)$$

with

$$w(t) = (kt + \alpha)^{\pm i} \sqrt{t^2 - (kt + \alpha - 1)^2}$$

and $t^2 - (kt + \alpha - 1)^2 \geq 0$.

P r o o f. From (2.6) we have that the values of $z\partial_q(R_q^\lambda f(z))/R_q^\lambda f(z)$ lie in $\Omega_{k,\alpha}$. Therefore

$$\frac{z\partial_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} \neq (kt + \alpha)^{\pm i} \sqrt{t^2 - (kt + \alpha - 1)^2} = w(t) \quad (2.27)$$

with $z \in \mathbb{U}$, $t^2 - (kt + \alpha - 1)^2 \geq 0$.

Applying the definition of $R_q^\lambda f$ and the properties of Hadamard product, the condition (2.27) will hold if

$$f(z) * [z\partial_q(F_{q,\lambda+1}(z)) - w(t)F_{q,\lambda+1}(z)]/z \neq 0. \quad (2.28)$$

Making use of (1.15), it follows from (2.28), that $(f * H_{q,\lambda})(z)/z \neq 0$, where $H_{q,\lambda}(z)$ is given by (2.26).

Conversely, if $(f * H_{q,\lambda})(z)/z \neq 0$ in $\mathbb{U}$, then the values of $z\partial_q(R_q^\lambda f(z))/R_q^\lambda f(z)$ lie completely inside $\Omega_{k,\alpha}$ or its complement. Since

$$\left. \frac{z\partial_q(R_q^\lambda f(z))}{R_q^\lambda f(z)} \right|_{z=0} = 1 \in \Omega_{k,\alpha}$$

we obtain $z\partial_q(R_q^\lambda f(z))/R_q^\lambda f(z) \in \Omega_{k,\alpha}$ which shows that $f \in \mathcal{ST}(k, \alpha, \lambda, q)$. $\square$

THEOREM 2.6. *Let $0 \leq k < \infty$, $\lambda > -1$, $q \in (0, 1)$ and $0 \leq \alpha < 1$. The coefficients h_n of the function $H_{q,\lambda}$ given by (2.26) satisfy the inequality*

$$|h_n| \leq \frac{\Gamma_q(n + \lambda)[1 - \alpha + [n](k + 1)]}{(1 - \alpha)[n - 1]!\Gamma_q(1 + \lambda)}, \quad n \geq 2. \quad (2.29)$$

P r o o f. From the power series of the function $H_{q,\lambda}$ we have

$$h_n = \frac{\Gamma_q(n + \lambda)}{[n - 1]!\Gamma_q(1 + \lambda)} \frac{[n] - w(t)}{1 - w(t)}$$

and therefore

$$\begin{aligned} |h_n|^2 &= \left(\frac{\Gamma_q(n + \lambda)}{[n - 1]!\Gamma_q(1 + \lambda)} \right)^2 \left(1 - \frac{2k([n] - 1)}{t} + ([n] - 1) \frac{[n] + 1 - 2\alpha}{t^2} \right) \\ &=: \left(\frac{\Gamma_q(n + \lambda)}{[n - 1]!\Gamma_q(1 + \lambda)} \right)^2 V(t). \end{aligned}$$

The function $V(t)$ is decreasing in the interval $\langle \frac{1-\alpha}{1+k}, t_0 \rangle$ and increasing in (t_0, ∞) , where $t_0 = \frac{[n]+1-2\alpha}{k}$, with its minimum at t_0 . The limit of $V(t)$ as t tends to infinity is equal to 1 and

$$V\left(\frac{1-\alpha}{k+1}\right) = 1 - 2k([n] - 1) \frac{1+k}{1-\alpha} + ([n] - 1)([n] + 1 - 2\alpha) \frac{(1+k)^2}{(1-\alpha)^2} \geq 1.$$

Thus, the maximum value of $V(t)$ is attained at the point $\frac{1-\alpha}{k+1}$. Since

$$V\left(\frac{1-\alpha}{k+1}\right) \leq \left[\frac{1-\alpha+[n](k+1)}{1-\alpha}\right]^2$$

the coefficients of $H_{q,\lambda}$ satisfy the inequality (2.29). $\square$

COROLLARY 2.2. Let $g(z) = z + a_n z^n$. If

$$|a_n| \leq \frac{(1-\alpha)[n-1]!\Gamma_q(1+\lambda)}{(1-\alpha+[n](k+1))\Gamma_q(n+\lambda)} \quad (n \geq 2),$$

then $g \in \mathcal{ST}(k, \alpha, \lambda, q)$

Proof. Since

$$\left| \frac{(g * H_{q,\lambda})(z)}{z} \right| = |1 + h_n a_n z^{n-1}| \geq 1 - |h_n| |a_n| |z| \geq 1 - |z| > 0 \quad (z \in \mathbb{U})$$

it follows that $g \in \mathcal{ST}(k, \alpha, \lambda, q)$. $\square$

Remark 2.1. The q -analogue of the Leibniz rule is

$$\partial_q(f(z)g(z)) = g(z)\partial_q f(z) + f(qz)\partial_q g(z).$$

Replacing $R_q^\lambda f(z)$ in (1.17) by $z\partial_q(R_q^\lambda f(z))$ we can obtain a new class of functions which is the analogue of the class $\mathcal{UCV}(k, \alpha)$ of k -uniformly convex functions of order α .

DEFINITION 2.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{UCV}(k, \alpha, \lambda, q)$ if it satisfies the inequality

$$\operatorname{Re} \left\{ 1 + q \frac{z\partial_q^2(R_q^\lambda f(z))}{\partial_q(R_q^\lambda f(z))} \right\} > k \left| q \frac{z\partial_q^2(R_q^\lambda f(z))}{\partial_q(R_q^\lambda f(z))} \right| + \alpha.$$

Note that if $\lambda = 0$ and $q \rightarrow 1^-$, the class $\mathcal{UCV}(k, \alpha, \lambda, q)$ reduces to the class $\mathcal{UCV}(k, \alpha)$.

Remark 2.2. Making use of the properties of the functions in $\mathcal{ST}(k, \alpha, \lambda, q)$ we can obtain easily the properties of the functions that belong to the class $\mathcal{UCV}(k, \alpha, \lambda, q)$.

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* *Institut of Mathematics*

University of Rzeszów

Al. Rejtana 16

PL-35-959 Rzeszów

POLAND

E-mail: skanas@ur.edu.pl

** *Department of Mathematics*

Faculty of Mathematics and Computer Science

Transilvania University of Braşov 50091

Iuliu Maniu 50

Braşov

ROMANIA

E-mail: dorinaraducanu@yahoo.com