Stability of the equilibrium positions of an engine with nonlinear quadratic springs

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Abstract: Our paper realizes a study of the equilibrium positions for an engine supported by four identical nonlinear springs of quadratic characteristic. The systems with quadratic characteristic are generally avoided because they lead to mathematical complications. Our goal is to realize such a study for an engine supported on quadratic springs. For the model purposed, we established the equations of motion and we discussed the possibilities for the equilibrium positions. Because of the quadratic characteristic of the springs and of the approximations made for the small rotations, the equations obtained for the equilibrium lead us to a paradox, which consists in the existence of an open neighborhood in which there exists an infinity of positions of indifferent equilibrium, or a curve where the equilibrium positions are situated. Moreover, the study of the stability shows that the stability is assured for the position at which the springs are not compressed. Finally, a numerical example is presented and completely solved.

Keywords: Engine • Suspension • Quadratic • Stability • Small oscillations

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1. Introduction

The vibrations of an engine are usually studied using models with linear elastic springs. These springs are always pre-tensioned by a sufficient large force so that the contact between the engine and springs is not lost [1–6]. This model is used due to its simplicity obtained by the linear relation between the force and the displacement given by the elastic springs. The study is also performed using CAE soft [7]. Other authors [8] identify causes of the engine vibrations and studied the relations between different parameters. The authors of this paper studied

the vibrations of the engine supported by neo-Hookean springs [9]. The study on nonlinear models [9] are performed using special characteristics of the springs. The use of quadratic characteristic is avoided because of the complications that appear (even if the second degree equation is easily solved) and the fact that the characteristic is not one with symmetries relative to origin. This absent of symmetries makes the written of the equations of motion a very complicate and laborious process.

Kilic *et al.* [10] discuss the problem of two quadratic springs antagonistically coupled obtaining a spring with linear mechanical adjustable characteristic. Luo *et al.* [11] study the leaf springs using a multibody approach and apply the theory to Hotchkiss rigid axle type, tires and a leaf-spring model. Champion and Champion [12] obtain a wave spring type equation for a small amplitude,

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undamped extensional oscillation of a spring mass system, where the spring is characterized by a quadratic force extension relation. Excellent overview for the synthesis of nonlinear springs and their design can be found in [13, 14]. Worden *et al.* [15] perform an overview of a number of nonlinear identification methods, specially to nonlinear automotive dampers. Poussot-Vassal *et al.* [16] perform an overview of the semi-active suspension control strategy performances. Yang *et al.* [17] treat the problem of the robust control of certain systems using DOBC (disturbance observer based control) with applications to MAGLEV suspension. Lee and Han [18] use CAE to automotive structure development. Huang *et al.* [19] discuss an algorithm for an active suspension for a half-car. Huang and Chen [20] use the fuzzy theory for the control of the automotive suspension. Koch *et al.* [21] study the control of the active suspension with continuously variable damper. Ibrahim [22] performs a review in nonlinear passive vibration isolators discussing the chaos, the influence of the random vibrations, vibration protection under vibro-impact etc. Wei and Yuling [23] study the nonlinear vibrations of the automotive with magneto-rheological damper. Surace *et al.* [24] present a shock absorber with nonlinear damping. Ando and Suzuki [25] discuss an active suspension system with nonlinear characteristic. Lin and Kanellakopoulos [26] present a nonlinear back-stepping design for the control of active suspension systems.

Our paper is organized as follows: first of all we described the model used in the paper, then we obtain the equations of motion. In the next paragraph we study the equilibrium positions and then the stability of these positions. A numerical example is also presented. The paper ends with conclusions.

2. The model

The model considered in this paper is drawn in Figure 1. It consists in the shell $ABDE$ which schematized the engine block. The shell is considered homogeneous of dimensions $2l_1$ and $2l_2$, respectively, and it is linked to the ground by four identical springs. We denote by C the center of the shell, which is also the center of gravity. The axes Cx , Cy and Cz are the principal central axes of inertia. the system is assumed to have three degrees of freedom: the vertical displacement z of the center of gravity C , the rotation of angle ψ about the Cx -axis, and the rotation of angle θ about the Cy -axis. Both rotations are assumed to be small, so that the following approximations take place: $\sin \psi \simeq \psi$, $\cos \psi \simeq 1$, $\sin \theta \simeq \theta$, $\cos \theta \simeq 1$. The mechanical parameters m - the mass of

Figure 1. Mathematical model.

the shell, and the principal central moments of inertia J_x , and J_y , respectively, with respect to the axes Cx , and Cy , respectively are known.

The elastic forces in the springs are given by the nonlinear expression

$$F = k_1 \delta + k_2 \delta^2, \quad (1)$$

where δ is the elongation of the spring.

With these assumptions the elastic forces which act at the points A , B , D and E are

$$\begin{aligned} F_A &= k_1 (z - l_1 \psi - l_2 \theta) + k_2 (z - l_1 \psi - l_2 \theta)^2, \\ F_B &= k_1 (z + l_1 \psi - l_2 \theta) + k_2 (z + l_1 \psi - l_2 \theta)^2, \\ F_D &= k_1 (z + l_1 \psi + l_2 \theta) + k_2 (z + l_1 \psi + l_2 \theta)^2, \\ F_E &= k_1 (z - l_1 \psi + l_2 \theta) + k_2 (z - l_1 \psi + l_2 \theta)^2. \end{aligned} \quad (2)$$

3. Equations of motion

The theorem of momentum leads to the equation

$$m\ddot{z} = F_A + F_B + F_D + F_E - mg, \quad (3)$$

where g is the gravitational acceleration, and the forces are taken with their signs corresponding to their projections onto the Cz -axis.

The theorem of moment of momentum offers the following two equations, relative to the Cx and Cy axes:

$$\begin{aligned} J_x \ddot{\psi} &= -F_A l_1 + F_B l_1 + F_D l_1 - F_E l_1, \\ J_y \ddot{\theta} &= -F_A l_2 - F_B l_2 + F_D l_2 + F_E l_2. \end{aligned} \quad (4)$$

Figure 2. The equilibrium positions.

The equations (3) and (4) hold true if and only if the forces F_A , F_B , F_C and F_D are vertical ascendent. If the sense of one or more of these forces is vertical descendent the equations must be modified correspondingly. The problem appears because the expressions of the elastic forces in the springs are quadratic and they are not odd, that is the origin is not a point of symmetry.

4. The equilibrium positions

4.1. All the elastic forces are vertical ascendent

Considering that the elastic forces are vertical ascendent (Figure 2), the equations from which one determines the equilibrium positions read

$$\begin{aligned} F_A + F_B + F_D + F_E &= mg, \\ F_A l_1 + F_E l_1 &= F_B l_1 + F_D l_1, \\ F_A l_2 + F_B l_2 &= F_D l_2 + F_E l_2. \end{aligned} \quad (5)$$

The last equations (5) can be written as

$$\begin{aligned} F_A + F_E &= F_B + F_D, \\ F_A + F_B &= F_D + F_E, \end{aligned} \quad (6)$$

wherefrom, by subtracting, we get

$$\begin{aligned} F_B &= F_E, \\ F_A &= F_D. \end{aligned} \quad (7)$$

The relations (7) lead to

$$\begin{aligned} (l_1 \psi - l_2 \theta) (k_1 + 2k_2 z) &= 0, \\ (l_1 \psi + l_2 \theta) (k_1 + 2k_2 z) &= 0. \end{aligned} \quad (8)$$

4.1.1. The case $k_1 + 2k_2 z = 0$

In this situation, one obtains $z = -k_1/(2k_2)$. Replacing in the equation (3), one gets

$$4k_2 (l_1^2 \psi^2 + l_2^2 \theta^2) = \frac{mgk_2 + k_1^2}{k_2}. \quad (9)$$

Since $l_1^2 \psi^2 + l_2^2 \theta^2 \geq 0$ it results that $mgk_2 + k_1^2 \geq 0$, wherefrom $k_2 \geq -k_1^2/(mg)$. But $l_1^2 \psi^2 + l_2^2 \theta^2$ is a small value (from the hypothesis that the angles ψ and θ are small) and from here we deduce that $k_2 \simeq -k_1^2/(mg)$. It means that there exists a relation between the parameters k_1 , k_2 and m . Moreover, the expression (9) defines an ellipse in the $\psi\theta$ plan. In this case, there is a domain around the point $(z, \psi, \theta) = (-k_1/(2k_2), 0, 0)$ in which the system is at equilibrium. We can say that in this situation there exist an infinity of equilibrium positions which defines an open neighborhood around the point $(-k_1/(2k_2), 0, 0)$. These equilibrium positions are positions of indifferent equilibrium.

4.1.2. The case $k_1 + 2k_2 z \neq 0$

This case is characterized by $l_1 \psi - l_2 \theta = 0$ and $l_1 \psi + l_2 \theta = 0$, wherefrom $\psi = 0$ and $\theta = 0$. From the equation (3) we obtain

$$4k_2 z^2 + 4k_1 z - mg = 0, \quad (10)$$

the discriminant of which is $\Delta = \sqrt{16k_1^2 + 16k_2 mg}$. There exist equilibrium positions if and only if $\Delta \geq 0$, which implies $k_2 \geq -k_1^2/(mg)$. In addition, the equilibrium positions are described by

$$\begin{aligned} z_1 &= \frac{-k_1 + \sqrt{k_1^2 + k_2 mg}}{2k_2}, \\ z_2 &= \frac{-k_1 - \sqrt{k_1^2 + k_2 mg}}{2k_2}. \end{aligned} \quad (11)$$

If $k_2 > 0$, then $z_1 > 0$ and $z_2 < 0$, and if $k_2 < 0$, then $z_1 > 0$ and $z_2 > 0$.

4.2. All the elastic forces are vertical descendent

The relations (5) and (9) become now

$$\begin{aligned} F_A + F_B + F_D + F_E + mg &= 0, \\ F_A l_1 + F_E l_1 &= F_B l_1 + F_D l_1, \\ F_A l_2 + F_B l_2 &= F_D l_2 + F_E l_2 \end{aligned} \quad (12)$$

and

$$4k_2(l_1^2\psi^2 + l_2^2\theta^2) = \frac{-mgk_2 + k_1^2}{k_2}, \quad (13)$$

respectively.

Since the left-hand side term of the equation (13) is positive, it results the expression

$$\frac{1}{k_2} \geq \frac{mg}{k_1^2} > 0. \quad (14)$$

The rest of discussion is similar to that from the paragraph 4.1.1.

4.3. Three elastic forces have the same sense and one opposite sense

For the clarity we will assume that the forces F_A , F_B and F_D are vertical ascendent, while the force F_E is vertical descendent. The rest of cases are similar to this one. The second and the third equations (5) read now

$$\begin{aligned} -F_A + F_B + F_D + F_E &= 0, \\ -F_A - F_B + F_D - F_E &= 0, \end{aligned} \quad (15)$$

wherefrom

$$\begin{aligned} F_A &= F_D, \\ F_B &= -F_E. \end{aligned} \quad (16)$$

The first relation (16) leads to

$$(l_1\psi + l_2\theta)(k_1 + 2k_2z) = 0, \quad (17)$$

while the second one offers

$$k_2z^2 + k_1z + (l_1\psi - l_2\theta)^2 = 0. \quad (18)$$

The first equation (5) becomes now

$$F_A + F_B + F_D - F_E = mg, \quad (19)$$

so that we get

$$(k_1 + k_2z)(z + l_1\psi - l_2\theta) + k_2(l_1\psi + l_2\theta)^2 = mg. \quad (20)$$

4.3.1. The case $k_1 + 2k_2 = 0$

Equation (18) reads now

$$\left(l_1\psi - l_2\theta + \sqrt{\frac{k_1^2}{4k_2}}\right) \left(l_1\psi - l_2\theta - \sqrt{\frac{k_1^2}{4k_2}}\right) = 0, \quad (21)$$

which defines two parallel lines in the $\psi - \theta$ plane. It also results

$$k_1 + k_2z = \frac{k_1}{2}; \quad (22)$$

hence, the expression (20) may be put in the form

$$\frac{k_1}{2}(z + l_1\psi - l_2\theta) + k_2(l_1\psi + l_2\theta)^2 = mg \quad (23)$$

and since $(l_1\psi + l_2\theta)^2 \simeq 0$, we deduce the relation

$$l_1\psi - l_2\theta = \frac{2mgk_2 + k_1^2}{2k_1k_2}, \quad (24)$$

that is, a straight line in the $\psi - \theta$ plane.

4.3.2. The case $l_1\psi + l_2\theta = 0$

In this situation, one obtains the relation

$$k_2z^2 + k_1z + 4l_1^2\psi^2 = 0, \quad (25)$$

which may be also written as

$$\left(z + \frac{k_1}{2k_2}\right)^2 + \frac{4l_1^2}{k_2}\psi^2 = \frac{k_1^2}{4k_2^2}. \quad (26)$$

The last equation defines in the $\psi - \theta$ plane an ellipse if $k_2 > 0$ or a hyperbola if $k_2 < 0$.

Moreover, the expression (20) leads now to

$$(k_1 + k_2z)(z + l_1\psi - l_2\theta) = mg. \quad (27)$$

Since $\theta = -l_1\psi/l_2$, that is

$$l_1\psi - l_2\theta = 2l_1\psi \quad (28)$$

and making the approximation

$$z + l_1\psi - l_2\theta \simeq z, \quad (29)$$

one obtains the equation

$$k_2z^2 + k_1z = mg, \quad (30)$$

with 0, 1 or 2 real solutions depending on the sign of $\Delta = \sqrt{k_1^2 + 4k_2mg}$.

4.4. Two elastic forces are vertical ascendent, while the other two are vertical descendent

According to the theorem of moment of momentum, the only possibility for the equilibrium is that the forces situated on the same diagonal have the same sense. For the clarity, we will assume that the forces F_A and F_D are vertical ascendent, while the forces F_B and F_E are vertical descendent. The second and the third relations (5) leads now to

$$\begin{aligned} F_A + F_B &= F_D + F_E, \\ F_A + F_E &= F_B + F_D, \end{aligned} \quad (31)$$

wherefrom

$$\begin{aligned} F_B &= F_E, \\ F_A &= F_D. \end{aligned} \quad (32)$$

The first equation (31) leads to

$$(l_1\psi - l_2\theta)(k_1 + 2k_2z) = 0, \quad (33)$$

while the second one gets

$$(l_1\psi + l_2\theta)(k_1 + 2k_2z) = 0. \quad (34)$$

From the first equation (5) results

$$F_A + F_D = F_B + F_E + mg, \quad (35)$$

wherefrom one obtains

$$8k_2l_1l_2\psi\theta = mg, \quad (36)$$

that is a hyperbola in the $\psi - \theta$ plane.

5. Stability of the equilibrium positions

5.1. All the elastic forces are vertical ascendent

Denoting $z = \xi_1$, $\psi = \xi_2$, $\theta = \xi_3$, $\dot{z} = \xi_4$, $\dot{\psi} = \xi_5$, $\dot{\theta} = \xi_6$, the equations of motion may be put in the form of a six first order nonlinear differential equations

$$\begin{aligned} \dot{\xi}_1 &= \xi_4, \quad \dot{\xi}_2 = \xi_5, \quad \dot{\xi}_3 = \xi_6, \\ \dot{\xi}_4 &= -\frac{4k_1\xi_1}{m} - \frac{4k_2(\xi_1^2 + l_1^2\xi_2^2 + l_2^2\xi_3^2)}{m} - g, \\ \dot{\xi}_5 &= -\frac{4k_1l_1^2\xi_2}{J_x} - \frac{8k_2\xi_1l_1^2\xi_2}{J_x}, \\ \dot{\xi}_6 &= -\frac{4k_1l_2^2\xi_3}{J_y} - \frac{8k_2\xi_1l_2^2\xi_3}{J_y}. \end{aligned} \quad (37)$$

Further on, we denote by j_{kl} the partial derivatives $j_{kl} = \partial f_j / \partial \xi_l$, where f_j marks the right-hand terms of the expressions (37), and $k, l = \overline{1, 6}$. One calculates

$$\begin{aligned} j_{14} &= 1, \\ j_{25} &= 1, \\ j_{36} &= 1, \end{aligned} \quad (38)$$

$$\begin{aligned} j_{41} &= -\frac{4k_1}{m} - \frac{8k_2\xi_1}{m}, \\ j_{42} &= -\frac{8k_2l_1^2\xi_2}{m}, \\ j_{43} &= -\frac{8k_2l_2^2\xi_3}{m}, \end{aligned} \quad (39)$$

$$\begin{aligned} j_{51} &= -\frac{8k_2l_1^2\xi_2}{J_x}, \\ j_{52} &= -\frac{4k_1l_1^2}{J_x} - \frac{8k_2l_1^2\xi_1}{J_x}, \end{aligned} \quad (40)$$

$$\begin{aligned} j_{61} &= -\frac{8k_2l_2^2\xi_3}{J_y}, \\ j_{63} &= -\frac{4k_1l_2^2}{J_y} - \frac{8k_2l_2^2\xi_1}{J_y}, \end{aligned} \quad (41)$$

the rest of j_{kl} being zero.

The characteristic equation reads (λ being the complex rate of growth)

$$\begin{vmatrix} -\lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 1 \\ j_{41} & j_{42} & j_{43} & -\lambda & 0 & 0 \\ j_{51} & j_{52} & 0 & 0 & -\lambda & 0 \\ j_{61} & 0 & j_{63} & 0 & 0 & -\lambda \end{vmatrix} = 0, \quad (42)$$

wherefrom

$$\lambda^6 + A\lambda^4 + B\lambda^2 + C = 0, \quad (43)$$

with

$$\begin{aligned} A &= -(j_{41} + j_{52} + j_{63}), \\ B &= (j_{52}j_{63} + j_{41}j_{63} + j_{41} + j_{41}j_{52} - j_{43}j_{61} - j_{42}j_{51}), \\ C &= -j_{41}j_{52}j_{63} + j_{43}j_{52}j_{61} + j_{42}j_{51}j_{63}. \end{aligned} \quad (44)$$

Putting $u = \lambda^2$, one obtains the third degree equation

$$u^3 + Au^2 + Bu + C = 0. \quad (45)$$

According to the Descartes theorem [27, 28], this equations has no positive solution if and only $A > 0$,

$B > 0$ and $C > 0$. With the change of variable $u = v - A/3$, the equation becomes

$$v^3 + av + b = 0, \quad (46)$$

with

$$\begin{aligned} a &= B - \frac{A^2}{3}, \\ b &= \frac{2A^3}{27} - \frac{AB}{3} + C. \end{aligned} \quad (47)$$

The equation (46) has three real roots if and only if [28] its discriminant $\Delta = 4a^3 + 27b^2$ is negative, that is

$$\Delta = 4B^3 - A^2B^2 + 27C^2 + 4A^3C - 18ABC \leq 0. \quad (48)$$

If we consider the equilibrium positions described by $\psi = \xi_2 = 0$, $\theta = \xi_3 = 0$, while z is given by the relations (11), then the characteristic equation takes the simpler form

$$(j_{41} - \lambda^2)(j_{52} - \lambda^2)(j_{63} - \lambda^2) = 0 \quad (49)$$

and the solutions are $\lambda = \pm\sqrt{j_{41}}$, $\lambda = \pm\sqrt{j_{52}}$, and $\lambda = \pm\sqrt{j_{63}}$.

The equilibrium position is stable (in fact, simply stable) if and only if $j_{41} < 0$, $j_{52} < 0$, and $j_{63} < 0$. These conditions are equivalent with the condition $k_1 + 2k_2 z_{eq} > 0$, where z_{eq} stays for the value of the coordinate z in the equilibrium position, value given by one of the relations (11). Replacing the values given by (11) in the stability condition, one gets $\pm\sqrt{k_1^2 + k_2 mg} > 0$, which holds true only for the value z_1 . In conclusion, the only stable position is given by z_1 .

The small oscillations around the equilibrium position are harmonic motions given by [27]

$$\begin{aligned} z &= z_0 \cos(\sqrt{|j_{41}|}t + \varphi_1), \\ \psi &= \psi_0 \cos(\sqrt{|j_{52}|}t + \varphi_2), \\ \theta &= \theta_0 \cos(\sqrt{|j_{63}|}t + \varphi_3), \end{aligned} \quad (50)$$

where z_0 , ψ_0 , θ_0 , φ_1 , φ_2 , and φ_3 are constants of integration with the values determined from the initial conditions.

The situation defined by all vertical descendent force is completely similar; we only have to change the sign in the previous expressions.

5.2. Three elastic forces are vertical ascendent, while one is vertical descendent

Considering again that the forces F_A , F_B and F_D are vertical ascendent, and the force F_E is a vertical descendent one, one obtains the equations of motion

$$\begin{aligned} m\ddot{z} &= F_A + F_B + F_D - F_E - mg, \\ J_x\ddot{\psi} &= -F_A l_1 + F_B l_1 + F_D l_1 + F_E l_1, \\ J_y\ddot{\theta} &= -F_A l_2 - F_B l_2 + F_D l_2 - F_E l_2, \end{aligned} \quad (51)$$

wherefrom, using the same notations as in paragraph 5.1,

$$\begin{aligned} \dot{\xi}_1 &= \xi_4, \quad \dot{\xi}_2 = \xi_5, \quad \dot{\xi}_3 = \xi_6, \\ \dot{\xi}_4 &= \frac{2k_1(\xi_1 + l_1\xi_2 - l_2\xi_3)}{m} + \frac{2k_2\xi_1^2}{m} + \frac{4k_2\xi_1(l_1\xi_2 - l_2\xi_3)}{m} \\ &\quad + \frac{2k_2(l_1\xi_2 + l_2\xi_3)}{m} - g, \\ \dot{\xi}_5 &= \frac{2k_1(\xi_1 + l_1\xi_2 - l_2\xi_3)}{J_x} + \frac{2k_2\xi_1^2}{J_x} \\ &\quad + \frac{4k_2\xi_1(l_1\xi_2 - l_2\xi_3)}{J_x} + \frac{2k_2(l_1\xi_2 + l_2\xi_3)^2}{J_x}, \\ \dot{\xi}_6 &= -\frac{2k_1(\xi_1 - l_1\xi_2 - l_2\xi_3)}{J_y} - \frac{2k_3\xi_1^2}{J_y} + \frac{4k_2\xi_1(l_1\xi_2 + l_2\xi_3)}{J_y} \\ &\quad - \frac{2k_2(l_1\xi_2 - l_2\xi_3)^2}{J_y}. \end{aligned} \quad (52)$$

The discussion is completely similar to that at the paragraph 5.1.

6. The influence of damping

In this case, the system (53) becomes

$$\begin{aligned} \dot{\xi}_1 &= \xi_4, \quad \dot{\xi}_2 = \xi_5, \quad \dot{\xi}_3 = \xi_6, \\ \dot{\xi}_4 &= -\frac{4k_1\xi_1}{m} - \frac{4k_2(\xi_1^2 + l_1^2\xi_2^2 + l_2^2\xi_3^2)}{m} - \frac{4c\xi_4}{m} - g, \\ \dot{\xi}_5 &= -\frac{4k_1l_1^2\xi_2}{J_x} - \frac{8k_2\xi_1l_1^2\xi_2}{J_x}, \\ \dot{\xi}_6 &= -\frac{4k_1l_2^2\xi_3}{J_y} - \frac{8k_2\xi_1l_2^2\xi_3}{J_y}, \end{aligned} \quad (53)$$

where c is the damping coefficient.

The only parameter that changes its value is

$$j_{44} = -\frac{4c}{m} < 0. \quad (54)$$

The characteristic equation becomes now

$$\begin{vmatrix} -\lambda & 0 & 0 & 1 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 1 \\ j_{41} & j_{42} & j_{43} & j_{44} - \lambda & 0 & 0 \\ j_{51} & j_{52} & 0 & 0 & -\lambda & 0 \\ j_{61} & 0 & j_{63} & 0 & 0 & -\lambda \end{vmatrix} = 0, \quad (55)$$

wherefrom

$$\begin{vmatrix} j_{41} + j_{44}\lambda - \lambda^2 & j_{42} & j_{43} \\ j_{51} & j_{52} - \lambda^2 & 0 \\ j_{61} & 0 & j_{63} - \lambda^2 \end{vmatrix} = 0, \quad (56)$$

so that one gets

$$\lambda^6 + A\lambda^5 + B\lambda^4 + C\lambda^3 + D\lambda^2 + E\lambda + F = 0. \quad (57)$$

The roots of the last equation (and therefore, the stability of the equilibrium positions) may be discussed with the aid of the Routh-Hurwitz criterion.

In the real cases, the damper is not presented and the damping appears from the natural characteristic of the spring. Hence, the values of the coefficient c are very small and the terms with odd powers in equation (57) can be neglected.

7. Numerical example

As a numerical example we consider the following realistic values: $m = 200$ kg, $g = 10$ m/s², $l_1 = 0.45$ m, $l_2 = 0.3$ m, $k_1 = 10^5$ N/m, $k_2 = 10^7$ N/m², $J_x = 13.5$ kgm², $J_y = 6$ kgm².

One obtains the equation

$$20000z^2 + 200z - 1 = 0, \quad (58)$$

with the solutions $z_1 = 0.00366$ m and $z_2 = -0.01366$ m. The partial derivatives are $j_{41} = -3464$, $j_{52} = -10392$, and $j_{63} = -10392$ for z_1 , respective $j_{41} = 3464$, $j_{52} = 10392$, and $j_{63} = 10392$ for z_2 . The eigenpulsations for the small oscillations around the simply stable position of equilibrium defined by z_1 are $\omega_1 = 58.86$ Hz for z , $\omega_2 = 101.94$ Hz for ψ , and $\omega_3 = 101.94$ Hz for θ .

8. Conclusions

Our paper presented a model for the suspension of an engine using four identic springs of quadratic characteristic. We proved that there exist situations,

described by particular values of the parameters k_1 , k_2 , and m (linking therefore the elastic and the inertial properties of the system), situation for which the systems presents an infinite number of equilibrium positions. This result is a paradox having as causes the linear approximations for the rotational angles about the Cx - and Cy -axes, and the quadratic characteristic of the springs. The second paradox is defined by the equilibrium position, in fact by their stability. The present study showed that the simply stable position of equilibrium is that for which the springs are stretched, not the position in which the springs are compressed. The reader can easily proved that for both positions of equilibrium the elastic forces are positive, i.e. the forces in the springs are vertical ascendent (as they have to be to equilibrate the weight of the engine). The small oscillations around the equilibrium position are harmonic ones and we determined the eigenpulsations of these oscillations. The method presented here, based on dynamical systems for the study of the equilibrium, is equivalent with the method of the normal forms, not detailed in this paper. The equivalence is assured by the fact that the eigenvalues of a matrix A are the same but with signs changed as those of the matrix $-A$.

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