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# On the global stability of time delayed CNNs

**Review Article**

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**Abstract:** The objective of this paper is the concise presentation of the most important and recent lemmas and theorems associated with the global asymptotic and exponential stability of the equilibrium point of time delayed cellular neural networks. For each theorem a short proof is given, so that the reader can understand its features and its relationships to other theorems. In the last section, the presented theorems are grouped according to their characteristics and the way they relate to one another, and some of them are demonstrated, in order to draw conclusions about their use.

**Keywords:** cellular neural networks • asymptotic and exponential stability • Lyapunov functional • linear matrix inequality



## 1. Introduction

Great attention has been paid to cellular neural networks (CNNs) in recent years, since they can be used to solve a great variety of problems associated mainly with the field of signal and image processing. These structures are closely related to linear as well as nonlinear features – their electronic implementations are based on linear capacitors and resistors as well as linear and nonlinear controlled and independent sources – and therefore, the study of their stability is a central problem associated with this field. A special type of cellular neural network is the one associated with fixed and time varying delays; these networks are used in various types of motion delayed applications, such as speed detection of moving objects, processing of moving images, and pattern classification. The stability analysis of this type of CNNs is much more difficult compared with the one associated with the traditional CNNs and it is based on Lyapunov method as well as the Linear Matrix Inequality (LMI) technique. In the next sections, a set of useful and recent theorems concerning the global asymptotic and exponential stability of the time delayed CNNs are proven and related to each other. Regarding other stability types (e.g complete and robust stability) see [8, 10, 22, 28], while for the stability type of other variations of CNNs, such as stochastic CNNs, see [9, 17]. The above references as well as the theorems presented in this paper are associated with continuous time CNNs, while for the stability of discrete time CNNs, see for example [21].

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In recent years, many theorems associated with the exponential and asymptotic stability of cellular neural networks have been proven. All these theorems use different notation to describe the same quantities, have analytic proofs that are too lengthy, and include a lot of mathematical details. The purpose of this paper is to collect all these theorems, to present the essential parts of their proof by omitting all technicalities and the unnecessary details using the same style of writing, as well as to find associations between them and demonstrate their use via specific examples. The last section of the paper includes a table that summarizes the main theorems and the Lyapunov functions used for their proof as well a diagram that depicts the associations between them.

## 2. Model Description

As it is well known from the literature [11–13], a time delayed cellular neural network (TDCNN) of dimensions $M \times N$ is described by the vector equation

$$\frac{d\boldsymbol{x}(t)}{dt} = -\boldsymbol{\Gamma}\boldsymbol{x}(t) + A\boldsymbol{y}[\boldsymbol{x}(t)] + B\boldsymbol{y}[\boldsymbol{x}(t-\boldsymbol{\tau}(t))] + \boldsymbol{u}(t) \tag{1}$$

which can be expanded in a system of nonlinear equations in the form

$$\frac{dx_i(t)}{dt} = -\gamma_i x_i(t) + \sum_{j=1}^{n} \alpha_{ij} y_j[x_j(t)] + \sum_{j=1}^{n} \beta_{ij} y_j[x_j(t-\tau_j(t))] + u_i \quad (i = 1, 2, \dots, n).$$

In the above equation, $\boldsymbol{x}(t) = [x_1(t), x_2(t), \dots, x_n(t)]^T \in R^n$ $(n = MN)$ is the state vector of the system with initial conditions $x_i(s) = \varphi_i(s)$ $(s \in [-\tau, 0]$, $i = 1, 2, \dots, n)$, $\boldsymbol{\Gamma} = \text{diag}\,\{\gamma_1, \gamma_2, \dots, \gamma_n\}$ is a positive diagonal matrix containing the state coefficients, with the coefficient $\gamma_i > 0$ $(i = 1, 2, \dots, n)$ to represent the rate in which the $i_{th}$ unit will reset its potential to the resting state in isolation, when disconnected from the network and external inputs, $\boldsymbol{y}[\boldsymbol{x}(t)] = [y_1(x_1(t)), y_2(x_2(t)), \dots, y_n(x_n(t))]^T \in R^n$ is the output vector, $\boldsymbol{u}(t)$ is the constant control input vector with the element $u_i(t)$ $(i = 1, 2, \dots, n)$ to represent the external bias of the $i_{th}$ unit at time $t$, and $A = \{a_{ij}\}$ and $B = \{\beta_{ij}\}$ $(i = 1, 2, \dots, M$ and $j = 1, 2, \dots, N)$ are the feedback and the delayed feedback matrices, respectively. In most cases, the coefficients $\gamma_i$ are equal to unity, a convention that will be mostly adopted in the presented theorems. Regarding the vector of the transmission delays along the axis of the $n$ neurons, it is defined as $\boldsymbol{\tau}(t) = [\tau_1(t), \tau_2(t), \dots, \tau_n(t)]^T \in R^n$ with $0 \leqslant \tau_i(t) \leqslant h_i$ and $\dot{\tau}_i(t) \leqslant d_i$ $(i = 1, 2, \dots, n)$, while the nonlinear functions $y_i[x_i(t)]$ $(i = 1, 2, \dots, n)$ are considered as continuous and bounded on $R$ monotonically increasing functions, satisfying the Lipschitz condition

$$|y_i(\xi_1) - y_i(\xi_2)| \leqslant \sigma_i |\xi_1 - \xi_2| \quad (i = 1, 2, \dots, n) \quad \forall\, \xi_1, \xi_2 \in R : \xi_1 \neq \xi_2.$$

These functions are clearly more general than both the usual sigmoidal function and the piecewise linear function $f(x) = 0.5(|x+1| - |x-1|)$ used in standard CNNs. Under those circumstances, it can be proven that there is an equilibrium point $\boldsymbol{x}^*$ that is globally asymptotically stable.

To prove the existence of the equilibrium point, let us consider such a point $\boldsymbol{x}^* = \{x_1^*, x_2^*, \dots, x_n^*\}$; it is clear that this point satisfies the equation

$$x_i^*(t) = \sum_{i=1}^{n} \left( \frac{\alpha_{ij} + \beta_{ij}}{\gamma_i} \right) y_j(x_j^*) + \frac{u_i}{\gamma_i} \quad (i = 1, 2, \dots, n). \tag{2}$$

By defining the quantities

$$C = \left[ \frac{\alpha_{ij} + \beta_{ij}}{\gamma_i} \right]_{n \times n}, \quad \boldsymbol{\upsilon} = \left[ \frac{u_1}{\gamma_1}, \dots, \frac{u_n}{\gamma_n} \right]^T, \quad \boldsymbol{f}(\boldsymbol{x}^*) = [y_1(x_1^*), \dots, y_n(x_n^*)]^T$$

the equation (2) can be written in a vector-matrix notation in the form

$$\boldsymbol{x}^* = \boldsymbol{F}(\boldsymbol{x}^*) = C\boldsymbol{f}(\boldsymbol{x}^*) + \boldsymbol{\upsilon}$$

meaning that the equilibrium point $\boldsymbol{x}^*$ is a fixed point of the map $\boldsymbol{F}: R^n \rightarrow R^n$. In this case, the existence of a fixed point of the map $\boldsymbol{F}$ can be shown by using the well known Brouwer's fixed point theorem. To use this theorem, we note that the $i_{th}$ component of the function $\boldsymbol{F}(\boldsymbol{x})$ satisfies the inequality

$$|\boldsymbol{F}(\boldsymbol{x})_i| \leqslant \sum_{j=1}^{n} \left| \frac{\alpha_{ij} + \beta_{ij}}{\gamma_i} \right| W + \frac{|u_i|}{\gamma_i} \quad (i = 1, 2, \dots, n),$$

where $\boldsymbol{x} = [x_1, x_2, \dots, x_n]^T$ and $W = \max_{1 \leqslant i \leqslant n} \sup_s |f_i(s)|$. By defining the quantity

$$K = \max_{1 \leqslant i \leqslant n} \left( \sum_{j=1}^{n} \left| \frac{\alpha_{ij} + \beta_{ij}}{\gamma_i} \right| W + \frac{|u_i|}{\gamma_i} \right)$$

it is clear that $\boldsymbol{F}(R^n) \subset Q = \{(x_1, x_2, \dots, x_n) \in R^n : |x_i| \leqslant K, i = 1, 2, \dots, n\}$ from which it is obvious that $\boldsymbol{F}$ is a continuous map. Furthermore, if we restrict the map $\boldsymbol{F}$ on $Q$, namely, $\boldsymbol{F}|_Q : Q \rightarrow Q$, then the map $\boldsymbol{F}|_Q$ maps the bounded closed and convex set $Q$ into itself, and therefore, according to Brouwer's theorem, it has at least one fixed point $\boldsymbol{x}^*$. We have to mention, however, that the Brouwer's theorem does not guarantee the uniqueness of the identified fixed point of the map $\boldsymbol{F}$.

Returning to (1) it is not difficult to note, that all the solutions $x_i(t)$ (for the values $i = 1, 2, \dots, n$), satisfy differential inequalities of the form

$$-\gamma_i x_i(t) - \alpha_i \leqslant \frac{dx_i(t)}{dt} \leqslant -\gamma_i x_i(t) + \alpha_i \quad (i = 1, 2, \dots, n),$$

where

$$\alpha_i = \sum_{j=1}^{n} \left( |\alpha_{ij}| + |\beta_{ij}| \right) \sup_{s \in R} |f_i(s)| + |u_i|.$$

By using the last equation, one can prove that the solutions $x_i(t)$ are bounded in the interval $[0, \infty)$.

There are many papers in the literature dealing with the time delayed cellular neural networks and in the next sections the most important and recent theorems associated with the global stability of TDCNNs will be presented. Even though the approaches and the stability criteria proposed by the authors are different in some extent, the starting point and the adopted methodology are, in fact, the same: in the first step, for the sake of simplicity, the equilibrium point $\boldsymbol{x}^*$ is shifted to the origin of the system by defining the variable $\boldsymbol{z}(t) = \boldsymbol{x}(t) - \boldsymbol{x}^*$; in this way, the defining equation of the TDCNN gets the form

$$\frac{\boldsymbol{z}(t)}{dt} = -\boldsymbol{z}(t) + A\boldsymbol{\Psi}[\boldsymbol{z}(t)] + B\boldsymbol{\Psi}[\boldsymbol{z}(\boldsymbol{t}) - \boldsymbol{\tau}(t)] \tag{3}$$

with the vector $\boldsymbol{z}(t) = [z_1(t), z_2(t), \dots, z_n(t)]^T$ to be the new state vector of the shifted system. Regarding the vector function $\boldsymbol{\Psi}[\boldsymbol{z}(t)]$ it has the form

$$\boldsymbol{\Psi}[\boldsymbol{z}(t)] = \{\Psi_1[z_1(t)], \Psi_2[z_2(t)], \dots, \Psi_n[z_n(t)]\}^T$$

and

$$\boldsymbol{\Psi}[\boldsymbol{z}(t - \boldsymbol{\tau}(t))] = \{\Psi_1(z_1[t - \tau_1(t)]), \Psi_2(z_2[t - \tau_2(t)]), \dots, \Psi_n(z_n[t - \tau_n(t)])\}^T$$

with the function components to be defined as

$$\Psi_i[z_i(t)] = y_i[z_i(t) + x_i^*] - y_i(x_i^*)$$

and

$$\Psi_i[z_i(t - \tau_i(t))] = y_i[z_i(t - \tau_i(t)) + x_i^*] - y_i(x_i^*).$$

These components satisfy the properties

$$|\Psi_i[z_i(t)]| \leqslant M_i|z_i(t)| \text{ and } \Psi_i^2[z_i(t)] \leqslant z_i\Psi_i[z_i(t)] \quad (i = 1, 2, \ldots, n) \tag{4}$$

while the vector function itself is characterized by the additional properties

$$\boldsymbol{\Psi}^T[\boldsymbol{z}(t)]\boldsymbol{z}(t)\boldsymbol{\Psi}[\boldsymbol{z}(t)] = |\boldsymbol{\Psi}(\boldsymbol{z}(t))|^2 \leqslant \boldsymbol{z}^{\,T}(t)\Sigma^2\boldsymbol{z}(\boldsymbol{t}),$$
$$\boldsymbol{\Psi}^T(\boldsymbol{z}[t-\boldsymbol{\tau}(t)])\boldsymbol{z}(t)\boldsymbol{\Psi}(\boldsymbol{z}[t-\boldsymbol{\tau}(\boldsymbol{t})]) = |\boldsymbol{\Psi}(\boldsymbol{z}[t-\boldsymbol{\tau}t])|^2 \leqslant \boldsymbol{z}^T[t-\boldsymbol{\tau}(t)]\Sigma^2\boldsymbol{z}[t-\tau(t)].$$

Table 1 summarizes the symbols used throughout the paper and the meaning for each one of them.

**Table 1.** The notation and the symbols used throughout the paper.

| Symbol | Meaning |
|---|---|
| $M, N$ | dimensions of the CNN ($M \times N$) |
| $n = MN$ | size of the CNN |
| $A = \{\alpha_{ij}\} \begin{pmatrix} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{pmatrix}$ | feedback matrix |
| $B = \{\beta_{ij}\} \begin{pmatrix} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{pmatrix}$ | delayed feedback matrix |
| $\Gamma = \text{diag}\ \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ | state coefficient matrix |
| $\boldsymbol{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ | state vector |
| $\boldsymbol{y}(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T$ | output vector |
| $y_i(t) = y_i[x_i(t)]$ (i=1,2,...,n) | activation function of the $i_{th}$ output unit |
| $u_i(t)\ \ (i = 1, 2, \ldots, n)$ | external bias of the $i_{th}$ output unit |
| $\varphi_i(s),\ s \in [-\tau, 0]\ \ (i = 1, 2, \ldots, n)$ | initial conditions for the state vector |
| $\boldsymbol{\tau}(t) = [\tau_1(t), \tau_2(t), \ldots, \tau_n(t)]^T$ | time delay vector |
| $\boldsymbol{x}^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ | equilibrium point of the original system |
| $\boldsymbol{z}(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^T$ | state vector of the shifted system |
| $\boldsymbol{z}^* = [z_1^*, z_2^*, \ldots, z_n^*]^T$ | equilibrium point of the shifted system |
| $\Psi_i[z_i(t)] = y_i[z_i(t) + x_i^*] - y_i(x_i^*)$ | activation functions of the shifted system |
| $\sigma_i\ (i = 1, 2, \ldots, n)$ | constants of the Lipschitz conditions (original system) |
| $\Sigma = \text{diag}\ \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$ | diagonal table of Lipschitz constants |
| $M_i\ (i = 1, 2, \ldots, n)$ | constants of the Lipschitz conditions (shifted system) |
| $\mathcal{M} = \text{diag}\ \{M_1, M_2, \ldots, M_n\}$ | diagonal table of Lipschitz constants |
| $\lambda_M(C), \lambda_m(C)$ | maximum and minimum eigenvalue of $C$ |
| $k$ | degree of exponential stability |

## 3. Asymptotic and exponential stability

To describe the next step of the method, let us give at this point some definitions regarding the stability of dynamical systems. By considering the equilibrium point $x = 0$ of an autonomous system $\dot{x}(t) = V(x)$ – where $V : D \to R^n$ is a locally Lipschitz map $f : (D \subset R^n) \to R^n$ – it is characterized as stable if

$$\forall \varepsilon > 0 \quad \exists \delta = \delta(\varepsilon) > 0 : |x(0)| < \delta \Rightarrow |x(t)| < \varepsilon \quad (\forall t \geqslant 0)$$

as locally stable if for all $\varepsilon > 0$ there exists a constant $\delta > 0$ such as for every solution $\boldsymbol{x}(t)$ with any initial value $\boldsymbol{\varphi}(s)$ such that

$$|\boldsymbol{\varphi} - \boldsymbol{x}^*| = \max_{1 \leqslant j \leqslant n} \left\{ \sup_{-\infty \leqslant t \leqslant 0} |\ \varphi_j - x_j^*\ | \right\} < \delta.$$

($x^*$ being the equilibrium point of the system) there holds

$$| x_i(t) - x_i^* | < \varepsilon \qquad \text{for all } t \geqslant 0, \ \ (i = 1, 2, \ldots, n)$$

(see [23]) and as asymptotically stable if it is stable and the parameter $\delta$ can be chosen such that

$$|x(0)| < \delta \Rightarrow \lim_{t\to\infty} x(t) = 0.$$

As it has been shown by Lyapunov in 1892, if the function $V(x)$ is a continuously differentiable function in $D$ and the domain $D$ contains the origin $x = 0$, then the equilibrium point $x = 0$ is (i) stable, if it satisfies the properties (a) $V(0) = 0$ and $V(x) > 0$ in $D - \{0\}$ and (b) $\dot{D}(x) \leqslant 0$ in $D$ and (ii) asymptotically stable, if the property $\dot{V}(x) < 0$ in $D - \{0\}$ is satisfied at the same time. Based on this description, the second step of the characterization of a TDCNN with respect to the global asymptotical stability is performed as follows: a set of hypotheses together with a Lyapunov functional are proposed, and then, based on those hypotheses, it has to be proven that the Lyapunov functional satisfies the fundamental properties of the asymptotic stability described above. More specifically, due to the fact that the Lyapunov functional is a function of the new state vector $\boldsymbol{z}(t)$ and furthermore, the defining equation of the TDCNN contains the first derivative of that function, the method is based on the differentiation of the Lyapunov functional and to the proof that this derivative is less or equal to zero.
Another very important stability type associated with the origin $\boldsymbol{z}$ of a time delayed CNN, is the exponential stability [18] defined as follows:

***Definition 3.1.***
Consider the TDCNN defined by the equation (3). If there exist positive constants $k > 0$ and $\gamma > 0$ such that

$$|\boldsymbol{z}(t)| \leqslant \gamma e^{-kt} \sup_{-\tau(t)\leqslant\vartheta\leq 0} |\boldsymbol{z}(\vartheta)| \quad (\forall t > 0),$$

then the origin of equation (3) is exponentially stable where $k$ is called the degree of exponential stability.

An equivalent definition can be postulated for the non shifted TDCNN described by the equation (1) as follows:

***Definition 3.2.***
The equilibrium point $\boldsymbol{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ is said to be globally exponentially stable if there exist constants $\varepsilon > 0$ and $k \geqslant 1$ such that

$$\sqrt[p]{\sum_{i=1}^{n} |x_i(t) - x_i^*|^p} \leqslant k e^{-\varepsilon t} \sup_{-\tau\leqslant t\leqslant 0} \sqrt[p]{\sum_{i=1}^{n} |\varphi_i(t) - x_i^*|^p}.$$

In the literature, the simplified form of the above definition emerging by setting $p = 1$ is also used for the proof of the global exponential stability of the equilibrium point.

## 4. A Collection of useful lemmas

The proofs of the theorems presented in the next sections are based on a set of useful lemmas, some of them are fundamental issues from the matrix algebra, and some others are related to specific theorems regarding the stability of cellular neural networks. These lemmas have been collected and presented in this section without proof; these proofs can be found in any textbook regarding matrix algebra or in the paper that presents the associated theorem (see the citations in the References section).

**Lemma 4.1.**

*Suppose $f : C \to R^n$ where $C = C([-\tau, 0], R^n)$ is continuous and maps bounded sets in $C$ in bounded sets in $R^n$ and $f(0) = 0$. Assume that there exists a Lyapunov function $V(x)$ and a constant $N$ such that (a) $V(0) = 0$, (b) $V(x) > 0$ $(\forall 0 \neq |x| < N)$, (c) $\dot{V}(0) = 0$, (d) $\dot{V}(\varphi) < 0$ $(\forall 0 \neq |\varphi| < N)$, and (e) $\max V[\varphi(s)] = V[\varphi(0)]$ for $-\tau \leqslant s \leqslant 0$ where $\dot{V}(\varphi)$ denotes the upper right-hand derivative of $V$ along a solution $x(t, \varphi)$ of the equation $\dot{x}(t) = f(x_t)$ with $x_t(s) = x(t + s)$. Then, the solution $x = 0$ of the last equation is asymptotically stable. In addition, for each solution such that $|x_t(\varphi)| < N$ $(\forall t \geqslant 0)$, it is $x_t(\varphi) \to 0$ in $C$ as $t \to \infty$.*

**Lemma 4.2.**

*For any two vectors $x, y \in R^n$ and any positive definite matrix $Q \in R^{n\times n}$, the following matrix inequality holds:*

$$2\boldsymbol{x}^T\boldsymbol{y} \leqslant \boldsymbol{x}^T Q\boldsymbol{x} + \boldsymbol{y}^T Q^{-1}\boldsymbol{y}.$$

**Lemma 4.3.**

*(Schur complement) The following linear matrix inequality*

$$\begin{pmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{pmatrix} > 0,$$

*where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$ depend affinely on $x$, is equivalent to each of the following conditions (these conditions are also equivalent to each other):*

$$\begin{aligned} &(1) \quad Q(x) > 0, \quad R(x) - S(x)^T Q(x)^{-1} S(x) > 0, \\ &(2) \quad R(x) > 0, \quad Q(x) - S(x) R(x)^{-1} S(x)^T > 0. \end{aligned}$$

**Lemma 4.4.**

*Assuming that the nonlinear activation functions $y_i[x_i(t)]$ satisfy the Lipschitz condition, it can be proven that*

$$\int_{\upsilon}^{u} [\Psi_i(s) - \Psi_i(\upsilon)]ds \geqslant \frac{1}{2\sigma_i}[\Psi_i(u) - \Psi_i(\upsilon)]^2 \quad (i = 1, 2, \dots, n).$$

**Lemma 4.5.**

*Assuming that the nonlinear activation functions $y_i[x_i(t)]$ satisfy the Lipschitz condition, it can be proven that*

$$\int_{\upsilon}^{u} [\Psi_i(s) - \Psi_i(\upsilon)]ds \leqslant [u - \upsilon][\Psi_i(u) - \Psi_i(\upsilon)] \quad (i = 1, 2, \dots, n).$$

**Lemma 4.6.**

*Let us consider a $2n \times 2n$ Hamiltonian matrix*

$$H = \begin{bmatrix} -\hat{\Gamma} & DD^T \\ -(K_1 + K_2) - \varepsilon I_n & \hat{\Gamma} \end{bmatrix}.$$

*In the above equation, $\varepsilon$ is a sufficiently small positive constant, $I_n$ is a $n \times n$ identity matrix, $\hat{\Gamma} = diag\,\{\gamma_j - \alpha\}$, $D = [A \quad B]$, $K_1 = diag\,\{\sigma_j^2\}$, and*

$$K_2 = diag\left\{\frac{e^{2\alpha\tau_j^*}\sigma_j^2}{1 - r_j^*}\right\}$$

*$(j = 1, 2, \dots, n)$. If $-\hat{\Gamma}$ is a stable matrix and Hamiltonian matrix $H$ has no eigenvalues on the imaginary axis, then the following algebraic Riccati equation*

$$-\hat{\Gamma}^T P - P\hat{\Gamma} + PDD^T P + (K_1 + K_2) + \varepsilon I_n = 0$$

*has a symmetric and positive definite solution $P$ for a given $\alpha > 0$.*

# 5. Stability theorems for time delayed CNNs

After the presentation of the fundamental theory of TDCNNs, the definitions associated with their asymptotic and exponential stability, and the collection of the required lemmas, let us now describe the most important theorems associated with this concept. These theorems are presented in two sections; the first one presents the theorems associated with asymptotic stability, while the second one presents the theorems associated with the exponential stability. For each theorem a short proof is given that summarizes the main points of it.

## 5.1. Theorems for asymptotic stability

The most recent and important theorems associated with the asymptotic stability of time delayed cellular neural networks are presented below:

***Theorem 5.1.***
*For the TDCNN described by equation (1), let us suppose that the cell outputs $y_i(x_i)$ $(i = 1, 2, , \ldots, n)$ satisfy the Lipschitz conditions, and furthermore, the system parameters $\{\alpha_{ij}\}$ and $\{\beta_{ij}\}$ $(i, j = 1, 2, \ldots, n)$ satisfy one of the following conditions:*

$$\frac{\sigma_i}{\gamma_i}\sum_{i=1}^{n}\left(|\alpha_{ij}|+|\beta_{ij}|\right)<1, \tag{i}$$

$$\frac{1}{\gamma_i}\sum_{j=1}^{n}\left[\sigma_j\left(|\alpha_{ij}|+|\beta_{ij}|\right)+\sigma_i\left(|\alpha_{ji}|+|\beta_{ji}|\right)\right]<2, \tag{ii}$$

$$\frac{1}{\gamma_i}\sum_{j=1}^{n}\left(\sigma_j^2|\alpha_{ij}|+|\alpha_{ji}|+\sigma_j|\beta_{ij}|+\sigma_i|\beta_{ji}|\right)<2, \tag{iii}$$

$$\frac{1}{\gamma_i}\sum_{j=1}^{n}\left(|\alpha_{ij}|+\sigma_i^2|\alpha_{ji}|+\sigma_j|\beta_{ij}|+\sigma_i|\beta_{ji}|\right)<2, \tag{iv}$$

$$\frac{1}{\gamma_i}\sum_{j=1}^{n}\left(\sigma_j|\alpha_{ij}|+\sigma_i|\alpha_{ji}|+\sigma_j^2|\beta_{ij}|+|\beta_{ji}|\right)<2. \tag{v}$$

*Then the equilibrium point $\boldsymbol{x}^*$ of the TDCNN described by the equation (1) is globally asymptotically stable, independent of the delays [3].*

***Proof.*** Let us consider the differential equation of the shifted point of the system $\boldsymbol{z}(t) = \boldsymbol{x}(t) - \boldsymbol{x}^*$ in its expanded form

$$\frac{dz_i(t)}{dt} = -\gamma_i z_i(t) + \sum_{j=1}^{n}\alpha_{ij}\left[y_i\left[x_j^*+z_j(t)\right]-y_j(x_j^*)\right]+\sum_{j=1}^{n}\beta_{ij}\left[y_i\left[x_j^*+z_j(t-\tau_j)\right]-y_j(x_j^*)\right] \quad (i=1,2,\ldots,n).$$

To prove the theorem, we have to choose the appropriate Lyapunov function for each one of the conditions defined above. In a more detailed description, the proof of theorem is made as follows:
To use the condition (i), we have to use a Lyapunov function in the form

$$V_1(t)=\sum_{i=1}^{n}\left(|z_i(t)|+\sum_{j=1}^{n}|\beta_{ij}|\sigma_j\int_{t-\tau_j}^{t}y_j(s)ds\right).$$

In this case, the upper right derivative $D^+V(t)$ it is proven to satisfy the inequality

$$D^+V_1(t)\leqslant -\xi_1\sum_{i=1}^{n}|z_i(t)|,$$

where

$$\xi_1 = \min_{1\leqslant i\leqslant n}\left\{\gamma_i\left[1-\frac{\sigma_i}{\gamma_i}\sum_{j=1}^{n}\left(|\alpha_{ji}|+|\beta_{ji}|\right)\right]\right\} > 0.$$

As a consequence of the last relation we can write that

$$V_1(t)+\xi_1\int_0^t\sum_{i=1}^{n}|z_i(s)|ds \leqslant V_1(0)$$

from which it follows that

$$\int_0^{+\infty}\sum_{i=1}^{n}|z_i(t)|dt < \infty$$

and

$$\lim_{t\to\infty}\sum_{i=1}^{n}|z_i(t)| = 0.$$

Therefore, we have $V(t) < V(0)$ and the equilibrium point is asymptotically stable. On the other hand, to use the conditions (ii), (iii) and (iv), the chosen Lyapunov function has the form

$$V_2(t)=\sum_{i=1}^{n}\left[\frac{1}{2}z_i^2(t)+\frac{1}{2}\sum_{j=1}^{n}\left(\sigma_j|\beta_{ij}|\int_{t-\tau_j}^{t}z_j^2(s)ds\right)\right]. \tag{5}$$

In this case, by using the inequality $2\alpha\beta \leqslant \alpha^2+\beta^2$ and the well known condition $|y_i(u)-y_i(v)| \leqslant \sigma_i|u-v|$, the time derivative of the Lyapunov function it is proven to satisfy the inequalities

$$\frac{dV_2[\mathbf{z}(t)]}{dt} \leqslant -\xi_i\sum_{j=1}^{n}z_j^2(t) \quad (i=2,3,4)$$

with the parameters $\xi_2, \xi_3$ and $\xi_4$ for the conditions (ii), (iii) and (iv) respectively, to be estimated as

$$\begin{aligned}
\xi_2 &= \min_{1\leqslant i\leqslant n}\left\{\gamma_i\left[1-\frac{1}{2\gamma_i}\sum_{j=1}^{n}\left[\sigma_j(|\alpha_{ij}|+|\beta_{ij}|)+\sigma_i(|\alpha_{ji}|+|\beta_{ji}|)\right]\right]\right\} > 0,\\
\xi_3 &= \min_{1\leqslant i\leqslant n}\left\{\gamma_i\left[1-\frac{1}{2\gamma_i}\sum_{j=1}^{n}\left(\sigma_j^2|\alpha_{ij}|+|\alpha_{ji}|+\sigma_j|\beta_{ij}|+\sigma_i|\beta_{ji}|\right)\right]\right\} > 0,\\
\xi_4 &= \min_{1\leqslant i\leqslant n}\left\{\gamma_i\left[1-\frac{1}{2\gamma_i}\sum_{j=1}^{n}\left(|\alpha_{ij}|+\sigma_i^2|\alpha_{ji}|+\sigma_j|\beta_{ji}|+\sigma_i|\beta_{ji}|\right)\right]\right\} > 0.
\end{aligned}$$

Finally, to use the condition (v), the chosen Lyapunov function has the form

$$V_3(t)=\sum_{i=1}^{n}\left[\frac{1}{2}z_i^2(t)+\frac{1}{2}\sum_{j=1}^{n}\left(|\beta_{ij}|\int_{t-\tau_j}^{t}z_i^2(s)ds\right)\right]$$

with its time derivative to satisfy the inequality $\dot{V}_3(t) \leqslant -\xi_5\sum_{j=1}^{n}z_i^2(t)$ where

$$\xi_5 = \min_{1\leqslant i\leqslant n}\left\{\gamma_i\left[1-\frac{1}{2\gamma_i}\sum_{j=1}^{n}\left(\sigma_j|\alpha_{ij}|+\sigma_i|\alpha_{ji}|+\sigma_j^2|\beta_{ij}|+|\beta_{ji}|\right)\right]\right\} > 0.$$

By using the same analysis as in condition (i), it is not difficult to prove that the equilibrium point of the system is asymptotically stable if any of the conditions (ii), (iii), (iv), and (v) is satisfied by the system. □

**Theorem 5.2.**
*Let us consider the TDCNN described by equation (3). In this case, there is a unique equilibrium point $\mathbf{z}^*$ characterized by global asymptotical stability if the following properties are satisfied: (a) the matrix $-(A+A^T)$ is a positive definite matrix and (b) the matrix $B$ satisfies the property $\|B\|_2 \leqslant 1$, where $\|B\|_2 = \sqrt{\lambda_M(B^TB)}$ [1].*

**Proof.** To prove the uniqueness of the equilibrium point the contradiction method is used; if we multiply the defining equation of that point – $\mathbf{z}^* - A\Psi(\mathbf{z}^*) - B\Psi(\mathbf{z}^*) = 0$ – by $\Psi^T(\mathbf{z}^*)$ and use the property $\Psi_i^2[z_i(t)] \leqslant z_i\Psi_i[z_i(t)]$ $(i = 1, 2, \ldots, n)$ we finally get

$$\Psi^T(\mathbf{z}^*)\Psi(\mathbf{z}^*) - \frac{1}{2}\Psi^T(\mathbf{z}^*)(A+A^T)\Psi(\mathbf{z}^*) - \Psi^T(\mathbf{z}^*)B\Psi(\mathbf{z}^*) \leqslant 0$$

or equivalently, $\Psi^T(\mathbf{z}^*)\Psi(\mathbf{z}^*) - \Psi^T(\mathbf{z}^*)B\Psi(\mathbf{z}^*) \leqslant 0$ since the matrix $-(A+A^T)$ is a positive definite matrix. On the other hand, by using the well known inequality $\Psi^T(\mathbf{z}^*)B\Psi(\mathbf{z}^*) \leqslant \|B\|_2\|\Psi(\mathbf{z}^*)\|_2^2$ it can be proven that

$$(1 - \|B\|_2)\|\Psi(\mathbf{z}^*)\|_2^2 < 0 \quad (\forall \Psi(\mathbf{z}^*) \neq 0),$$

and

$$\|\Psi(\mathbf{z}^*)\|_2^2 = \sum_{i=1}^{n} \Psi_i^2(z_i^*).$$

But the first of the above conditions is in contradiction with the equation $(1 - \|B\|_2)\|\Psi(\mathbf{z}^*)\|_2^2 \geqslant 0$ implied by the property (b) of the theorem. Therefore, the origin $\mathbf{z}^* = 0$ is the only equilibrium point of the equation $\mathbf{z}^* - A\Psi(\mathbf{z}^*) - B\Psi(\mathbf{z}^*) = 0$. In the next step, it has to be proven that this point is characterized by global asymptotic stability. In order to do this, we use the positive definite Lyapunov function

$$V[\mathbf{z}(t)] = \mathbf{z}^T(t)\mathbf{z}(t) + 2\alpha\sum_{i=1}^{n}\int_0^{z_i}\varphi(s)ds + (\alpha+\beta)\sum_{i=1}^{n}\int_{t-\tau}^{t}\varphi^2[z_i(\zeta)]d\zeta$$

(where $\alpha > 0$ and $\beta > 0$). If we estimate the first derivative of this function and use the inequality $\Psi^T[\mathbf{z}(t)]\Psi[\mathbf{z}(t)] \leqslant \Psi^T[\mathbf{z}(t)]\mathbf{z}(t)$, we finally get

$$\begin{aligned}\dot{V}[\mathbf{z}(t)] \leqslant& \beta\Psi^T[\mathbf{z}(t-\tau)]\Psi[\mathbf{z}(t-\tau)] \leqslant \Psi^T[\mathbf{z}(t)]A^TA\Psi[\mathbf{z}(t)] + \Psi^T[\mathbf{z}(t-\tau)]B^TB\Psi[\mathbf{z}(t-\tau)] + 2\alpha\Psi^T[\mathbf{z}(t)]A\Psi[\mathbf{z}(t)] \\ &- \alpha\Psi^T[\mathbf{z}(t)]\Psi[\mathbf{z}(t)] + \alpha\Psi^T[\mathbf{z}(t)]B^TB\Psi[\mathbf{z}(t)] + \beta\Psi^T[\mathbf{z}(t)]\Psi[\mathbf{z}(t)] - \beta\Psi^T[\mathbf{z}(t-\tau)]\Psi[\mathbf{z}(t-\tau)],\end{aligned}$$

where $\beta = \lambda_M(B^TB)$. On the other hand, by using the inequalities

$$\begin{aligned}\Psi^T[\mathbf{z}(t)]A^TA\Psi[\mathbf{z}(t)] &\leqslant \lambda_M(A^TA)\|\Psi[\mathbf{z}(t)]\|_2^2, \\ 2\Psi^T[\mathbf{z}(t)]A\Psi[\mathbf{z}(t)] = \Psi^T[\mathbf{z}(t)](A+A^T)\Psi[\mathbf{z}(t)] &\geqslant \lambda_m(A+A^T)\|\Psi[\mathbf{z}(t)]\|_2^2\end{aligned}$$

– the first inequality holds for any matrix $A$, while the second one is valid under the condition that the matrix $A+A^T$ is positive definite – the derivative $\dot{V}[\mathbf{z}(t)]$ finally gets the form

$$\dot{V}[\mathbf{z}(t)] \leqslant \left\{\lambda_M(A^TA) + \lambda_M(B^TB) - \alpha\lambda_m[-(A+A^T)] - \alpha[1-\lambda_M(BB^T)]\right\}\|\Psi[\mathbf{z}(b)]\|_2^2.$$

Having estimated analytically the derivative of the Lyapunov functional, it can be proven that this is always negative definite, by tuning appropriately the parameters of the above equations as follows: (a) in the case $\Psi[\mathbf{z}(t)] \neq 0$ and $\mathbf{z}(t) \neq 0$, $\alpha$ should be selected such that

$$\alpha > \left\{\frac{\lambda_M(A^TA) + \lambda_M(B^TB)}{\lambda_m(-AA^T)}\right\},$$

(b) in the case $\Psi[\mathbf{z}(t)] = 0$ and $\mathbf{z}(t) \neq 0$ we must set $\alpha > 1$, and (c) in the case $\Psi[\mathbf{z}(t)] = 0$ and $\mathbf{z}(t) = 0$ the property $\Psi[\mathbf{z}(t-\tau)] \neq 0$ must hold. Furthermore, it is proven that for $|\mathbf{z}(t)| \to \infty$ it is $V[\mathbf{z}(t)] \to \infty$ and therefore, the equilibrium point $\mathbf{x}^*$ is globally asymptotically stable. □

**Theorem 5.3.**
*For a TDCNN described by the equation (3) with the time delay $\tau_i(t)$ to satisfy the inequality $0 \leqslant \tau_i(t) \leqslant h_i$ $(i = 1, 2, \dots, n)$, the origin is a unique and globally asymptotically stable equilibrium point if the following linear matrix inequality holds*

$$\begin{bmatrix} A^T A + (2I + h)\Sigma + 2k - 2I & e^{kh} B \\ e^{kh} B^T & -\mu \end{bmatrix} < 0,$$

*where $h = diag\,\{h_1, h_2, \dots, h_n\}$, $k$ is the degree of exponential stability (see Definition 3.1), $\mu = 1 - d$ and $I$ is the appropriate identity matrix [26].*

***Proof.*** To prove the above theorem, we use a Lyapunov function of the form

$$V[\mathbf{z}(t)] = e^{2kt}\mathbf{z}^{\,T}(t)\mathbf{z}(t) + \int_{t-\tau(t)}^{t} e^{2k\vartheta}\Psi^T[\mathbf{z}(\vartheta)]\Psi[\mathbf{z}(\vartheta)]d\vartheta + \int_{t-\tau(t)}^{t} [s - t - \tau(t)]e^{2ks}\Psi^T[\mathbf{z}(s)]\Psi[\mathbf{z}(s)]ds.$$

If we estimate the derivative $\dot V[\mathbf{z}(t)]$ along the solution of the equation (3) and use the inequality $2Z^T Y \leqslant Z^T Z + \varepsilon^{-1} Y^T Y$ (where $Y$ and $Z$ are vectors or matrices with appropriate dimensions and $\varepsilon > 0$ a positive constant) we get the result

$$\dot V(\mathbf{z}) \leqslant e^{2kt}\Big\{\mathbf{z}^{\,T}\big[2(k-1)I + A^T A + \mu^{-1} e^{2kh} B L^T + (2+h)\Sigma^T\Sigma\big]\mathbf{z}(t)\Big\} + (1-d)\int_{\ell-\tau(\ell)}^{\ell} e^{2ks}\Psi^T[\mathbf{z}(s)]\Psi[\mathbf{z}(s)]ds.$$

But the last term vanishes since it is identified as a positive definite matrix, while the remaining part can be written in the form $\dot V \leqslant X^T \Omega X$ where

$$X = [\Psi^T[\mathbf{z}(t)]\Psi^T(\mathbf{z}[t - \tau(t))]]^T$$

and

$$\Omega = \begin{bmatrix} AA^T + (2I + h)\Sigma + 2k - 2I & e^{kh} B \\ e^{kh} B^T & -Mu \end{bmatrix}.$$

From the last equation it can be easily seen that the given linear matrix inequality guarantees the validity of the condition $\dot V < 0$, which immediately implies the global asymptotic stability of equation (3). □

**Theorem 5.4.**
*The equilibrium point of the equation (1) is globally asymptotically stable if there exist two positive constants $\varepsilon_1$ and $\varepsilon_2$ such that [29]*

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \sum_{j=1}^{n}\left(\frac{\sigma_j|a_{ij}|}{4\varepsilon_1} + \frac{\sigma_j|b_{ij}|}{4\varepsilon_2} + \varepsilon_1\sigma_i|a_{ji}|\right)\right\} > \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\varepsilon_2\sigma_i|b_{ji}|\right\}. \tag{6}$$

***Proof.*** The proof of this theorem is based on the Lemma 4.1. Let us consider the candidate Lyapunov function $V[\mathbf{z}(t)] = \frac{1}{2}\sum_{i=1}^{n} z_i^2(t)$. If we calculate the upper-hand derivative of the function $V$ along a solution of the equation (3) we get

$$D^+V[\mathbf{z}(t)] = \sum_{i=1}^{n} z_i(t)\dot z_i(t) \leqslant \sum_{i=1}^{n}\Big\{-\gamma_i z_i^2(t) + \sum_{j=1}^{n}|a_{ij}|\sigma_j|z_i(t)||z_j(t)| + \sum_{j=1}^{n}|b_{ij}||z_i(t)|\sigma_j|z_j(t-\tau(t))|\Big\}.$$

By using the inequality

$$\alpha\beta \leqslant \frac{\alpha^2}{4\varepsilon} + \varepsilon\beta^2$$

that holds for any $\alpha, \beta \in R$ and any $\varepsilon > 0$ it is clear that

$$|z_i(t)||z_j(t)| \leqslant \frac{z_i^2(t)}{4\varepsilon_1} + \varepsilon_1 z_j^2(t)$$

and

$$|z_i(t)||z_j(t-\tau(t))| \leqslant \frac{z_i^2(t)}{4\varepsilon_2} + \varepsilon_2 z_j^2(t-\tau(t))$$

and by substituting in the last equation, it can be written as

$$D^+V[\boldsymbol{z}(t)] \leqslant -\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \sum_{j=1}^n\left(\frac{|a_{ij}|\sigma_j}{4\varepsilon_1} + \frac{|b_{ij}|\sigma_j}{4\varepsilon_2} + \varepsilon_1\sigma_i|a_{ji}|\right)\right\}\sum_{i=1}^n z_i^2(t) + \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n \varepsilon_2\sigma_i|b_{ji}|\right\}\sum_{i=1}^n z_i^2(t-\tau(t)).$$

For those $t$ satisfying the conditions $\boldsymbol{z}(t) \neq 0$ and $\max_{s\in[-\tau,0]}\|\boldsymbol{z}(t+s)\|_2 = \|\boldsymbol{z}(t)\|_2$ we have

$$D^+V[\boldsymbol{z}(t)] \leqslant -\left[\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \sum_{j=1}^n\left(\frac{|a_{ij}|\sigma_j}{4\varepsilon_1} + \frac{|b_{ij}|\sigma_j}{4\varepsilon_2} + \varepsilon_1\sigma_i|a_{ji}|\right)\right\} - \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n \varepsilon_2\sigma_i|b_{ji}|\right\}\right]\sum_{i=1}^n z_i^2(t) < 0$$

and therefore, according to Lemma 4.1, the solution of equation (3) is globally asymptotically stable. □

By slightly changing the process of proof of the above theorem, some analogous stability conditions can be obtained; these conditions are stated in the next theorem.

### Theorem 5.5.

*The equilibrium point of the equation (1) is globally asymptotically stable if there exist two positive constants $\varepsilon_1$ and $\varepsilon_2$ such that any one of the following conditions holds:*

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \sum_{j=1}^n\left(\frac{\sigma_j^2|a_{ij}|}{4\varepsilon_1} + \frac{\sigma_j^2|b_{ij}|}{4\varepsilon_2} + \varepsilon_1|a_{ji}|\right)\right\} > \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n \varepsilon_2\sigma_i|b_{ji}|\right\}, \tag{7}$$

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \sum_{j=1}^n\left(\frac{|a_{ij}|}{4\varepsilon_1} + \frac{|b_{ij}|}{4\varepsilon_2} + \varepsilon_1\sigma_i^2|a_{ji}|\right)\right\} > \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n \varepsilon_2\sigma_i^2|b_{ji}|\right\}, \tag{8}$$

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \sum_{j=1}^n\left(\frac{\sigma_j^2|a_{ij}|}{4\varepsilon_1} + \frac{|b_{ij}|}{4\varepsilon_2} + \varepsilon_1|a_{ji}|\right)\right\} > \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n \varepsilon_2\sigma_i^2|b_{ji}|\right\}, \tag{9}$$

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \sum_{j=1}^n\left(\frac{|a_{ij}|}{4\varepsilon_1} + \frac{\sigma_j^2|b_{ij}|}{4\varepsilon_2} + \varepsilon_1\sigma_i^2|a_{ji}|\right)\right\} > \max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^n \varepsilon_2|b_{ji}|\right\}. \tag{10}$$

The proofs of these inequalities are similar to the one presented above, but the inequalities used in each case are different, and more specifically:

- For the first equation:

  $$\sigma_j|z_i(t)||z_j(t)| \leqslant \frac{\sigma_j^2 z_i^2(t)}{4\varepsilon_1} + \varepsilon_1 z_j^2(t)$$

  and

  $$\sigma_j|z_i(t)||z_j[t-\tau(t)]| \leqslant \frac{\sigma_j^2 z_i^2(t)}{4\varepsilon_2} + \varepsilon_2 z_j^2[t-\tau(t)].$$

- For the second equation:

$$\sigma_j|z_i(t)||z_j(t)| \leqslant \frac{z_i^2(t)}{4\varepsilon_1} + \varepsilon_1\sigma_j^2 z_j^2(t)$$

and

$$\sigma_j|z_i(t)||z_j[t-\tau(t)]| \leqslant \frac{z_i^2(t)}{4\varepsilon_2} + \varepsilon_2\sigma_j^2 z_j^2[t-\tau(t)].$$

- For the third equation: same as the first, except using $\sigma_j|z_i(t)||z_j[t-\tau(t)]| \leqslant \dfrac{z_i^2(t)}{4\varepsilon_2} + \varepsilon_2\sigma_j z_j^2[t-\tau(t)]$.

- For the fourth equation: same as the second, except using $\sigma_j|z_i(t)||z_j[t-\tau(t)]| \leqslant \dfrac{\sigma_j^2 z_i^2(t)}{4\varepsilon_2} + \varepsilon_2 z_j^2[t-\tau(t)]$.

***Theorem 5.6.***
*For the TDCNN described by equation (1), let us suppose that the cell outputs $y_i(x_i)$ $(i = 1, 2, , \ldots, n)$ satisfy the Lipschitz conditions, and there exists constants $w_i > 0$, $\ell^*_{ij}, v^*_{ij}, \eta^*_{ij}, \zeta^*_{ij}, \ell_{ij}, v_{ij}, \eta_{ij}, \zeta_{ij} \in R$ $(i, j = 1, 2, \ldots, n)$ such that*

$$\sum_{j=1}^{n}\left(|\alpha_{ij}|^{2\eta^*_{ij}}\,\sigma_j^{2\ell^*_{ij}} + \frac{w_j}{w_i}\,|\alpha_{ji}|^{2\zeta^*_{ji}}\,\sigma_i^{2v^*_{ji}} + |\beta_{ij}|^{2\eta_{ij}}\,\sigma_j^{2\ell_{ij}} + \frac{w_j}{w_i}\,|\beta_{ji}|^{2\zeta_{ji}}\,\sigma_i^{2v_{ji}}\right) < 2\gamma_i \tag{11}$$

*$(i, j = 1, 2, \ldots, n)$ in which, $\ell^*_{ij}, v^*_{ij}, \eta^*_{ij}, \zeta^*_{ij}, \ell_{ij}, v_{ij}, \eta_{ij}, \zeta_{ij}$ are real constant numbers satisfying the properties $\ell^*_{ij} + v^*_{ij} = 1$, $\ell_{ij} + v_{ij} = 1$, $\eta^*_{ij} + \zeta^*_{ij} = 1$ and $\eta_{ij} + \zeta_{ij} = 1$, and $\sigma_i$ $(i = 1, 2, \ldots, n)$ are the coefficients of the Lipschitz condition. Then, the equilibrium point $x^*$ of the TDCNN (1) is globally asymptotically stable, independent of delays [5].*

***Proof.*** The proof of this theorem is based on the hypothesis of the bound solutions in the interval $[0, \infty)$ and the fact that a nonnegative function $f(t)$ defined in the above interval, satisfies the equation $f(t) \to 0$ in the limit $t \to \infty$ if it is integrable and uniformly continuous on that interval. For sake of convenience, the proof that follows is associated with the global asymptotic stability of the trivial solution of the TDCNN described by the equation (3); this equation in expanded form it is written as

$$\frac{dz_i(t)}{dt} = -\gamma_i z_i(t) + \sum_{j=1}^{n}\alpha_{ij}\left\{y_j\left[x_j^* + z_j(t)\right] - y_j(x_j^*)\right\} + \sum_{j=1}^{n}\beta_{ij}\left\{y_j\left[x_j^* + z_j(t-\tau_j)\right] - y_j(x_j^*)\right\} \quad (i = 1, 2, \ldots, n)$$

with the indices and the parameters of this equation to be defined as in the introductory section. Following the methodology of the theorems described so far, let us consider the Lyapunov function

$$V(t) = \frac{1}{2}\sum_{i=1}^{n} w_i\left[z_i^2(t) + \sum_{j=1}^{n}|\beta_{ij}|^{2\zeta_{ij}}\,\sigma_j^{2v_{ij}}\int_{t-\tau_j}^{t} z_j^2(s)ds\right], \tag{12}$$

where $w_i > 0$, and $v_{ij}, \zeta_{ij}$ $(i, j = 1, 2, \ldots, n)$ any real constant numbers. The time derivative of this function, is easily estimated as

$$\begin{aligned}\dot V[z(t)] = \sum_{i=1}^{n} w_i\Bigg[ &-\gamma_i z_i^2(t) + \sum_{j=1}^{n}\left(|\alpha_{ij}|^{\eta^*_{ij}}\,\sigma_j^{\ell^*_{ij}}\,|z_i(t)|\right)\left(|\alpha_{ij}|^{\zeta^*_{ij}}\,\sigma_j^{v^*_{ij}}\,|z_j(t)|\right) \\ &+ \sum_{j=1}^{n}\left(|\beta_{ij}|^{\eta_{ij}}\,\sigma_j^{\ell_{ij}}\,|z_i(t)|\right)\left(|\beta_{ij}|^{\zeta_{ij}}\,\sigma_j^{v_{ij}}\,|z_j(t-\tau_j)|\right) + \frac{1}{2}\sum_{j=1}^{n}|\beta_{ij}|^{2\zeta_{ij}}\,\sigma_j^{2v_{ij}}\left[z_j^2(t) - z_j^2(t-\tau_j)\right]\Bigg].\end{aligned}$$

If we estimate the right hand side of the above equation and use the well known inequality $2\alpha\beta \leqslant \alpha^2 + \beta^2$ it can be proven that

$$\begin{aligned}\dot{V}[\boldsymbol{z}(t)] &\leqslant \sum_{i=1}^{n} w_i \Bigg[ -\gamma_i + \frac{1}{2}\sum_{j=1}^{n}\Bigg( |\,\alpha_{ij}\,|^{2\eta^*_{ij}}\ \sigma_j^{2\ell^*_{ij}} + \frac{w_j}{w_i}\,|\,\alpha_{ji}\,|^{2\zeta^*_{ji}}\ \sigma_i^{2\nu^*_{ji}} + |\,\beta_{ij}\,|^{2\eta_{ij}}\ \sigma_j^{2\ell_{ij}} + \frac{w_j}{w_i}\,|\,\beta_{ji}\,|^{2\zeta_{ji}}\ \sigma_i^{2\nu_{ji}}\Bigg)\Bigg] z_i^2(t) \\ &\leqslant -r\sum_{i=1}^{n} z_i^2(t)\end{aligned}$$

with the parameter $r$ to be defined as

$$r = \min_{1\leqslant i\leqslant n}\ w_i\Bigg[\gamma_i - \frac{1}{2}\sum_{j=1}^{n}\Bigg( |\,\alpha_{ij}\,|^{2\eta^*_{ij}}\ \sigma_j^{2\ell^*_{ij}} + \frac{w_j}{w_i}\,|\,\alpha_{ji}\,|^{2\zeta^*_{ji}}\ \sigma_i^{2\nu^*_{ji}} + |\,\beta_{ij}\,|^{2\eta_{ij}}\ \sigma_j^{2\ell_{ij}} + \frac{w_j}{w_i}\,|\,\beta_{ji}\,|^{2\zeta_{ji}}\ \sigma_i^{2\nu_{ji}}\Bigg)\Bigg] > 0.$$

A consequence of the above result is that

$$V(t) + r\int_0^t \sum_{i=1}^{n} z_i^2(s)ds \leqslant V(0)$$

and therefore

$$\int_0^\infty \sum_{i=1}^{n} z_i^2(t)dt < \infty$$

and

$$\lim_{t\to\infty}\sum_{i=1}^{n} z_i^2(t) = 0.$$

From the above equation it is drawn the conclusion that the equilibrium point $z_i(t)$ is globally asymptotically stable for any delay, and therefore, the same is true for the equilibrium point of (1). □

### Theorem 5.7.

*For the TDCNN described by equation (1), let us suppose that the cell outputs $y_i(x_i)$ $(i = 1, 2, , \dots, n)$ satisfy the Lipschitz conditions, and there are positive definite diagonal matrix $P = diag\,\{p_1, p_2, \dots, p_n\}$ and positive definite matrix $D > 0$ such that any one of the following conditions hold:*

$$\begin{pmatrix} 2P\Sigma^{-1}\Gamma - D - PA - A^TP & B^TP \\ PB & D \end{pmatrix} > 0, \tag{i}$$

$$\begin{pmatrix} 2P\Sigma^{-1}\Gamma - D - PA - A^TP & PB \\ B^TP & D \end{pmatrix} > 0, \tag{ii}$$

$$2P\Sigma^{-1}\Gamma - (D + PA + A^TP + PBD^{-1}B^TP) > 0, \tag{iii}$$

$$2P\Sigma^{-1}\Gamma - (D + PA + A^TP + B^TPD^{-1}PB) > 0. \tag{iv}$$

*Then, the equilibrium point $x^*$ of the TDCNN (1) is globally asymptotically stable, independent of delays [8].*

**Proof.** To prove the theorem we start from the Schur complement lemma for the positive definite matrix

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22}\end{bmatrix}$$

with $S_{11} = S_{11}^T$ and $S_{22} = S_{22}^T$; then we know that the inequalities $S_{22} < 0$ and $S_{11} - S_{12}S_{22}^{-1}S_{12}^T < 0$ hold. If we apply these properties for the matrices $S_{11} = 2P\Sigma^{-1}\Gamma - D - PA - A^TP$, $S_{12} = B^TP$ and $S_{22} = D$, it is proven directly that the

condition (i) is equivalent to the condition (iv). In a similar way, if we make the assignments $S_{11} = 2P\Sigma^{-1}\Gamma - D - PA - A^T P$, $S_{12} = PB$ and $S_{22} = D$ and use the same lemma, it is proven that the conditions (ii) and (iii) are equivalent, too.
It can be proven that the theorem is true if the condition (iii) is satisfied. To prove this theorem we have to prove by contradiction that the equilibrium point $z^*$ of the shifted system (3) is unique and then to establish the global stability analysis condition. Starting from the first task, the equation (3) for the equilibrium point of the shifted system (3) has the form

$$\boldsymbol{\Gamma z}^* - A\Psi(\boldsymbol{z}^*) - B\Psi(\boldsymbol{z}^*) = 0 \tag{13}$$

and it is evident that if the condition $\Psi(z^*) = 0$, then it should be $\boldsymbol{z}^* = 0$, too.
To setup the contradiction, let us suppose that $\Psi(\boldsymbol{z}^*) \neq 0$. In this case, multiplying the above equation with the quantity $2\boldsymbol{\Psi}^T(\boldsymbol{z}^*)P$ it can be written as

$$2\boldsymbol{\Psi}^T(\boldsymbol{z}^*)P\Gamma\boldsymbol{z}^* - \boldsymbol{\Psi}^T(\boldsymbol{z}^*)PA\boldsymbol{\Psi}(\boldsymbol{z}^*) - \boldsymbol{\Psi}^T(\boldsymbol{z}^*)A^T P\boldsymbol{\Psi}(\boldsymbol{z}^*) = 2\boldsymbol{\Psi}^T(\boldsymbol{z}^*)PB\boldsymbol{\Psi}(\boldsymbol{z}^*).$$

By using the second property of the equation (4) in its more general form

$$\Psi_i^2[z_i(t)] \leqslant \mu_i z_i\{\Psi_i[z_i(t)]\}, \quad i = 1, 2, \ldots, n$$

the above equation is transformed to the inequality

$$\boldsymbol{\Psi}^T(\boldsymbol{z}^*)P\Gamma\boldsymbol{z}^* = \sum_{i=1}^{n} \Psi_i(z_i^*)p_i c_i z_i^* \geqslant \sum_{i=1}^{n} \frac{p_i c_i}{\mu_i}\Psi_i^2(z_i^*) = \boldsymbol{\Psi}^T(\boldsymbol{z}^*)P\Lambda^{-1}\Gamma\Psi(\boldsymbol{z}^*).$$

This inequality, together with the fundamental matrix inequality $2\boldsymbol{x}^T\boldsymbol{y} \leqslant \boldsymbol{x}^T Q\boldsymbol{x} + \boldsymbol{y}^T Q^{-1}\boldsymbol{y}$ that holds for any vectors $x, y \in R^n$ and positive definite matrix $Q \in R^{n\times n}$ gives

$$\begin{aligned}2\boldsymbol{\Psi}^T(\boldsymbol{z}^*)P\Sigma^{-1}\Gamma\Psi(\boldsymbol{z}^*) - \boldsymbol{\Psi}^T(\boldsymbol{z}^*)PA\Psi(\boldsymbol{z}^*) - \boldsymbol{\Psi}^T(\boldsymbol{z}^*)A^T P\boldsymbol{\Psi}(\boldsymbol{z}^*) &\leqslant 2\boldsymbol{\Psi}^T(\boldsymbol{z}^*)PB\boldsymbol{\Psi}(\boldsymbol{z})^* \\ &\leqslant \boldsymbol{\Psi}^T(\boldsymbol{z})^* D\boldsymbol{\Psi}(\boldsymbol{z}^*) + \boldsymbol{\Psi}^T(\boldsymbol{z}^*)PBD^{-1}B^T P\boldsymbol{\Psi}(\boldsymbol{z}^*).\end{aligned}$$

This inequality implies that

$$\boldsymbol{\Psi}^T(\boldsymbol{z}^*)\left[2P\Sigma^{-1}\Gamma - (\Gamma + PA + A^T P + PBD^{-1}B^T P)\right]\boldsymbol{\Psi}(\boldsymbol{z}^*) \leqslant 0$$

which, in turn, leads to the conclusion that at the equilibrium point is $\boldsymbol{\Psi}(\boldsymbol{z}^*) = 0$ and also $\boldsymbol{z}^* = 0$. But this result contradicts the inequality

$$\boldsymbol{\Psi}^T(\boldsymbol{z}^*)\left[2P\Sigma^{-1}\Gamma - (\Gamma + PA + A^T P + PBD^{-1}B^T P)\right]\boldsymbol{\Psi}(\boldsymbol{z}^*) > 0$$

emerging from the fact that the matrix $2P\Sigma^{-1}\Gamma - (\Gamma + PA + A^T P + PBD^{-1}B^T P)$ is a positive definite matrix, and therefore, the origin $\boldsymbol{z}^*$ of the system (3) and the equivalent point $x^*$ of the system (1) are unique equilibrium points of the corresponding systems.
On the other hand, to establish the condition for the global asymptotic stability, we use the Lyapunov function

$$V[\boldsymbol{z}(t)] = 2\sum_{i=1}^{n} p_i \int_0^{z_i(t)} \Psi_i(s)ds + \int_{t-\tau}^{t} \Psi^T[z(\vartheta)]D\boldsymbol{\Psi}[z(\vartheta)]d\vartheta$$

whose derivative is estimated as

$$\begin{aligned}\dot{V}[\boldsymbol{z}(t)] = &-2\boldsymbol{\Psi}^T[\boldsymbol{z}(t)]P\Gamma\boldsymbol{z}(t) + \boldsymbol{\Psi}^T[\boldsymbol{z}(t)]PA\boldsymbol{\Psi}[\boldsymbol{z}(t)] + \boldsymbol{\Psi}^T[\boldsymbol{z}(t)]A^TP\boldsymbol{\Psi}[\boldsymbol{z}(t)] + 2\boldsymbol{\Psi}^T[\boldsymbol{z}(t)]PB\boldsymbol{\Psi}[\boldsymbol{z}(t-\tau)]\\ &+ \boldsymbol{\Psi}^T[\boldsymbol{z}(t)]D\boldsymbol{\Psi}[\boldsymbol{z}(t)] - \boldsymbol{\Psi}^T[\boldsymbol{z}(t-\tau)]D\boldsymbol{\Psi}[\boldsymbol{z}(t-\tau)].\end{aligned}$$

By using again the fundamental matrix inequality $2\boldsymbol{x}^T\boldsymbol{y} \leqslant \boldsymbol{x}^TQ\boldsymbol{x} + \boldsymbol{y}^TQ^{-1}\boldsymbol{y}$ we get

$$2\boldsymbol{\Psi}^T[\boldsymbol{z}(t)]PB\boldsymbol{\Psi}[\boldsymbol{z}(t-\tau)] \leqslant \boldsymbol{\Psi}^T[\boldsymbol{z}(t-\tau)]D\boldsymbol{\Psi}[\boldsymbol{z}(t-\tau)] + \boldsymbol{\Psi}^T[\boldsymbol{z}(t)]PBD^{-1}B^TP\boldsymbol{\Psi}[\boldsymbol{z}(t)]$$

and the time derivative of the Lyapunov function gets the form

$$\dot{V}[\boldsymbol{z}(t)] \leqslant -\boldsymbol{\Psi}^T[\boldsymbol{z}(t)]W\boldsymbol{\Psi}[\boldsymbol{z}(t)],$$

where $W = 2P\Sigma^{-1}\Gamma - (D + PA + A^TP + PBD^{-1}B^TP)$ a positive definite matrix. Hence we have $\dot{V}[\boldsymbol{z}(t)] < 0$ $(\forall\boldsymbol{\Psi}[\boldsymbol{z}(t)] \neq 0)$, and therefore, the $\boldsymbol{z}^*$ and the equivalent $\boldsymbol{x}^*$ equilibrium points are globally asymptotically stable. The proof of the condition (iv) is based on the same methodology, but in this case we have to use the Lyapunov function

$$V[\boldsymbol{z}(t)] = 2\sum_{i=1}^{n} p_i \int_0^{z_i(t)} \Psi_i(s)ds + \int_{t-\tau}^{t} \boldsymbol{\Psi}^T[\boldsymbol{z}(\vartheta)]B^TPD^{-1}PB\Psi[\boldsymbol{z}(\vartheta)]d\vartheta$$

and the inequalities

$$\begin{aligned}2\boldsymbol{\Psi}^T(\boldsymbol{z}^*)PB\boldsymbol{\Psi}(\boldsymbol{z}^*) &\leqslant \boldsymbol{\Psi}^T(\boldsymbol{z}^*)D\boldsymbol{\Psi}(\boldsymbol{z}^*),\\ 2\boldsymbol{\Psi}^T[\boldsymbol{z}(t)]PB\boldsymbol{\Psi}[\boldsymbol{z}(t-\tau)] &\leqslant \boldsymbol{\Psi}^T[\boldsymbol{z}(t-\tau)]B^TPD^{-1}PB\boldsymbol{\Psi}[\boldsymbol{z}(t-\tau)] + \boldsymbol{\Psi}^T[\boldsymbol{z}(t)]D\boldsymbol{\Psi}[\boldsymbol{z}(t)].\end{aligned}$$

□

Based on the above theorem, the following corollaries can be proven:

### Corollary 5.1.

*The equilibrium point of system (1) is globally asymptotically stable if there are positive definite diagonal matrix $\boldsymbol{P} = diag\{p_1, p_2, \ldots, p_n\}$, a positive definite matrix $D > 0$ and a constant $\beta \geqslant 0$ such that*

*(1) The matrix $-(PA + A^TP + \beta I)$ is positive definite,*

*(2) $\lambda_M(B^TPD^{-1}PB) < \alpha + \beta + 2\lambda_m(P\Sigma^{-1}\Gamma) - \lambda_M(D)$,*

*where $I$ is the identity matrix and $\alpha = \lambda_m[-(PA + A^TP + \beta I)]$.*

### Corollary 5.2.

*The equilibrium point of (1) is globally asymptocally stable if any one of the following conditions hold:*

*(1)* $\begin{pmatrix} 2\Sigma^{-1}\Gamma - I - A - A^T & B^T \\ B & I \end{pmatrix} > 0,$ *(2)* $\begin{pmatrix} 2\Sigma^{-1}\Gamma - I - A - A^T & B \\ B^T & I \end{pmatrix} > 0,$

*(3)* $2\Sigma^{-1}\Gamma - (I + A + A^T + BB^T) > 0,$ *(4)* $2\Sigma^{-1}\Gamma - (I + A + A^T + B^TB) > 0,$

*where $I$ is the identity matrix.*

### Corollary 5.3.

*The equilibrium point of (1) is globally asymptotically stable if any one of the following conditions hold:*

$$(1)\quad \begin{pmatrix} I-A-A^T & B^T \\ B & I \end{pmatrix} > 0, \qquad (2)\quad \begin{pmatrix} I-A-A^T & B \\ B^T & I \end{pmatrix} > 0,$$

$$(3)\quad I-(A+A^T)-BB^T > 0, \qquad (4)\quad I-(A+A^T)-B^TB > 0.$$

### Corollary 5.4.

*The equilibrium point of (1) is globally asymptotically stable if*

*(1) The matrix $A+A^T+\beta I$ is negative definite,*

*(2) $\|B\|_2 < \sqrt{1+\beta+\alpha}$, where $\|B\|_2 = \sqrt{\lambda_M(B^TB)}$ and $\alpha = \lambda_m[-(A+A^T+\beta I)] > 0$.*

### Theorem 5.8.

*For the TDCNN described by equation (1), let us suppose that the cell outputs $y_i(x_i)$ $(i = 1, 2, , \dots, n)$ satisfy the Lipschitz conditions, and there are positive definite symmetric matrices $D, R \in R^{n\times n}$ and positive definite diagonal matrix $P = diag\ \{p_1, p_2, \dots, p_n\}$ such that any of the following conditions hold:*

$$\begin{pmatrix} H & -RA & -RB \\ -A^TR & 2P\Sigma^{-1}\Gamma - D - PA - A^TP & -PB \\ -B^TR & -B^TP & [1-\dot{\tau}(t)]D \end{pmatrix} > 0, \tag{i}$$

$$\begin{pmatrix} 2P\Sigma^{-1}\Gamma - D - PA - A^TP - A^TRH^{-1}RA & -PB - A^TRH^{-1}RB \\ B^TP - B^TRH^{-1}RA & [1-\dot{\tau}(t)]D - B^TRH^{-1}RB \end{pmatrix} > 0, \tag{ii}$$

*where $H = RC + CR$. Then, the equilibrium point $x^*$ of the TDCNN (1) is globally asymptotically stable [6].*

***Proof.*** This theorem is very similar to the previous one and the proof is exactly the same. In the first step we can prove that the conditions (i) and (ii) are, in fact, equivalent if we use the Schur complement lemma; in this case we can verify easily that

$$\begin{pmatrix} 2P\Sigma^{-1}\Gamma - D - PA - A^TP & -PB \\ -B^TP & [1-\dot{\tau}(t)]D \end{pmatrix} - \begin{pmatrix} A^TR \\ B^TR \end{pmatrix} H^{-1} \begin{pmatrix} RA & RB \end{pmatrix} > 0$$

an expression that becomes the condition (ii) after a simplification procedure. In the next step, we can see that the condition (i) implies that

$$\begin{pmatrix} 2P\Sigma^{-1}\Gamma - D - PA - A^TP & -PB \\ -B^TP & [1-\dot{\tau}(t)]D \end{pmatrix} > 0$$

or, by using again the Schur complement

$$2P\Sigma^{-1}\Gamma - D - PA - A^TP - PB\Big\{[1-\dot{\tau}(t)]D\Big\}^{-1} B^TD = 2P\Sigma^{-1}\Gamma - D - PA - A^TP - \frac{1}{1-\dot{\tau}(t)}PBD^{-1}B^TP > 0.$$

Following the proof of the previous theorem, we can prove the uniqueness of the equilibrium point $z^*$ of the shifted system, by a contradiction. Multiplying the equation (13) by the quantity $2\mathbf{\Psi}^T(z^*)P$ and using the properties of the functions $\Psi_i(z^*)$ $(i = 1, 2, \dots, n)$ (see the proof of the previous theorem), we get the inequality

$$\mathbf{\Psi}^T(z^*)\left[2P\Sigma^{-1}\Gamma - \left(D + PA + A^TP + \frac{1}{1-\dot{\tau}(t)}PBD^{-1}B^TP\right)\right]\mathbf{\Psi}(z^*) \leqslant 0$$

that contradicts with the inequality

$$\boldsymbol{\Psi}^T(\boldsymbol{z}^*)\left[2P\Sigma^{-1}\Gamma - \left(D + PA + A^T P + \frac{1}{1-\dot{\tau}(t)}PBD^{-1}B^T P\right)\right]\boldsymbol{\Psi}(\boldsymbol{z}^*) > 0$$

$[\forall\boldsymbol{\Psi}(\boldsymbol{z}^*)]$ emerging from the fact that the matrix

$$D + PA + A^T P + \frac{1}{1-\dot{\tau}(t)}PBD^{-1}B^T P$$

is a positive definite matrix. Therefore, by following the arguments of the previous theorem, we can draw the conclusion that the equilibrium point $\boldsymbol{z}^*$ of the shifted system, or the equivalent point $\boldsymbol{x}^*$ of the original system, is unique. Regarding the proof of the global asymptotic stability, let us use the Lyapunov function

$$V[\boldsymbol{z}(t)] = \boldsymbol{z}^T(t)R\boldsymbol{z}(t) + 2\sum_{i=1}^{n} p_i \int_0^{z_i(t)} \Psi_i(s)ds + \int_{t-\tau}^{t} \Psi^T[\boldsymbol{z}(\vartheta)]D\boldsymbol{\Psi}[\boldsymbol{z}(\vartheta)]d\vartheta$$

whose time derivative is easily estimated as

$$\dot{V}[\boldsymbol{z}(t)] = -\{\boldsymbol{z}^T(t)\Psi^T[\boldsymbol{z}(t)]\}\Psi^T(\boldsymbol{z}[t-\tau(t)])F\begin{pmatrix}\boldsymbol{z}(t)\\ \boldsymbol{\Psi}[\boldsymbol{z}(t)]\\ \Psi(\boldsymbol{z}[t-\tau(t)])\end{pmatrix},$$

where $F$ is the matrix of the condition (i). Since $F > 0$, we have $\dot{V}[\boldsymbol{z}(t)] < 0 \ \forall\boldsymbol{z}(t) \neq 0$ and also $\dot{V}[\boldsymbol{z}(t)] = 0$ if and only if $\boldsymbol{z}(t) = 0$, $\boldsymbol{\Psi}[\boldsymbol{z}(t)] = 0$ as well as $\Psi(\boldsymbol{z}[t-\tau(t)]) = 0$. On the other hand, the function $V(z)$ is radially unbounded since $V[\boldsymbol{z}(t)] \to \infty$ in the limit $\|\boldsymbol{z}(t)\| \to \infty$. Therefore, by following the arguments of the previous theorem, we can draw the conclusion that the equilibrium point $\boldsymbol{z}^*$ of the shifted system, or the equivalent point $\boldsymbol{x}^*$ of the original system, is globally asymptotically stable. □

Based on the above theorem, the following corollaries can be proven [6]:

### *Corollary 5.5.*

*Under the assumptions of the previous theorem, the equilibrium point of* (1) *is globally asymptotically stable if any one of the following conditions hold:*

$$\begin{pmatrix} 2\Gamma & -A & -B \\ -A^T & 2\Lambda^{-1}\Gamma - I - A - A^T & -B \\ -B^T & -B^T & \left[1-\dot{\tau}(t)\right]I \end{pmatrix} > 0, \qquad \text{(i)}$$

$$\begin{pmatrix} 2\Lambda^{-1}\Gamma - I - A - A^T - \frac{1}{2}A^T\Gamma^{-1}A & -B - \frac{1}{2}A^T\Gamma^{-1}B \\ B^T - \frac{1}{2}B^T\Gamma^{-1}A & \left[1-\dot{\tau}(t)\right]I - \frac{1}{2}B^T\Gamma^{-1}B \end{pmatrix} > 0, \qquad \text{(ii)}$$

*with the matrices in the above equations to be defined as previously.*

### Corollary 5.6.

*Under the assumptions of the previous theorem and for a TDCNN with a constant time delay, the equilibrium point of (1) is globally asymptotically stable if any one of the following conditions hold:*

$$\begin{pmatrix} H & -RA & -RB \\ -A^T R & 2P\Lambda^{-1}\Gamma - D - PA - A^T P & -PB \\ -B^T R & -B^T P & D \end{pmatrix} > 0, \quad \text{(i)}$$

$$\begin{pmatrix} 2P\Lambda^{-1}\Gamma - D - PA - A^T P - A^T RH^{-1}RA & -PB - A^T RH^{-1}RB \\ B^T P - B^T RH^{-1}RA & D - B^T RH^{-1}RB \end{pmatrix} > 0, \quad \text{(ii)}$$

*with the matrices in the above equations to be defined as previously.*

### Corollary 5.7.

*Under the assumptions of the previous theorem and for a TDCNN with a constant time delay, the equilibrium point of (1) is globally asymptotically stable if any one of the following conditions hold:*

$$\begin{pmatrix} 2\Gamma & -A & -B \\ -A^T & 2\Lambda^{-1}\Gamma - I - A - A^T & -B \\ -B^T & -B^T & I \end{pmatrix} > 0, \quad \text{(i)}$$

$$\begin{pmatrix} 2\Lambda^{-1}\Gamma - I - A - A^T - \frac{1}{2}A^T\Gamma^{-1}A & -B^T - \frac{1}{2}B^T\Gamma^{-1}A \\ B^T - \frac{1}{2}B^T\Gamma^{-1}A & I - \frac{1}{2}B^T\Gamma^{-1}B \end{pmatrix} > 0, \quad \text{(ii)}$$

*with the matrices in the above equations to be defined as previously.*

### Corollary 5.8.

*Under the assumptions of the previous theorem and for a TDCNN with a constant time delay, the equilibrium point of (1) is globally asymptotically stable if any one of the following conditions hold:*

$$\begin{pmatrix} 2P\Lambda^{-1}\Gamma - D - PA - A^T P - 2A^T RH^{-1}RA & -PB \\ B^T P & D - 2B^T RH^{-1}RB \end{pmatrix} > 0,$$

*with the matrices in the above equations to be defined as previously.*

### Corollary 5.9.

*Under the assumptions of the previous theorem and for a TDCNN with a constant time delay, the equilibrium point of (1) is globally asymptotically stable if any one of the following conditions hold:*

$$\begin{pmatrix} 2P\Lambda^{-1}\Gamma - I - PA - A^T P - 2A^T RH^{-1}RA - 2B^T RH^{-1}RB & -PB \\ B^T P & I \end{pmatrix} > 0, \quad \text{(i)}$$

$$2P\Lambda^{-1}\Gamma - I - PA - A^T P - 2A^T RH^{-1}RA - 2B^T RH^{-1}RB - PBB^T P > 0, \quad \text{(ii)}$$

*with the matrices in the above equations to be defined as previously.*

***Theorem 5.9.***
*For the TDCNN described by equation (1) and using the piecewise linear function $f(x) = 0.5(|x+1| - |x-1|)$, let us suppose that the cell outputs $y_i(x_i)$ $(i = 1, 2, , \dots, n)$ satisfy the Lipschitz conditions, and there are positive definite symmetric matrices $D, R \in R^{n\times n}$ and positive definite diagonal matrix $P = diag\ \{p_1, p_2, \dots, p_n\}$ such that any of the following conditions hold:*

$$\begin{pmatrix} 2R & -RA & -RB \\ -A^TR & 2P - D - PA - A^TP & -PB \\ -B^TR & -B^TP & D \end{pmatrix} > 0, \tag{i}$$

$$\begin{pmatrix} 2P - D - PA - A^TP - \frac{1}{2}A^TRA & -PB - \frac{1}{2}A^TRB \\ -B^TP - \frac{1}{2}B^TRA & D - \frac{1}{2}B^TRB \end{pmatrix} > 0, \tag{ii}$$

$$\begin{pmatrix} 2P - I - PA - A^TP - A^TRA - B^TRB & -PB \\ -B^TP & I \end{pmatrix} > 0, \tag{iii}$$

$$2P - I - PA - A^TP - A^TRA - B^TRB - PBB^TP > 0, \tag{iv}$$

*where $\Lambda = diag\ \{\mu_1, \mu_2, \dots, \mu_n\}$ and $H = RC + CR$. Then, the equilibrium point $x^*$ of the TDCNN (1) is globally asymptotically stable [6].*

The proof of this theorem is based on the use of Corollaries 5.5 and 5.8.

## 5.2. Theorems for exponential stability

The most recent and important theorems associated with the exponential stability of time delayed cellular neural networks are presented below:

***Theorem 5.10.***
*For the TDCNN described by equation (1), let us suppose that the cell outputs $y_i(x_i)$ $(i = 1, 2, , \dots, n)$ satisfy the Lipschitz conditions, and there exist constants $w_i > 0$, $q^*_{ij}, r^*_{ij}, q_{ij}, r_{ij} \in R$, such that*

$$-\gamma_i + \frac{1}{2}\sum_{j=1}^{n}\left(\sigma_j^{2-q^*_{ij}} \mid \alpha_{ij} \mid^{2-r^*_{ij}} + \frac{w_j}{w_i}\sigma_i^{q^*_{ij}} \mid \alpha_{ji} \mid^{r^*_{ji}}\right) + \frac{1}{2}\sum_{j=1}^{n}\left(\sigma_j^{2-q_{ij}} \mid \beta_{ij} \mid^{2-r_{ij}} + \frac{w_j}{w_i}\sigma_i^{q_{ij}} \mid \beta_{ji} \mid^{r_{ij}}\right) < 0, \tag{14}$$

*where $\sigma_i$ $(i = 1, 2, \dots, n)$ are the coefficients of the Lipschitz condition. Then, the equilibrium point $x^*$ of the TDCNN (1) is globally exponentially stable [7].*

***Proof.*** To prove the theorem, let us consider the Lyapunov function

$$V(t) = \frac{1}{2}\sum_{i=1}^{n} w_i\left[z_i^2(t)e^{\varepsilon t} + \sum_{j=1}^{n}\sigma_j^{q_{ij}}|\beta_{ij}|^{r_{ij}}\int_{t-\tau_j}^{t} z_j^2(s)e^{\varepsilon(s+\tau_j)}ds\right], \tag{15}$$

where $\varepsilon$ is a small positive constant such that

$$\frac{\varepsilon}{2} - \gamma_i + \frac{1}{2}\sum_{j=1}^{n}\left(\sigma_j^{2-q^*_{ij}} \mid \alpha_{ij} \mid^{2-r^*_{ij}} + \frac{w_j}{w_i}\sigma^{q^*_{ij}} \mid \alpha_{ji} \mid^{r^*_{ji}}\right) + \frac{1}{2}\sum_{j=1}^{n}\left(\sigma_j^{2-q_{ij}} \mid b_{ij} \mid^{2-r_{ij}} + \frac{w_j}{w_i}\sigma_i^{q_{ij}} \mid \beta ji \mid^{r_{ij}}\right) < 0.$$

The time derivative of the above function is equal to

$$\dot{V}(t)=\sum_{i=1}^{n} w_i\Big[\frac{1}{2}z_i^2(t)\varepsilon e^{\varepsilon t}+z_i(t)\dot{z}_i(t)e^{\varepsilon t}+\frac{1}{2}\sum_{j=1}^{n}\sigma_j^{q_{ij}}\mid\beta_{ij}\mid^{r_{ij}} z_j^2(t)e^{\varepsilon(t+\tau_j)}-\frac{1}{2}\sum_{j=1}^{n}\sigma_j^{q_{ij}}\mid\beta_{ij}\mid^{r_{ij}} z^2(t-\tau_j)e^{\varepsilon t}\Big].$$

By using the elementary mathematical inequality $2\alpha\beta\leqslant\alpha^2+\beta^2$ and performing a lot of but simple mathematical operations it can be proven that

$$\begin{aligned}\dot{V}(t) &\leqslant e^{\varepsilon t}\sum_{i=1}^{n} w_i\Big[\frac{\varepsilon}{2}-\gamma_i+\frac{1}{2}\sum_{j=1}^{n}\Big(\mid\alpha_{ij}\mid^{2-r_{ij}^*}\sigma_j^{2-q_{ij}^*}+\frac{w_j}{w_i}\mid\alpha_{ij}\mid^{r_{ij}^*}\sigma_i^{q_{ij}^*}\Big)\\ &+\frac{1}{2}\sum_{j=1}^{n}\Big(\mid\beta_{ij}\mid^{2-r_{ij}}\sigma_j^{2-q_{ij}}+e^{\varepsilon t}\frac{w_j}{w_i}\mid\beta_{ij}\mid^{r_{ij}}\sigma_i^{q_{ij}}\Big)\Big]z_i^2(t)\leqslant 0\end{aligned}$$

and therefore $V(t)\leqslant V(0)$ $(t\geqslant 0)$, since

$$e^{\varepsilon t}\Big(\min_{1\leqslant j\leqslant n} w_j\Big)\frac{1}{2}\sum_{i=1}^{n} z_i^2(t)\leqslant V(t)\quad (t\geqslant 0)$$

and

$$\begin{aligned}V(0) &= \frac{1}{2}\sum_{i=1}^{n} w_i\Big[(\varphi_i(0)-x_i^*)^2+\sum_{j=1}^{n}\sigma_j^{q_{ij}}\mid\beta_{ij}\mid^{r_{ij}}\int_{-\tau_j}^{0} z_j^2(s)e^{\varepsilon(s+\tau_j)}ds\Big]\\ &\leqslant \frac{1}{2}\Big[\max_{1\leqslant i\leqslant n} a_i+\tau e^{\varepsilon\tau}\sum_{i=1}^{n} w_i\max_{1\leqslant j\leqslant n}\Big(\sigma_j^{q_{ij}}\mid\beta_{ij}\mid^{r_{ij}}\Big)\Big]|\varphi-x^*|.\end{aligned}$$

Then, we easily get

$$\sum_{i=1}^{n} z_i^2(t)\leqslant M|\varphi-x^*|e^{-\varepsilon t}\quad \forall t\leqslant 0,$$

where $M\geqslant 1$ a constant, and therefore, the equilibrium point $z^*$ of the shifted system or the equilibrium point $x^*$ of the original system, is globally exponentially stable. □

Based on the above theorem, we can prove the following corollary:

### *Corollary 5.10.*

*Under the assumptions of the previous theorem and by assuming that the output and the state of each cell is described by a piecewise linear function $f(x)=0.5(|x+1|-|x-1|)$, then if there exist constants $r_{ij}^*, r_{ij}\in R$ $(i,j=1,2,\ldots,n)$ such that*

$$\frac{1}{2}\sum_{i=1}^{n}\Big(\mid\alpha_{ij}\mid^{2-r_{ij}^*}+\mid\alpha_{ji}\mid^{r_{ji}^*}\Big)+\frac{1}{2}\sum_{i=1}^{n}\Big(\mid\beta_{ij}\mid^{2-r_{ij}}+\mid\beta_{ji}\mid^{r_{ji}}\Big)<\gamma_i\quad (i=1,2,\ldots,n)\tag{16}$$

*it is proven that the equilibrium point $\mathbf{x}^*$ of the system (1), is globally exponentially stable.*

### *Theorem 5.11.*

*Suppose that in system (3), the time delay function $\tau(t)$ satisfies the conditions $\dot{\tau}(t)\leqslant\nu<1$ and $1\leqslant\tau(t)\leqslant\bar{\tau}$ where $\bar{\tau}$ is a constant. If there exist positive diagonal matrices $P$ and $Q$ and a positive constant $k$ such that*

$$\Omega=2P\Gamma-2kP-(P|A|\Sigma+\Sigma|A^T|P)-\Sigma Q\Sigma-(1-\dot{\tau}(t))^{-1}e^{2k\bar{\tau}}|PBQ^{-1}B^TP|>0,\tag{17}$$

*then the origin of (3) is exponentially stable [2].*

***Proof.*** The existence of the equilibrium point is guaranteed by the form and the properties of the nonlinear activation functions $y_i[x_i(t)]$ $(i = 1, 2, \dots, n)$, while to examine its stability, we define the Lyapunov function

$$V[z(t)] = e^{2kt} \sum_{i=1}^{n} p_i z_i^2(t) + \int_{t-\tau(t)}^{t} e^{2k\zeta} \Psi^T[z(\zeta)] Q \Psi[z(\zeta)] d\zeta,$$

where $Q = \text{diag}\{q_1, q_2, \dots, q_n\}$ is a positive diagonal matrix, $p_i > 0$ (for the values $i = 1, 2, \dots, n$) and $k$ is a positive constant.

It can be easily proven after simple mathematical manipulations, that the time derivative of the function $V[z(t)]$ along the trajectory of the system (3) satisfies the inequality

$$\begin{aligned} \dot{V}[z(t)] \;&\leqslant\; e^{2kt}\Big\{2kz^T(t)Pz(t) - 2z^T(t)PAz(t) + 2\sum_{i=1}^{n}\sum_{j=1}^{n} p_i|a_{ij}|\sigma_j|z_i(t)||z_j(t)| + 2z^T(t)PB\Psi(z[t-\tau(t)]) \\ &+\Psi^T[z(t)]\Omega\Psi[z(t)] - [1-\dot{\tau}(t)]e^{-2k\tau(t)}\Psi^T(z[t-\tau(t)])\Omega\Psi(z[t-\tau(t)])\Big\}. \end{aligned}$$

By using the appropriate matrix inequalities, this function can be written in the form

$$\begin{aligned} \dot{V}[z(t)] \;&\leqslant\; -e^{2kt}|z^T|\big\{2P\Gamma - 2kP - (P|A|\Sigma + \Sigma|A^T|P) - \Sigma Q\Sigma - [1-\dot{\tau}(t)]^{-1}e^{2k\bar{\tau}}|PBQ^{-1}B^TP|\big\}|z(t)| \\ &= \; e^{-2kt}|z^T(t)|\Omega|z(t)|, \end{aligned}$$

where

$$\Omega = 2P\Gamma - 2kP - (P|A|\Sigma + \Sigma|A^T|P) - \Sigma Q\Sigma - [1-\dot{\tau}(t)]^{-1}e^{2k\bar{\tau}}|PBQ^{-1}B^TP|.$$

Since $\Omega > 0$, we can conclude that $\dot{V}[z(t)] \leqslant 0$ $(\forall z(t) \neq 0)$ and therefore the Lyapunov function satisfies the property $V[z(t)] \leqslant V[z(0)]$. For $t = 0$, the Lyapunov function gets the form

$$V[z(0)] = z^T(0)Pz(0) + \int_{-\tau(t)}^{0} e^{2k\zeta}\Psi^T[z(\zeta)]Q\Psi[z(\zeta)]d\zeta \leqslant \left\{\lambda_M(P) + \lambda_M(Q)\sigma_M^2\frac{1-e^{-2k\bar{\tau}}}{2k}\right\}|\varphi|^2,$$

where $\sigma_M = max(\sigma_i), (i = 1, 2, \dots, n)$ and $|\varphi| = \sup\limits_{-\tau(t)\leqslant\vartheta\leqslant 0} |z(\vartheta)|$. But at the same time we know that the condition $V[z(t)] \geqslant e^{2kt}\lambda_m(P)|z(t)|^2$ holds, implying that

$$e^{2kt}\lambda_m(P)|z(t)|^2 \leqslant \left\{\lambda_M(P) + \lambda_M(Q)\sigma_M^2\frac{1-e^{-2k\bar{\tau}}}{2k}\right\}|\varphi|^2$$

from which we obtain

$$|z(t)| \leqslant \sqrt{\frac{\lambda_M(P) + \lambda_M(Q)\sigma_M^2\dfrac{1-e^{-2k\bar{\tau}}}{2k}}{\lambda_m(P)}}\,|\varphi|e^{-kt}.$$

Since $|z(t)|$ satisfies the property defined in the Definition 1, the origin of (3) is exponentially stable. □

For constant time delay, the previous theorem is particularized as follows:

***Theorem 5.12.***

*Suppose that in system (3), $\tau(t) = \tau$ is a constant delay. If there exist positive diagonal matrices $P$ and $Q$ and a positive constant $k$ such that*

$$\Omega^* = 2P\Gamma - 2kP - (P|A|\Sigma + \Sigma|A^T|P) - \Sigma Q\Sigma - e^{2k\tau}|PBQ^{-1}B^TP| > 0 \tag{18}$$

*then the origin of (3) is exponentially stable [2].*

A similar theorem to Theorem 5.11 regarding the functional form of the associated Lyapunov function can be found in [24]. This theorem uses large matrices of dimensions $7 \times 7$ to establish the LMI condition for exponential stability and it is too lengthy to be presented here; however its proof is similar to the ones of the above theorems and it is based on the proof of the condition $\dot{V}(t) < 0$ for the first time derivative of the selected Luapunov functional.

### Theorem 5.13.

*The equilibrium point of (1) is globally exponentially stable if there exist positive constants $0 < k < \min\{\gamma_i\}$, $D = diag\ \{d_1, d_2, \ldots, d_n\}$ $(d_i > 0, i = 1, 2, \ldots, n)$ and positive definite matrices $Q > 0$ and $P \geqslant 0$ such that*

$$\tau P - 2D(\Gamma - kI)\Sigma^{-1} + DA + A^T D + DQD + e^{2k\tau} B^T Q^{-1} B < 0. \tag{19}$$

*This expression is equivalent to the matrix inequality*

$$\begin{bmatrix} \tau P - 2D(\Gamma - kI)\Sigma^{-1} + DA + A^T D & e^{k\tau} B^T Q^{-1} & DQ \\ e^{k\tau} Q^{-1} B & -Q^{-1} & 0 \\ QD & 0 & -Q \end{bmatrix} < 0,$$

*where $\tau \leqslant 0$, and $I$ is the appropriate identity matrix (in this description the $\tau$ parameter is the upper bound of the bounded function $\tau(t)$) [30].*

**Proof.** The proof of the above theorem is based on Lemma 4.2. Let us define the positive definite Lyapunov functional $V(t) = V_1(t) + V_2(t) + V_3(t)$ where

$$V_1(t) = 2\sum_{i=1}^{n} d_i e^{kt} \int_0^{z_i(t)} \Psi_i(s) ds,$$

$$V_2(t) = e^{2k\tau} \int_{t-\tau t}^{t} e^{ks} \Psi^T(z(s)) B^T Q^{-1} B \Psi(z(t)) ds$$

and

$$V_3(t) = \int_{-\tau(t)}^{0} \int_{t+\beta}^{t} e^{k\alpha} \Psi^T(z(\alpha)) P \Psi(z(\alpha)) d\alpha d\beta.$$

The derivatives of those functions are proven – after mathematical manipulation and by using the lemmas defined above – to satisfy the inequalities

$$\begin{aligned}
\dot{V}_1(t) &\leqslant e^{kt}\Psi^T[z(t)][D(2kI - 2\Gamma)\Sigma^{-1} + DA + A^T D + DQD]\Psi[z(t)] + e^{kt}\Psi^T(z(t - \tau(t)))B^T Q^{-1} B\Psi(z[t - \tau(t)]), \\
\dot{V}_2(t) &\leqslant e^{k(t+2\tau)}\Psi^T[z(t)]B^T Q^{-1} B\Psi[z(t)] - e^{kt}\Psi^T(z[t - \tau(t)])B^T Q^{-1} B\Psi(z[t - \tau(t)]), \\
\dot{V}_3(t) &\leqslant \tau e^{kt}\Psi^T[z(t)]P\Psi[z(t)]
\end{aligned}$$

and therefore the derivative $\dot{V}(t)$ satisfies the inequality

$$\dot{V}(t) \leqslant e^{kt}\left\{\Psi^T[z(t)]\left[D(2kI - 2\Gamma)\Sigma^{-1} + DA + A^T D + DQD + e^{2k\tau}B^T Q^{-1} B + \tau P\right]\Psi[z(t)]\right\} < 0.$$

From the last equation it is clear that $V(t) \leqslant V(0)$ where

$$\begin{aligned}
V(0) &= 2\sum_{i=1}^{n} d_i \int_0^{z_i(0)} \Psi_i(s)ds + e^{2kt}\int_{-\tau(0)}^{0} e^{ks}\Psi^T[z(s)]B^T Q^{-1} B\Psi[z(s)]ds + \int_{-\tau(0)}^{0}\int_{\beta}^{0} e^{k\alpha}\Psi^T[z(\alpha)]P\Psi[z(\alpha)]d\alpha d\beta \\
&\leqslant \left(2\lambda_M(D\Sigma) + \lambda_M(B^T Q^{-1} B)e^{2k\tau}\lambda_M(\Sigma^2)\tau + \lambda_M(P)\lambda_M(\Sigma^2)\tau^2\right)|z(0)|^2.
\end{aligned}$$

If we combine the above inequality with the following one

$$V(t) \leqslant 2\sum_{i=1}^{n} d_i e^{kt}\int_0^{z_i(t)} \Psi_i(s)ds \leqslant e^{kt}2d_i \int_0^{z_i(t)} \Psi_i(s)ds \geqslant e^{kt}\frac{di}{\sigma_i}\Psi_i^2[z_i(t)]$$

by using the result $V(t) < V(0)$ proved in the previous analysis, we get

$$|\Psi_i[z_i(t)]| \leq \sqrt{\frac{\sigma_i(2\lambda_M(D\Sigma)+\lambda_M(B^TQ^{-1}B)e^{2k\tau}\lambda_M(\Sigma^2)\tau+\lambda_M(P)\lambda_M(\Sigma^2)\tau^2)}{d_i}}\exp\left(-\frac{kt}{2}\right)|\boldsymbol{z}(0)| \triangleq \delta_i \exp\left(-\frac{kt}{2}\right)|\boldsymbol{z}(0)|.$$

In the last step we use this result in conjunction with equation (3) to finally get

$$|z_i(t)| \leqslant \left[1+\frac{1}{\gamma_i-(k/2)}\sum_{j=1}^{n}\delta_j\left(|a_{ij}|+|b_{ij}|e^{kt/2}\right)\right]|z(0)|\exp\left(-\frac{kt}{2}\right)$$

proving that the equilibrium point of the system is globally exponentially stable. □

There is a more recent theorem (see [16] and also [25] for a slight variation of it) that can be considered as the generalization of the previous one, in the sense that the lower bound of the time delay is not equal to zero, as in most cases, but this delay is represented in two parts, a constant part $h_1$ and a time varying part $h(t)$ such that $d(t) = h_1 + h(t)$ under the condition $0 \leqslant h(t) \leqslant h_2 - h_1$. In this notation, we consider for simplicity that all the processing units have the same time delay function $h(t)$, and that the time delay gets values in the interval $h_1 \leqslant h(t) \leqslant h_2$. By following the methodology used in the previous theorem, we define the Lyapunov-Krasovskii functional candidate $V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t)$ where

$$\begin{aligned}
V_1(t) &= \boldsymbol{x}^T(t)P\boldsymbol{x}(t) + 2\sum_{i=1}^{n}\lambda_i\int_0^{x_i(t)}\boldsymbol{\Psi}_i(s)ds,\\
V_2(t) &= \int_{t-\frac{h_1}{m}}^{t}\boldsymbol{\xi}^T(s)Q_1\boldsymbol{\xi}(s)ds + \int_{t-d(t)}^{t-h_1}\boldsymbol{x}^T(s)Q_2\boldsymbol{x}(s)ds + \int_{t-h_2}^{t}\boldsymbol{x}^T(s)Q_3\boldsymbol{x}(s)ds,\\
V_3(t) &= \int_{t-\frac{h_1}{m}}^{t}\boldsymbol{y}^T[\boldsymbol{\xi}(s)]M_1\boldsymbol{y}[\boldsymbol{\xi}(s)]ds + \int_{t-d(t)}^{t-h_1}\boldsymbol{y}^T[\boldsymbol{x}(s)]M_2\boldsymbol{y}[\boldsymbol{x}(s)]ds + \int_{t-h_2}^{t}\boldsymbol{y}^T[\boldsymbol{x}(s)]M_3\boldsymbol{y}[\boldsymbol{x}(s)]ds,\\
V_4(t) &= \int_{-\frac{h_1}{m}}^{0}\int_{t+\vartheta}^{t}\dot{\boldsymbol{x}}^T(s)Z_1\dot{\boldsymbol{x}}(s)dsd\vartheta + \int_{-h_1}^{h_2}\int_{t+\vartheta}^{t}\dot{\boldsymbol{x}}^T(s)Z_2\dot{\boldsymbol{x}}(s)dsd\vartheta
\end{aligned}$$

with the matrix function $\boldsymbol{\xi}(s)$ to be defined as

$$\boldsymbol{\xi}(s) = \left[\boldsymbol{x}(s)\ \ \boldsymbol{x}\left(s-\frac{h_1}{m}\right)\ \ \ldots\ \ \boldsymbol{x}\left(s-\frac{m-1}{m}h_1\right)\right]^T.$$

In the above relations, $m \geqslant 1$ is an integer number, $P, Q_1, Q_2, Q_3, M_1, M_2, M_3, Z_1, Z_2 > 0$ are positive definite matrices, while the matrix $\Lambda$ is defined as $\Lambda = \text{diag}\ \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. If we estimate the time derivative of the function $V(t)$ and work as in the previous case by using the appropriate LMI inequalities (too lengthy to present here), it is proven that $\dot{V}(t) < 0$, meaning that the equilibrium point of the system (1) is asymptotically stable. A similar theorem regarding global robust asymptotic stability can be found in [27], while in [19] a Lyapunov-Krasovskii functional of the same type as the one presented here, is used to setup the exponential stability of a special class of CNNs characterized by impulsive effects.

**Theorem 5.14.**
*The equilibrium point of (1) is globally exponentially stable if there exist positive constants $0 < k < \min\{\gamma_i\}$, $D = diag\ \{d_1, d_2, \ldots, d_n\}$ $(d_i > 0, i = 1, 2, \ldots, n)$ and positive definite matrices $Q > 0$ and $P \geqslant 0$ such that*

$$\tau P - 2D(\Gamma - kI)\Sigma^{-1} + DA + A^T D + DQD + \frac{e^{2k\tau}}{1-\tau^*} B^T Q^{-1} B < 0. \tag{20}$$

*This expression is equivalent to the matrix inequality*

$$\begin{bmatrix} \tau P - 2D(\Gamma - kI)\Sigma^{-1} + DA + A^T D & \frac{e^{k\tau}}{\sqrt{1-\tau^*}} B^T Q^{-1} & DQ \\ \frac{e^{k\tau}}{\sqrt{1-\tau^*}} Q^{-1} B & -Q^{-1} & 0 \\ QD & 0 & -Q \end{bmatrix} < 0,$$

*where $0 < \tau^* < 1$ [30].*

***Proof.*** To prove the theorem, let us define a Lyapunov function in the form $V(t) = V_1(t) + \tilde{V}_2(t) + V_3(t)$ where $V_1(t)$ and $V_3(t)$ are the same as in previous theorem, while $\tilde{V}_2(t)$ is defined as

$$\tilde{V}_2(t) = \frac{e^{2k\tau}}{1-\tau^*} \int_{t-\tau(t)}^{t} e^{ks} \Psi^T[\mathbf{z}(t)] B^T Q^{-1} B \Psi[\mathbf{z}(t)] ds$$

with derivative along the trajectories of the system (3) to satisfy the inequality

$$\dot{\tilde{V}}_2(t) \leqslant \frac{e^{k(t+2\tau)}}{1-\tau^*} \Psi^T(\mathbf{z}(t)) B^T Q^{-1} B \Psi(\mathbf{z}(t)) - e^{kt} \Psi^T(\mathbf{z}(t-\tau(t))) B^T Q^{-1} B \Psi(\mathbf{z}(t-\tau(t))).$$

The remaining part of the proof is similar to that of the previous theorem, so it is omitted here. □

By applying the theorem currently proven, one can easily define the following corollaries:

**Corollary 5.11.**
*The equilibrium point of (1) is globally exponentially stable if there exists a positive constant $0 < k < \min\{\gamma_i\}$ such that*

$$(\tau + 1)I + 2(\Gamma - kI)\Sigma^{-1} + A + A^T + \frac{e^{2k\tau}}{1-\tau^*} B^T B < 0.$$

*For $\Gamma = I$ the above equation gets the simplified form*

$$(\tau + 1)I + 2(1 - k)\Sigma^{-1} + A + A^T + \frac{e^{2k\tau}}{1-\tau^*} B^T B < 0.$$

**Theorem 5.15.**
*For the TDCNN described by the equation (1), suppose that the outputs of the $i_{th}$ cell are Lipschitz continuous functions and there exist constants $\alpha_j > 0$ $(j = 1, 2, \ldots, n)$ such that*

$$-\frac{\gamma_j}{\sigma_j} + a_{jj} + \frac{1}{\alpha_j} \sum_{\substack{i=1 \\ i \neq j}}^{n} \alpha_i |a_{ij}| + \frac{1}{\alpha_j} \sum_{i=1}^{n} \alpha_i |b_{ij}| < 0 \quad (j = 1, 2, \ldots, n). \tag{21}$$

*Then the equilibrium point $\mathbf{x}^*$ is globally exponentially stable [4].*

***Proof.*** To prove the above theorem we start from the proposed condition and we choose a small $\varepsilon > 0$ such that $\varepsilon - \gamma_j < 0$ and

$$\frac{\varepsilon - \gamma_j}{\sigma_j} + \alpha_{jj} + \frac{1}{\alpha_j}\sum_{\substack{i=1\\ i\neq j}}^{n} \alpha_i |a_{ij}| + \frac{e^{\varepsilon\tau}}{\alpha_j}\sum_{i=1}^{n} \alpha_i |b_{ij}| < 0 \quad (j = 1, 2, \ldots, n).$$

Next, by considering the Lyapunov function

$$V(t) = \sum_{i=1}^{n} \alpha_i \left[ |x_i(t) - x_i^*| e^{\varepsilon t} + \sum_{j=1}^{n} |b_{ij}| \int_{t-\tau_j}^{t} \left| y_j[x_j(s)] - y_j(x_j^*) \right| e^{\varepsilon(s+\tau_j)} ds \right] \tag{22}$$

its upper right-hand derivative along the solution of the shifted state equation – which can be written as

$$[x_i(t) - x_i^*]' = -\gamma_i[x_i(t) - x_i^*] + \sum_{j=1}^{n} a_{ij}[y_j[x_j(t)] - y_j(x_j^*)] + \sum_{j=1}^{n} b_{ij}[f_j[x_j(t-\tau_j)] - f_j(x_j^*)]$$

it is proven to satisfy the inequality

$$D^+V(t) \leqslant e^{\varepsilon t}\sum_{j=1}^{n} \alpha_j \left[ \frac{\varepsilon - \gamma_i}{\sigma_j} + \alpha_{jj} + \frac{1}{\alpha_j}\sum_{\substack{i=1\\ i\neq j}}^{n} \alpha_i |a_{ij}| + \frac{e^{\varepsilon\tau}}{\alpha_j}\sum_{i=1}^{n} \alpha_i |b_{ij}| \right] |y_j(x_j(t)) - f_j(x_j^*)| \leqslant 0$$

and therefore $V(t) \leqslant V(0)$ $(\forall t \geqslant 0)$ where

$$\begin{aligned} V(0) &= \sum_{i=1}^{n} \alpha_i \left[ |\varphi_i(0) - x_i^*| + \sum_{j=1}^{n} |b_{ij}| \int_{-\tau_j}^{0} \left| y_j[x_j(s)] - y_j(x_j^*) \right| e^{\varepsilon(s+\tau_j)} ds \right] \\ &\leqslant \left[ \max_{1\leqslant i\leqslant n}\{\alpha_i\} + \max_{1\leqslant i\leqslant n}\{\gamma_i\}\tau e^{\varepsilon\tau}\sum_{j=1}^{n} \alpha_i \max_{1\leqslant j\leqslant n}\{|b_{ij}|\} \right] \sup_{-\tau\leqslant t\leqslant 0}\left[ \sum_{i=1}^{n} |\varphi_i(t) - x_i^*| \right]. \end{aligned}$$

By using the last equation we easily get

$$\sum_{i=1}^{n} |x_i(t) - x_i^*| \leqslant M \sup_{-\tau\leqslant t\leqslant 0}\left[ \sum_{i=1}^{n} |\varphi_i(t) - x_i^*| \right] e^{-\varepsilon t} \quad (\forall t \geqslant 0) \tag{23}$$

and, therefore, the equilibrium point of the system is globally exponentially stable. □

By applying this theorem, we can also easily prove the next corollary:

***Corollary 5.12.***
*For the TDCNN described by the equation (1) suppose that the nonlinear activation functions are Lipschitz continuous functions and*

$$\frac{\gamma_j}{\sigma_j} - \alpha_{jj} > \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}| + \sum_{i=1}^{n} |b_{ij}| \quad (i, j = 1, 2, \ldots, n). \tag{24}$$

*Then the equilibrium point $\mathbf{x}^*$ is also globally exponentially stable.*

***Theorem 5.16.***
*For the TDCNN described by the equation (1) suppose that the outputs of the $i_{th}$ cell are Lipschitz continuous and there exist constants $h_{ij}, h^*_{ij}, \ell_{ij}, \ell^*_{ij} \in R$, $\alpha_i > 0$ $(i, j = 1, 2, \ldots, n)$ and $p \geqslant 1$ such that*

$$-p\gamma_i + \sum_{j=1}^{n}\left[(p-1)|a_{ij}|^{(p-h^*_{ij})/(p-1)}\sigma_j^{(p-\ell^*_{ij})/(p-1)} + \frac{\alpha_j}{\alpha_i}|a_{ji}|^{h^*_{ji}}\sigma_i^{\ell^*_{ji}}\right] + \sum_{j=1}^{n}\left[(p-1)|b_{ij}|^{(p-h_{ij})/(p-1)}\sigma_j^{(p-\ell_{ij})/(p-1)} + \frac{\alpha_j}{\alpha_i}|b_{ji}|^{h_{ji}}\sigma_i^{\ell_{ji}}\right] < 0 \quad (i = 1, 2, \ldots, n). \tag{25}$$

*Then the equilibrium point of the system is globally exponentially stable [14].*

***Proof.*** To prove the theorem we start from the proposed inequality, and we choose a small $\varepsilon > 0$ such that

$$p(\varepsilon-\gamma_i) + \sum_{j=1}^{n}\left[(p-1)|a_{ij}|^{(p-h^*_{ij})/(p-1)}\sigma_j^{(p-\ell^*_{ij})/(p-1)} + \frac{\alpha_j}{\alpha_i}|a_{ji}|^{h^*_{ji}}\sigma_i^{\ell^*_{ji}}\right] + \sum_{j=1}^{n}\left[(p-1)|b_{ij}|^{(p-h_{ij})/(p-1)}\sigma_j^{(p-\ell_{ij})/(p-1)} + e^{p\varepsilon\tau}\frac{\alpha_j}{\alpha_i}|b_{ji}|^{h_{ji}}\sigma_i^{\ell_{ji}}\right] < 0 \quad (i = 1, 2, \ldots, n).$$

Now, by defining the Lyapunov function

$$V(t) = \sum_{i=1}^{n}\alpha_i\left[|x_i(t)-x_i^*|^p\, e^{p\varepsilon t} + \sum_{j=1}^{n}|b_{ij}|^{h_{ij}}\sigma_j^{\ell_{ij}}\int_{t-\tau_j}^{t}|x_j(s)-x_j^*|^p\, e^{p\varepsilon(s+\tau_j)}ds\right] \tag{26}$$

its upper right-hand derivative along the solution of the shifted state equation has the form

$$D^+V(t) \leqslant e^{p\varepsilon t}\sum_{i=1}^{n}\Bigg[p\alpha_i\sum_{j=1}^{n}|a_{ij}||y_i(x_j(t))-y_j(x_j^*)||x_i(t)-x_i^*|^{p-1} + p\alpha_i\sum_{j=1}^{n}|b_{ij}||y_i(x_j(t)-\tau_j)-y_j(x_j^*)||x_i(t)-x_i^*|^{p-1} - p\alpha_i\gamma_i|x_i(t)-x_i^*|^p + p\varepsilon\alpha_i|x_i(t)-x_i^*|^p\Bigg] + e^{p\varepsilon t}e^{p\varepsilon\tau}\sum_{i=1}^{n}\alpha_i\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}\sigma_j^{\ell_{ij}}|x_j(t)-x_j^*|^p - e^{p\varepsilon t}\sum_{i=1}^{n}\alpha_i\sum_{j=1}^{n}|b_{ij}|^{h_{ij}}\sigma^{\ell_{ij}}|x_j(t-\tau_j)-x_j^*|^p.$$

By using the inequality $\alpha^m\beta^{1-m} \leqslant m\alpha + (1-m)\beta$ holding for $0 < m < 1$ and $\alpha, \beta > 0$, it can be proven that

$$D^+V(t) \leqslant e^{p\varepsilon t}\sum_{i=1}^{n}\Bigg[p\alpha_i(\varepsilon-\gamma_i) + \alpha_i\sum_{j=1}^{n}\left((p-1)|a_{ij}|^{\xi_1^*}\sigma_j^{\xi_2^*} + \alpha_j|a_{ij}|^{h^*_{ji}}\sigma_i^{\ell^*_{ji}}\right) + \alpha_i\sum_{j=1}^{n}\left((p-1)|b_{ij}|^{\xi_1}\sigma_j^{\xi_2} + e^{p\varepsilon\tau}\alpha_j|b_{ij}|^{h_{ji}}\sigma_i^{\ell_{ji}}\right)\Bigg]|x_i(t)-x_i^*|^p \leqslant 0,$$

where

$$\xi_1 = \frac{p-h_{ij}}{p-1}, \quad \xi_2 = \frac{p-\ell_{ij}}{p-1}, \quad \xi_1^* = \frac{p-h^*_{ij}}{p-1}, \quad \xi_2^* = \frac{p-\ell^*_{ij}}{p-1}$$

and therefore $V(t) \leqslant V(0)$ $(\forall t \geqslant 0)$. But since the functions $V(x)$ and $V(0)$ are proven to satisfy the inequalities

$$V(0) = \sum_{i=1}^{n} \alpha_i \left[ |\varphi_i(0) - x_i^*|^p + \sum_{j=1}^{n} |b_{ij}|^{h_{ij}} \sigma_j^{\ell_{ij}} \int_{-\tau_j}^{0} |x_j(s) - x_j^*|^p e^{p\varepsilon(s+\tau_j)} ds \right]$$
$$\leqslant \left[ \max_{1 \leqslant i \leqslant n} \{\alpha_i\} + \tau e^{p\varepsilon\tau} \sum_{j=1}^{n} \alpha_j \max_{1 \leqslant i \leqslant n} \{|b_{ji}|^{h_{ij}} \sigma_i^{\ell_{ij}}\} \right] \sup_{-\tau \leqslant t \leqslant 0} \sum_{i=1}^{n} |\varphi_i(t) - x_i^*|^p$$

and

$$V(t) \geqslant \sum_{i=1}^{n} \alpha_i |x_i(t) - x_i^*|^p e^{p\varepsilon t} \geqslant e^{p\varepsilon t} \min_{1 \leqslant i \leqslant n} \{\alpha_i\} \sum_{i=1}^{n} |x_i(t) - x_i^*|^p \quad (\forall t \geqslant 0)$$

we have that

$$\sqrt[p]{\sum_{i=1}^{n} |x_i(t) - x_i^*|^p} \leqslant k e^{-\varepsilon t} \sup_{-\tau \leqslant t \leqslant 0} \sqrt[p]{\sum_{i=1}^{n} |\varphi_i(t) - x_i^*|^p} \quad (\forall t > 0),$$

where

$$k = \sqrt[p]{\frac{\max_{1 \leqslant i \leqslant n} \{\alpha_i\} + \tau e^{p\varepsilon\tau} \sum_{j=1}^{n} \alpha_j \max_{1 \leqslant i \leqslant n} \{|b_{ji}|^{h_{ij}} \sigma_i^{\ell_{ij}}\}}{\min_{1 \leqslant i \leqslant n} \{\alpha_i\}}} \leqslant 1$$

and therefore, the origin of the TDCNN is exponentially stable. □

Let us now return to the TDCNN described by the equation (3); this system can be transformed to a more appropriate form by defining the new variable $\hat{z}_i(t) = e^{\alpha t} z_i(t)$. Performing the substitutions $\hat{\varphi}_j[\hat{z}_j(t)] = e^{\alpha t} \varphi_j[z_j(t)]$ and $\tilde{\varphi}_j(\hat{z}_j[t - \tau_j(t)]) = e^{\alpha t} \varphi_j(z_j[t - \tau_j(t)])$, the state equation of the system can be written as

$$\frac{d\hat{z}_i}{dt} = -(\gamma_i - \alpha)\hat{z}_i + \sum_{j=1}^{n} \alpha_{ij} \hat{\varphi}_j[\hat{z}_j(t)] + \sum_{j=1}^{n} \beta_{ij} \tilde{\varphi}_j[\hat{z}_j(t - \tau_j(t))] \quad (i = 1, 2, \dots, n)$$

or, in compact form,

$$\frac{d\hat{\mathbf{z}}}{dt} = -\hat{\Gamma}\hat{\mathbf{z}}(t) + A\hat{\varphi}[\hat{\mathbf{z}}(t)] + B\tilde{\varphi}(\hat{\mathbf{z}}[t - \tau(t)]), \tag{27}$$

where

$$\hat{\mathbf{z}}(t) = [\hat{z}_1(t), \hat{z}_2(t), \dots, \hat{z}_n(t)]^T \in R^n, \quad \hat{\varphi}[\hat{\mathbf{z}}(t)] = \{\hat{\varphi}_1[\hat{z}_1(t)], \hat{\varphi}_2[\hat{z}_2(t)], \dots, \hat{\varphi}_n[\hat{z}_n(t)]\}^T \in R^n$$

and

$$\tilde{\varphi}[\hat{\mathbf{z}}(t) - \boldsymbol{\tau}] = \{\tilde{\varphi}_1(\hat{z}_1[t - \tau_1(t)]), \tilde{\varphi}_2(\hat{z}_2[t - \tau_2(t)])], \dots, \tilde{\varphi}_n(\hat{z}_n[t - \tau_n(t)])\}^T \in R^n.$$

In this case, based on Lemma 4.6, the following theorem can be postulated:

### *Theorem 5.17.*

*For the class of the TDCNNs defined by the equation (27), let us set $\tau_j^* = \max[\tau_j(t)]$ and $r_j^* = \max[\dot{\tau}_j(t)] < 1$. If $-\hat{\Gamma}$ is a stable matrix, and Hamiltonian matrix $H$ defined in Lemma 4.6 for a given $\alpha$, has no eigenvalues on the imaginary matrix, then the equilibrium point $\mathbf{x}^*$ of the system is globally exponentially stable with a stability degree of $\alpha$ [20].*

***Proof.*** To prove the above theorem we note that since $-\hat{\Gamma}$ is stable and Hamiltonian matrix $H$ has no eigenvalues on the imaginary axis, then according to Lemma 4.6, the algebraic Riccati equation has a symmetric and positive definite solution $P$. By defining the Lyapunov function

$$V(t) = \hat{z}^T(t)P\hat{z}(t) + \sum_{j=1}^{n} \frac{e^{2\alpha\tau_j^*}}{1-r_j^*} \int_{t-\tau_j(t)}^{t} \hat{\varphi}_j^2[\hat{z}_j(s)]ds$$

it can be easily verified that $V(t)$ is a non-negative function over $[-\tau, +\infty]$ and that is radially unbounded, namely, $V(t) \to \infty$ as $\hat{z}(t) \to \infty$. Furthermore, the following inequalities are satisfied:

$$|\varphi_j[z_j(t)]| \leqslant \sigma_j|z_j(t)| \quad \text{and} \quad |\hat{\varphi}_j[\hat{z}_j(t)]| = |e^{\alpha t}\varphi_j[z_j(t)]| \leqslant \sigma_j|e^{\alpha t}z_j(t)| = \sigma_j|\hat{z}_j(t)|.$$

By using the well known inequality $X^TY + Y^TX \leqslant X^TX + Y^TY$ holding for any matrices $X$ and $Y$ with appropriate dimensions, the time derivative of the Lyapunov function along the trajectory of the system, it is proven to satisfy the inequality

$$\dot{V}(t) \leqslant \hat{\boldsymbol{z}}(t)[-\hat{\Gamma}P - P\hat{\Gamma} + PDD^TP + (K_1+K_2)]\hat{\boldsymbol{z}}(t) = -\varepsilon\hat{\boldsymbol{z}}^T(t)\hat{\boldsymbol{z}}(t).$$

It can be proven that the inequality $\dot{V}(t) \leqslant -\varepsilon\|\hat{\boldsymbol{z}}(t)\|^2$ holds. This means that $V(t)$ converges to zero asymptotically, and therefore, the equilibrium point is globally asymptotically stable. □

## 6. Relationships and demonstrations

In this section we conclude the presentation of the collection of papers regarding the characterization of the global stability of time delayed CNNs, by identifying relationships between the presented theorems and giving illustrative examples that can help the reader see how they are applied in practice.

A first group of theorems includes Theorems 5.1, 5.6, 5.10, 5.15, and 5.16. The common feature of these theorems is the establishment of a stability criterion (regarding either asymptotic or exponential stability) by using a carefully designed inequality and an appropriately selected Lyapunov-Krasovskii functional. The most general of those theorems is Theorem 5.16, while all the remaining theorems can be deduced as special cases of it. To prove this relationship, let us consider the condition of the Theorem 5.16 given by the inequality (25). If we make the substitutions $p = 2$, $\alpha_i = w_i$, $\alpha_j = w_j$, $\ell_{ij} = q_{ij}$, $h_{ij} = r_{ij}$, $\ell_{ij}^* = q_{ij}^*$ and $h_{ij}^* = r_{ij}^*$, we get the inequality (14), namely, the stability condition of Theorem 5.10; therefore, the Theorem 5.10 is a special case of the Theorem 5.16.

Working in the same way, we note that if we express the inequality (11)) of the Theorem 5.6 in the form

$$-\gamma_i + \frac{1}{2}\sum_{j=1}^{n}\left[\sigma_j^{2\ell_{ij}^*}|\alpha_{ij}|^{2\eta_{ij}^*} + \frac{w_j}{w_i}\sigma_i^{2v_{ij}^*}|\alpha_{ji}|^{2\zeta_{ji}^*}\right] + \frac{1}{2}\sum_{j=1}^{n}\left[\sigma_j^{2\ell_{ij}}|\beta_{ij}|^{2\eta_{ij}} + \frac{w_j}{w_i}\sigma_i^{2v_{ji}}|\beta_{ji}|^{2\zeta_{ji}}\right] < 0 \tag{28}$$

it can be derived from the inequality (14) of Theorem 5.10, by making the substitutions $q_{ij} = 2(\ell_{ij}+1) = 2v_{ij}$, $r_{ij} = 2(\eta_{ij}+1) = \zeta_{ij}$, $q_{ij}^* = 2(\ell_{ij}^*+1) = 2v_{ij}^*$, and $r_{ij}^* = 2(\eta_{ij}^*+1) = \zeta_{ij}^*$ (therefore, Theorem 5.6 is a special case of Theorem 5.10). The last expression, in turn, can be simplified to the conditions (ii), (iii), (iv), and (v) of the Theorem 5.1 if they are expressed in the compatible form

$$-\gamma_i + \frac{1}{2}\sum_{j=1}^{n}\left[\sigma_j|\alpha_{ij}| + \sigma_i|\alpha_{ji}|\right] + \frac{1}{2}\left[\sum_{j=1}^{n}\sigma_j|\beta_{ij}| + \sigma_i|\beta_{ji}|\right] < 0, \tag{ii}$$

$$-\gamma_i + \frac{1}{2}\sum_{j=1}^{n}\left[\sigma_j^2|\alpha_{ij}| + |\alpha_{ji}|\right] + \frac{1}{2}\sum_{j=1}^{n}\left[\sigma_j|\beta_{ij}| + \sigma_i|\beta_{ji}|\right] < 0, \tag{iii}$$

$$-\gamma_i + \frac{1}{2}\sum_{j=1}^{n}\left[|\alpha_{ij}| + \sigma_i^2|\alpha_{ji}|\right] + \frac{1}{2}\sum_{j=1}^{n}\left[\sigma_j|\beta_{ij}| + \sigma_i|\beta_{ji}|\right] < 0, \tag{iv}$$

$$-\gamma_i + \frac{1}{2}\sum_{j=1}^{n}\left[\sigma_j|\alpha_{ij}| + \sigma_i|\alpha_{ji}|\right] + \frac{1}{2}\sum_{j=1}^{n}\left[\sigma_j^2|\beta_{ij}| + |\beta_{ji}|\right] < 0. \tag{v}$$

In this way, we can easily note that:

- The inequality (28) is simplified to the condition (ii) of Theorem (5.1) by making the following substitutions: $w_i = w_j = 1$, $\ell_{ij} = \eta_{ij} = \ell^*_{ij} = \eta^*_{ij} = \frac{1}{2}$ and $v_{ij} = \zeta_{ij} = v^*_{ij} = \zeta^*_{ij} = \frac{1}{2}$.
- The inequality (28) is simplified to the condition (iii) of Theorem (5.1) by making the following substitutions: $w_i = w_j = 1$, $\ell^*_{ij} = 1$, $\ell_{ij} = \eta_{ij} = \eta^*_{ij} = \frac{1}{2}$, $v^*_{ij} = 0$ and $v_{ij} = \zeta_{ij} = \zeta^*_{ij} = \frac{1}{2}$.
- The inequality (28) is simplified to the condition (iv) of Theorem (5.1) by making the following substitutions: $w_i = w_j = 1$, $\ell^*_{ij} = 0$, $\ell_{ij} = \eta_{ij} = \eta^*_{ij} = \frac{1}{2}$, $v^*_{ij} = 1$, and $v_{ij} = \zeta_{ij} = \zeta^*_{ij} = \frac{1}{2}$.
- The inequality (28) is simplified to the condition (v) of Theorem (5.1) by making the following substitutions: $w_i = w_j = 1$, $\ell_{ij} = 1$, $\eta_{ij} = \ell^*_{ij} = \eta^*_{ij} = \frac{1}{2}$, $v_{ij} = 0$, and $\zeta_{ij} = v^*_{ij} = \zeta^*_{ij} = \frac{1}{2}$.

Note, that in these substitutions, all restrictions imposed by Theorem 5.6 are completely safisfied.
On the other hand, Theorems 5.4 and 5.5 are closely related each other (Theorem 5.5 can be derived from/together with Theorem 5.4 after slight modifications) and use the inequality

$$\alpha\beta \leqslant \frac{\alpha^2}{4\varepsilon} + \varepsilon\beta^2.$$

We note that for the value $\varepsilon = \frac{1}{2}$, this inequality gets the form $2\alpha\beta \leqslant \alpha^2 + \beta^2$. Since this inequality is used in Theorems 5.1, 5.6, and 5.10, let us try to find if all these theorems are related in some way. By substituting the values $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ to the inequality (6) we easily get

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \frac{1}{2}\sum_{j=1}^{n}\Big(\sigma_j|\alpha_{ij}| + \sigma_j|\beta_{ij}| + \sigma_i|\alpha_{ji}|\Big)\right\} - \max_{1\leqslant i\leqslant n}\left\{\frac{1}{2}\sum_{j=1}^{n}\sigma_i|\beta_{ji}|\right\} > 0.$$

It is clear that if we substitute in the above inequality the maximum value of the second term with its minimum one, the resulting inequality will be a valid one; therefore we have

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \frac{1}{2}\sum_{j=1}^{n}\Big(\sigma_j|\alpha_{ij}| + \sigma_j|\beta_{ij}| + \sigma_i|\alpha_{ji}|\Big)\right\} - \min_{1\leqslant i\leqslant n}\left\{\frac{1}{2}\sum_{j=1}^{n}\sigma_i|\beta_{ji}|\right\} > 0$$

or equivalently

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \frac{1}{2}\sum_{j=1}^{n}\Big(\sigma_j|\alpha_{ij}| + \sigma_j|\beta_{ij}| + \sigma_i|\alpha_{ji}| + \sigma_i|\beta_{ji}|\Big)\right\} > 0.$$

But this is the defining equation of the parameter $\xi_2$ of the Theorem 5.1; therefore the condition ii of Theorem 5.1 is a special case of Theorem 5.4 for the value $\varepsilon = \frac{1}{2}$.
If we use the values $\varepsilon_1 = \varepsilon_2 = \frac{1}{2}$ in the Theorem 5.5 and work in the same way, we easily get

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \frac{1}{2}\sum_{j=1}^{n}\Big(\sigma_j^2|\alpha_{ij}| + \sigma_j^2|\beta_{ij}| + |\alpha_{ji}| + \sigma_i|\beta_{ji}|\Big)\right\} > 0,$$

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \frac{1}{2}\sum_{j=1}^{n}\Big(|\alpha_{ij}| + |\beta_{ij}| + \sigma_i^2|\alpha_{ji}| + \sigma_i^2|\beta_{ji}|\Big)\right\} > 0,$$

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \frac{1}{2}\sum_{j=1}^{n}\Big(\sigma_j^2|\alpha_{ij}| + |\beta_{ij}| + |\alpha_{ji}| + \sigma_i^2|\beta_{ji}|\Big)\right\} > 0,$$

$$\min_{1\leqslant i\leqslant n}\left\{\gamma_i - \frac{1}{2}\sum_{j=1}^{n}\Big(|\alpha_{ij}| + \sigma_j^2|\beta_{ij}| + \sigma_i^2|\alpha_{ji}| + |\beta_{ji}|\Big)\right\} > 0.$$

These conditions are not the same with the ones associated with the Theorem 5.1, and they can be used in conjunction with this theorem to give new stability results (all we have to do is select the appropriate Lyapunov function and work in the same way as Theorem 5.1). A generalization of Theorems 5.6 and 5.10 for a value of $\varepsilon \neq \frac{1}{2}$ (since these theorems are also based on the inequality $2\alpha\beta \leqslant \alpha^2 + \beta^2$) could be interesting too.
Finally, Theorem 5.15 can be considered as a variation of the condition (i) of Theorem 5.1 if we write the condition (21) in the more compact form

$$\frac{\sigma_j}{\gamma_j \alpha_j} \sum_{i=1}^{n} \alpha_i \left( |a_{ij}| + |b_{ij}| \right) < 1.$$

The main feature of this first group of theorems is that the stability condition they define is expressed as a system of $n$ three-term inequalities: the first term is a function of the coefficients $\gamma_i$ ($i = 1, 2, \ldots, n$), while the second and the third term have the form

$$\sum_{j=1}^{n} \left[ \sigma_j^{\xi_1} |\alpha_{ij}|^{\xi_2} + \sigma_i^{\xi_3} |\alpha_{ji}|^{\xi_4} \right]$$

and

$$\sum_{j=1}^{n} \left[ \sigma_j^{\psi_1} |\beta_{ij}|^{\psi_2} + \sigma_i^{\psi_3} |\beta_{ji}|^{\psi_4} \right]$$

respectively, with the parameters $\xi_1, \xi_2, \xi_3, \xi_4, \psi_1, \psi_2, \psi_3, \psi_4$ having the appropriate values. In other words, in this family of theorems the stability conditions contain linear combinations of powers of the absolute values of the elements of the feedback and the delayed feedback matrices, with the coefficients of those combinations to be the Lipschitz constants raised to the appropriate powers, too.
The second group of theorems includes the Theorems 5.7, 5.8, 5.9 and 5.2, since they are related to each other. More specifically, the condition (ii) – and therefore, the equivalent condition (iii) – of the Theorem 5.7 is the same with the one that emerges by applying the Schur complement lemma to the condition (i) of the Theorem 5.8 for the value $\dot{\tau}(t) = 0$ (therefore, it is associated with a fixed time delay). Regarding the Theorem 5.9 its proof is based on the Corollaries 5.5 and 5.8, which in turn, are associated with the Theorem 5.8. Since Theorems 5.7, 5.8 and 5.9 use $2 \times 2$ and $3 \times 3$ matrices to set up stability criteria for the asymptotic stability type, they can be considered as equivalent to each other, in the sense that they represent the same reality in different ways. The same comments are valid for the condition (iv) of the Theorem 5.7 which is an equivalent expression of condition (i) of the same theorem.
Theorem 5.2 can be associated with Theorem 5.7 via Corollary 5.4 which is a consequence of Theorem 5.7. If we use the value $\beta = 0$ in this corollary it is rephrased as follows:
*The equilibrium point of system (1) is globally asymptotically stable if*

(1) *The matrix $-(A + A^T)$ is positive definite,*

(2) $\sqrt{\lambda_M(B^T B)} < \sqrt{1 + \lambda_m[-(A + A^T)]}$.

Although Theorem 5.2 refers to the shifted and not to the original system as Corollary 5.4, the matrices $A$ and $B$ are not alterd by the conversion of the one system to the other and the above result is true. Since we know that $\lambda_m[-(A+A^T)] > 0$, the validity of the condition (2), ensures the validity of the property (b) of Theorem 5.2.
Regarding the remaining theorems, it can be easily seen that Theorem 5.12 is a special case of Theorem 5.11 with the inequality (18) to be extracted from the inequality (17) if in the last inequality we make the substitution $\dot{\tau}(t) = 0$ – corresponding, therefore, to a constant time delay. In the same way, the Theorem 5.13 is a special case of Theorem 5.14 with the equation (19) to be extracted from the equation (20) if in the last equation make the substitution $\tau^* = 0$. Theorems 5.11 and 5.14 do not seem to be associated in an obvious way even though their stability conditions share some common factors and therefeore, may exist some relationships between them. The same is true for Theorem 5.3 whose stabilily condition if transformed by using the Schur complement lemma, resembles the conditions of Theorems 5.11 and 5.14 (unfortunately Theorem 5.3 is not well documented and there is a confusion regarding the role of some parameters). Finally, Theorem 5.17 uses a Hamiltonian function and can not be related to any of the remaining theorems.
The associations between all these theorems described above are depicted graphically in Figure 1. In this figure, the highlighted theorems are the main theorems that can be simplified to or related to other theorems or they can not be associated with other theorems such as Theorem 5.17.

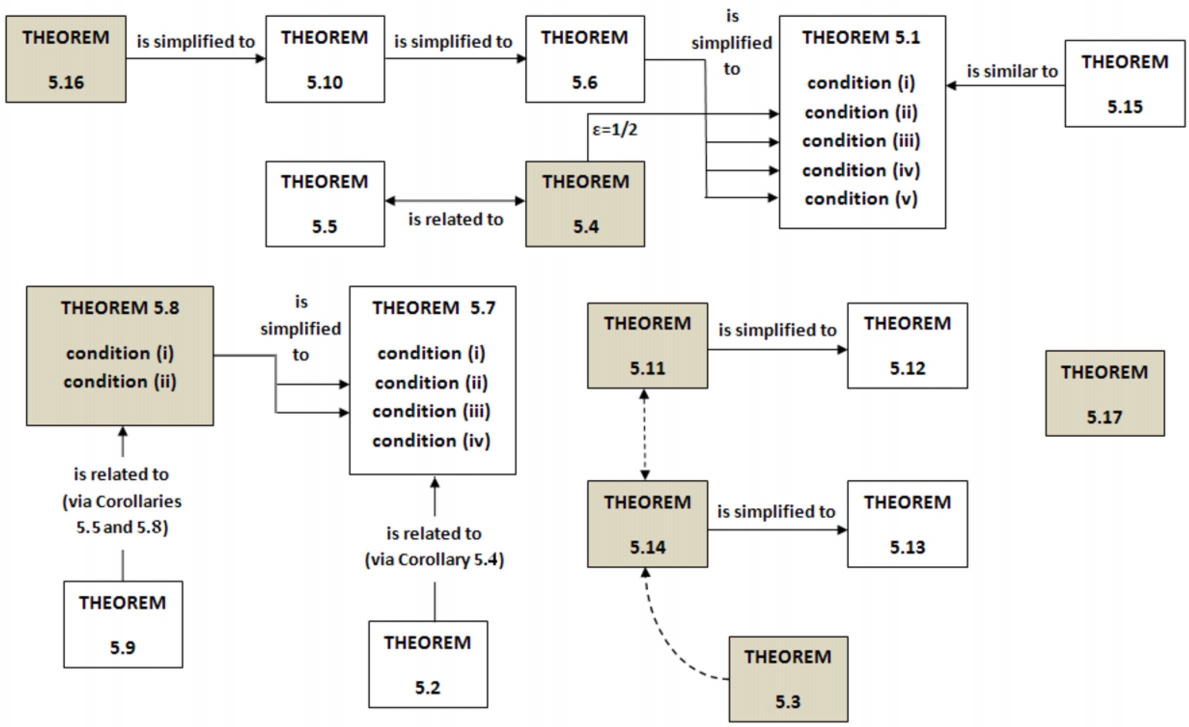


**Figure 1.** The associations between the various theorems describing the asymptotic and exponential stability of TDCNNs. The highligted theorems are the fundamental theorems that generate all the other theorems as special cases of them. The dashed lines indicate possible asscociations that may exist between the related theorems.

After the identification of the relationships between the presented theorems, let us now demonstrate some of them, by using a typical CNN model defined as

$$A = \begin{bmatrix} -0.1 & 0.3 \\ 0.0 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 0.4 \\ 0.15 & 0.5 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

To demonstrate Theorem 5.3, the LMI toolbox has to be used with the vector $\boldsymbol{h}$ as the matrix variable. In this case it is proven that $0 \leqslant \tau_1(t) \leqslant 0.3623$ and $0 \leqslant \tau_2(t) \leqslant 0.3623$, and therefore, according to this theorem, the CNN has a globally asymptotically stable equilibrium point. On the other hand, the application of Theorem 5.4 to the above CNN for which we suppose that $f_i(x) = 0.5(|x+1| - |x-1|)$ $(i = 1, 2)$ and $\tau(t) = |\, t+1 \,| - |\, t-1 \,|$ lead to the results

$$\min_{1 \leqslant i \leqslant 2} \left\{ \gamma_i - \sum_{j=1}^{2} \left( \frac{\sigma_j |a_{ij}|}{4\varepsilon_1} + \frac{\sigma_j |b_{ij}|}{4\varepsilon_2} + \varepsilon_1 \sigma_i |a_{ji}| \right) \right\} = 1.7125 > \max_{1 \leqslant i \leqslant 2} \left\{ \sum_{j=1}^{2} \varepsilon_2 \sigma_i |b_{ji}| \right\} = 0.1$$

and therefore, according to Theorem 5.4, the system described by the equation (1) has a globally asymptotically stable equilibrium point.
Finally, to demonstrate Theorem 5.10 we use the parameters $r_{ij} = r^*_{ij} = q_{ij} = q^*_{ij} = 1$ $(i, j = 1, 2)$ and $w_1 = w_2 = 1$. If we substitute those values in the condition of Theorem 5.10 and for the CNN defined above we can easily get

$$\gamma_1 = 2 > 0.05 \sum_{j=1}^{2} \left( \alpha_{1j} + |\alpha_{j1}| + |\beta_{1j}| + |\beta_{j1}| \right) = 0.1025,$$

$$\gamma_2 = 2 > 0.05 \sum_{j=1}^{2} \left( \alpha_{2j} + |\alpha_{j2}| + |\beta_{2j}| + |\beta_{j2}| \right) = 0.1025$$

and therefore the CNN under consideration is globally exponentially stable. For other demonstrations of these theorems and the remaining theorems, consult the cited references.

We conclude this section by noting that Theorems 5.2, 5.3, 5.7, 5.8, 5.9, 5.11, 5.12, 5.13, and 5.14 are based on the LMI approach. Theorems 5.1, 5.4, 5.5, 5.6, 5.10, 5.15, and 5.16 are based on the establishment of algebraic inequalities characterizing the CNN under consideration. Theorem 5.17 is associated with a Hamiltonian function and it cannot be put in either of those categories. It seems that the theorems of the second category can be used more easily than the theorems of the first category. For complex cases, the last theorems can be used only in conjuction with software tools as the LMI Matlab Control Toolbox. The stability condition and the Lyapunov functional of the most important theorems are summarized in Table 2.

**Table 2.** The stability condition and the Lyapunov function of the fundamental theorems regarding the asymptotic and exponential stability of time delayed CNNs.

| | |
|---|---|
| Theorem 5.4 | $\min_{1\leqslant i\leqslant n}\left\{\gamma_i-\sum_{j=1}^{n}\left(\frac{\sigma_j\lvert a_{ij}\rvert}{4\varepsilon_1}+\frac{\sigma_j\lvert b_{ij}\rvert}{4\varepsilon_2}+\varepsilon_1\sigma_i\lvert a_{ji}\rvert\right)\right\}>\max_{1\leqslant i\leqslant n}\left\{\sum_{j=1}^{n}\varepsilon_2\sigma_i\lvert b_{ji}\rvert\right\},$ <br> $V[\mathbf{z}(t)]=\frac{1}{2}\sum_{i=1}^{n}z_i^2(t)$ |
| Theorem 5.8 | $\begin{pmatrix} H & -RA & -RB \\ -A^TR & 2P\Sigma^{-1}\Gamma-D-PA-A^TP & -PB \\ -B^TR & -B^TP & [1-\dot{\tau}(t)]D \end{pmatrix}>0,$ <br> $\begin{pmatrix} 2P\Sigma^{-1}\Gamma-D-PA-A^TP-A^TRH^{-1}RA & -PB-A^TRH^{-1}RB \\ B^TP-B^TRH^{-1}RA & [1-\dot{\tau}(t)]D-B^TRH^{-1}RB \end{pmatrix}>0,$ <br> $V[\mathbf{z}(t)]=\mathbf{z}^T(t)R\mathbf{z}(t)+2\sum_{i=1}^{n}p_i\int_0^{z_i(t)}\Psi_i(s)ds+\int_{t-\tau}^{t}\Psi^T[\mathbf{z}(\vartheta)]D\Psi[\mathbf{z}(\vartheta)]d\vartheta$ |
| Theorem 5.11 | $\Omega=2P\Gamma-2kP-(P\lvert A\rvert\Sigma+\Sigma\lvert A^T\rvert P)-\Sigma Q\Sigma-(1-\dot{\tau}(t))^{-1}e^{2k\bar{\tau}}\lvert PBQ^{-1}B^TP\rvert>0,$ <br> $V[\mathbf{z}(t)]=e^{2kt}\sum_{i=1}^{n}p_iz_i^2(t)+\int_{t-\tau(t)}^{t}e^{2k\zeta}\Psi^T[\mathbf{z}(\zeta)]Q\Psi[\mathbf{z}(\zeta)]d\zeta$ |
| Theorem 5.14 | $\tau P-2D(\Gamma-kI)\Sigma^{-1}+DA+A^TD+DQD+\frac{e^{2k\tau}}{1-\tau^*}B^TQ^{-1}B<0,$ <br> $V(t)=2\sum_{i=1}^{n}d_ie^{kt}\int_0^{z_i(t)}\Psi_i(s)ds+\frac{e^{2k\tau}}{1-\tau^*}\int_{t-\tau(t)}^{t}e^{ks}\Psi^T[\mathbf{z}(t)]B^TQ^{-1}B\Psi[\mathbf{z}(t)]ds+$ <br> $+\int_{-\tau(t)}^{0}\int_{t+\beta}^{t}e^{k\alpha}\Psi^T(\mathbf{z}(\alpha))P\Psi(\mathbf{z}(\alpha))d\alpha d\beta$ |
| Theorem 5.16 | $-p\gamma_i+\sum_{j=1}^{n}\left[(p-1)\lvert a_{ij}\rvert^{(p-h_{ij}^*)/(p-1)}\sigma_j^{(p-\ell_{ij}^*)/(p-1)}+\frac{\alpha_j}{\alpha_i}\lvert a_{ji}\rvert^{h_{ji}^*}\sigma_i^{\ell_{ji}^*}\right]+$ <br> $+\sum_{j=1}^{n}\left[(p-1)\lvert b_{ij}\rvert^{(p-h_{ij})/(p-1)}\sigma_j^{(p-\ell_{ij})/(p-1)}+\frac{\alpha_j}{\alpha_i}\lvert b_{ji}\rvert^{h_{ji}}\sigma_i^{\ell_{ji}}\right]<0,$ <br> $V(t)=\sum_{i=1}^{n}\alpha_i\left[\lvert x_i(t)-x_i^*\rvert^p\,e^{p\varepsilon t}+\sum_{j=1}^{n}\lvert b_{ij}\rvert^{h_{ij}}\sigma_j^{\ell_{ij}}\int_{t-\tau_j}^{t}\lvert x_j(s)-x_j^*\rvert^p\,e^{p\varepsilon(s+\tau_j)}ds\right]$ |

## 7. Conclusions

The aim of this survey paper is to collect and present in a unified and concise way the most important and recent theorems that describe the global asymptotic and exponential stability of the equilibrium point of time delayed cellular neural networks. Even though the proofs of all these theorems can be found in the cited references, they are too lengthy, include a lot of technicalities and mathematical details and use different symbols to represent the same physical quantities. For these reasons all these theorems were collected and presented here, with short and essential proofs and with uniform notation. In most cases, the CNNs are characterized by a variable time delay $\tau(t)$, and their characterization with respect to their stability is performed by utilizing the direct Lyapunov method as well as the linear matrix inequality technique. According to the basic theory, there are two approaches in applying the first of these methods: the Lyapunov-Krasovskii approach, that usually requires the knowledge of the upper bound of the time-varying delay as well as additional information about the time derivative of the delay function, and the less conservative Lyapunov-Razumikhin approach, in which this information is not necessary. In this paper, the presented theorems are based on the first approach. The linear matrix inequality technique allows the formulation of the stability criteria in a matrix notation but its use is more complicated and may require some mathematical tools such the LMI Matlab Control Toolbox. Therefore, the problem of which theorem to choose and why depends on the specific problem, the parameters used, and maybe the available software tools for solving the problem.

In the last few years, cellular neural networks have been used in many applications and the study of their stability has been paid great attention. A lot of theorems have emerged with this as their motivation, to be related with the understanding of the way they behave in various circumstances. A well known example of applying CNNs is the detection of moving objects in images (see [15]); this technique is based on identifying differences between consecutive images in a sequence of image frames with the object speed usually measured in the frequency domain. In such applications, the role of CNN is to transform an input image to an appropriate output image, and since this is a mapping problem, the CNN must be completely stable in the sense that each trajectory must converge to an equilibrium point. Similar arguments can be formulated to support the use of TDCNNs in other scientific domains.

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