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RESEARCH PAPER

MULTIPLE SOLUTIONS TO BOUNDARY VALUE PROBLEM FOR IMPULSIVE FRACTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor Ivan Dimovski
on the occasion of his 80th Anniversary

Abstract

We study the multiplicity of solutions for fractional differential equations subject to boundary value conditions and impulses. After introducing the notions of classical and weak solutions, we prove the existence of at least three solutions to the impulsive problem considered.

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1. Introduction

The study of boundary value problems (BVP) for fractional differential equations is an intensively developed area. This kind of equations appears, for instance, in some mathematical models in rheology, viscoelasticity, electrochemistry, electromagnetism, and so forth. For details, see the monographs of Kilbas et al. [8], Kiryakova [9], Podlubny [12] and the papers of Mainardi et al. [10], Belmekki et al. [1], Chen and Tang [6], and Bonanno et al. [5].

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In this work, we consider the Dirichlet's boundary value problem for fractional differential equations with impulses

$$(P)\begin{cases} {}_tD_T^{\alpha}\ ({}_0^cD_t^{\alpha}u(t)) + a(t)u(t) = \lambda f(t,u(t)), & t \neq t_j,\ \text{a.e. } t \in [0,T], \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = \mu I_j(u(t_j)), & j = 1,2,\ldots,n, \\ u(0) = u(T) = 0, & \end{cases}$$

where $\lambda \in (0,+\infty), \mu \in \mathbb{R}$ are two parameters,

$0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} = T$,

$\Delta({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u))(t_j) = {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^+) - {}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^-)$, and

${}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^+) = \lim_{t\to t_j^+}({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t))$,

${}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t_j^-) = \lim_{t\to t_j^-}({}_tD_T^{\alpha-1}({}_0^cD_t^{\alpha}u)(t))$.

We impose the following restrictions on the functions involved:

$(H1)$ $f : [0,T]\times\mathbb{R} \to \mathbb{R}$ and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,2,\ldots,n$, are continuous functions, $a \in C([0,T])$, and there are two positive constants a_1 and a_2, such that $0 < a_1 \leq a(t) \leq a_2$.

Given a, b fixed real numbers with $a < b$, we denote by $AC\,([a,b])$ the space of absolutely continuous functions on $[a,b]$. For $0 < \alpha < 1$ and $f \in AC\,([a,b])$, the left and right Riemann-Liouville and Caputo fractional derivatives (see [8], pp. 69–93, [7]) are defined as follows:

$${}_aD_t^{\alpha}f(t) \equiv \frac{d}{dt}\,{}_aD_t^{\alpha-1}f(t) \equiv \frac{d}{dt}I_{a+}^{1-\alpha}f(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_a^t (t-s)^{-\alpha}f(s)ds\right),$$

$${}_tD_b^{\alpha}f(t) \equiv -\frac{d}{dt}\,{}_tD_b^{\alpha-1}f(t) \equiv \frac{d}{dt}I_{b-}^{1-\alpha}f(t)$$
$$:= -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\left(\int_t^b (s-t)^{-\alpha}f(s)ds\right),$$

$${}_a^cD_t^{\alpha}f(t) \equiv {}^cD_{a+}^{\alpha}f(t) := {}_aD_t^{\alpha-1}f'(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha}f'(s)ds\right),$$

$${}_t^cD_b^{\alpha}f(t) \equiv {}^cD_{b-}^{\alpha}f(t) := -\,{}_tD_b^{\alpha-1}f'(t) = -\frac{1}{\Gamma(1-\alpha)}\left(\int_t^b (s-t)^{-\alpha}f'(s)ds\right).$$

Note that, when $\alpha = 1$, ${}_a^cD_t^1f(t) = f'(t)$ and ${}_t^cD_b^1f(t) = -f'(t)$.

The problem (P) is already studied by Bonanno et al. [5] using variational methods. In this paper, we present a variant of Theorem 1.3 in [5], as well as several comments and an example. For reader's convenience, we repeat here several notations, definitions and statements, which are already considered in [5].

For $0 < \alpha < 1$ and $1 < p < \infty$, we consider $E_0^{\alpha,p}(a,b)$ the Banach space which is the closure of $C_0^\infty([a,b])$ with respect to the norm $||u||^p_{E_0^{\alpha,p}(a,b)} = || \, {}_a^c D_t^\alpha u(t)||^p_{L^p(a,b)} + ||u||^p_{L^p(a,b)}$. It is well known that $E_0^{\alpha,p}(a,b)$ is a reflexive and separable Banach space (see [7], Proposition 3.1]). For simplicity, we denote $E_{0,T}^{\alpha,2} = E^\alpha$, and we represent by $|| \cdot ||$ and $|| \cdot ||_\infty$ the norms in $L^2(0,T)$ and $C([0,T])$, respectively: $||u||^2 = \int_0^T |u(t)|^2 dt$, for $u \in L^2(0,T)$ and $||u||_\infty = \max_{t\in[0,T]} |u(t)|$, for $u \in C([0,T])$.

The space E^α has the structure of a Hilbert space where the inner product is given by $(u,v)_\alpha = \int_0^T (\, {}_0^c D_t^\alpha u(t) \, {}_0^c D_t^\alpha v(t) + u(t)v(t))dt$ and the corresponding norm by $||u||^2_\alpha = \int_0^T (| \, {}_0^c D_t^\alpha u(t)|^2 + |u(t)|^2)dt$. It is also easy to check that, if $a \in C([0,T])$ is such that $0 < a_1 \le a(t) \le a_2$, an equivalent norm in E^α is the following:

$$||u||^2_{a,\alpha} = \int_0^T (| \, {}_0^c D_t^\alpha u(t)|^2 + a(t)|u(t)|^2)dt,$$

which we also denote by $|| \cdot ||_\alpha$ for simplicity of the notation.

It is proved in [7] that, for $1/2 < \alpha < 1$,

$$\max_{t\in[0,T]} |u(t)|^2 \le R\|u\|^2_\alpha, \quad \forall u \in E^\alpha, \tag{1.1}$$

where

$$R = \frac{T^{2\alpha-1}}{\Gamma^2(\alpha)(2\alpha-1)}. \tag{1.2}$$

As indicated in [5], the problem of study (P) has a variational structure and its solutions are critical points of a functional φ defined on E^α. In the following, we consider the functional $\varphi : E^\alpha \to \mathbb{R}$ given by

$$\varphi(u) = \Phi(u) - \lambda\Psi(u), \tag{1.3}$$

where

$$\Phi(u) = \frac{1}{2}\int_0^T (| \, {}_0^c D_t^\alpha u(t)|^2 + a(t)u^2(t))dt,$$

$$\Psi(u) = \int_0^T F(t,u(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^n J_j(u(t_j))$$

and

$$F(t,u) = \int\limits_0^u f(t,s)ds, \quad J_j(u) = \int\limits_0^u I_j(t)dt.$$

The main result in this paper is the following:

THEOREM **1.1**. *Suppose that $1/2 < \alpha < 1$ and assumption $(H1)$ holds. Assume also that*

(H2) There exist positive constants $L,\ L_1,\dots,L_n,\ \beta,\ d_1,\dots,d_n$, with $\beta < 2$ and $d_j < 2$, $j = 1,\dots,n$, such that

$$F(t,x) \le L(1 + |x|^{\beta}), \quad \forall (t,x) \in [0,T] \times \mathbb{R} \tag{1.4}$$

and

$$-J_j(x) \le L_j(1 + |x|^{d_j}), \quad \forall x \in \mathbb{R}. \tag{1.5}$$

Suppose also that there exist $r > 0$ and $w \in E^{\alpha}$ such that $\frac{1}{2}||w||_{\alpha}^2 > r$,

$$\int\limits_0^T F(t,w(t))\,dt > 0, \quad \sum_{j=1}^n J_j(w(t_j)) > 0$$

and the following inequality holds:

$$A_l := \frac{\frac{1}{2}\int\limits_0^T (|\,{}_0^cD_t^{\alpha} w(t)|^2 + a(t)w^2(t))dt}{\int\limits_0^T F(t,w(t))\,dt} < A_r := \frac{r}{\int\limits_0^T \max\limits_{|x|\le\sqrt{2Rr}} F(t,x)\,dt}, \tag{1.6}$$

where $R > 0$ is such that (1.1) holds.

Then, for every λ in $\Lambda_r = (A_l, A_r)$, there exists

$$\gamma := \min\left\{ \frac{r - \lambda\int\limits_0^T \max\limits_{|x|\le\sqrt{2Rr}} F(t,x)\,dt}{\max\limits_{|x|\le\sqrt{2Rr}} \sum\limits_{j=1}^n (-J_j(x))}, \frac{\lambda\int\limits_0^T F(t,w(t))\,dt - \frac{1}{2}||w||_{\alpha}^2}{\sum\limits_{j=1}^n J_j(w(t_j))} \right\}$$

such that, for each $\mu \in [0,\gamma)$, the problem (P) has at least three classical solutions, which are critical points of the functional $\varphi : E^{\alpha} \to \mathbb{R}$.

To give an overview of the contents in this paper, in Section **2**, we recall the notions of weak and classical solutions and establish the variational setting of the problem considered. Afterwards, in Section **3** and using a three critical points theorem, we prove our main result. We conclude with some remarks, corollaries and an example.

2. Preliminaries

We represent by $L^p(a,b)$, with $p \geq 1$, the space of Lebesgue $p-$integrable functions over the interval (a,b), in which we consider the usual norm $||u||^p_{L^p(a,b)} = \int\limits_a^b |u(t)|^p dt$. It is well known that $u \in AC([a,b])$ if and only if $u' \in L^1(a,b)$ and $u(t) = u(a) + \int\limits_a^t u'(t)dt$ (see [13], pp. 148–149). On the other hand, for $f \in AC([a,b])$, the following properties hold (see [8],[7]):

$$ {}_a^cD_t^\alpha f(t) = {}_aD_t^\alpha f(t) - \frac{f(a)}{\Gamma(1-\alpha)}(t-a)^{-\alpha}, \; t \in [a,b], \tag{2.1}$$

$$ {}_t^cD_b^\alpha f(t) = {}_tD_b^\alpha f(t) - \frac{f(b)}{\Gamma(1-\alpha)}(b-t)^{-\alpha}, \; t \in [a,b]. \tag{2.2}$$

In particular, for $u \in E^\alpha$, using (2.1) and (2.2) we have

$$ {}_0^cD_t^\alpha u(t) = {}_0D_t^\alpha u(t), \; {}_t^cD_T^\alpha u(t) = {}_tD_T^\alpha u(t). $$

LEMMA **2.1** ([7], Proposition 3.2). *Let $0 < \alpha < 1$. For $u \in E^\alpha$, we have*

$$ ||u|| \leq \frac{T^\alpha}{\Gamma(\alpha+1)} ||{}_0^cD_t^\alpha u||. \tag{2.3}$$

Moreover, for $1/2 < \alpha < 1$,

$$ ||u||_\infty \leq \frac{T^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}} ||{}_0^cD_t^\alpha u||. \tag{2.4}$$

LEMMA **2.2** ([7], Proposition 3.3). *Let $1/2 < \alpha < 1$ and the sequence (u_k) converges weakly to u in E^α, i.e. $(u_k, v)_\alpha \to (u,v)_\alpha$ as $k \to \infty$ for every $v \in E^\alpha$. Then (u_k) converges strongly to u in $C([0,T])$, i.e., $||u_k - u||_\infty \to 0$ as $k \to \infty$.*

In relation with the variational structure of problem (P), and the fact that its solutions are critical points of a functional φ whose domain is the space E^α, we recall the following definitions and statements which are given and discussed by Bonanno et al. [5]:

Definition **2.1** ([5], Definition 2.1). A function $u \in \{u \in AC([0,T]) : \int_{t_j}^{t_{j+1}} (|\,{}_0^cD_t^\alpha u(t)|^2 + u^2(t))dt < \infty,\ j = 0, \ldots, n\}$ is said to be a classical solution of problem (P), if u satisfies the equation a.e. on $[0,T] \backslash \{t_1, t_2, \ldots, t_n\}$, the limits

$$ {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^+) \quad \text{and} \quad {}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u)(t_j^-) $$

exist and satisfy the impulsive condition

$$ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u))(t_j) = \mu I_j(u(t_j)) $$

and the boundary condition $u(0) = u(T) = 0$ holds.

Definition **2.2** ([5], Definition 2.2). A function $u \in E^\alpha$ is said to be a weak solution of problem (P), if for every $v \in E^\alpha$, the following identity holds:

$$ \int_0^T (({}_0^cD_t^\alpha u(t))({}_0^cD_t^\alpha v(t)) + a(t)u(t)v(t))dt + \mu \sum_{j=1}^n I_j(u(t_j))v(t_j) = \lambda \int_0^T f(t, u(t))v(t)dt. \tag{2.5} $$

Assuming the validity of condition $(H1)$, the functional $\varphi : E^\alpha \to \mathbb{R}$ defined in (1.3) is continuous, differentiable and, for any $v \in E^\alpha$, we have

$$ < \varphi'(u), v >= \int_0^T (({}_0^cD_t^\alpha u(t))({}_0^cD_t^\alpha v(t)) + a(t)u(t)v(t))dt + \mu \sum_{j=1}^n I_j(u(t_j))v(t_j) - \lambda \int_0^T f(t, u(t))v(t)dt. $$

From the previous expression and Definition **2.2**, it is clear that the critical points of φ are weak solutions of the problem (P). Moreover, we have also the following

Lemma **2.3** ([5], Lemma 2.1). *The function $u \in E^\alpha$ is a weak solution of (P) if and only if u is a classical solution of (P).*

3. Existence of multiple solutions

Consider X a reflexive real Banach space. A functional $\phi : X \to \mathbb{R}$ is called lower semi-continuous (resp., weakly lower semi-continuous (w.l.s.c.)) if $u_k \to u$ (resp., $u_k \rightharpoonup u$) in X implies $\liminf_{k\to\infty} \phi(u_k) \geq \phi(u)$ (see [11], pp. 3–4). We recall that the functional $\phi : X \to \mathbb{R}$ is coercive if $\phi(u) \to +\infty$ if $||u|| \to \infty$.

The following theorem provides the existence of multiple critical points for differentiable functionals.

THEOREM **3.1**. ([4], Theorem 3.6). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semi continuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^* , $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$ such that:

(i) $\sup\{\Psi(x) : \Phi(x) \leq r\} < r\dfrac{\Psi(\overline{x})}{\Phi(\overline{x})}$,

(ii) for each λ in $\Lambda_r = \left(\dfrac{\Phi(\overline{x})}{\Psi(\overline{x})}, \dfrac{r}{\sup\{\Psi(x) : \Phi(x) \leq r\}}\right)$,
the functional $\Phi - \lambda\Psi$ is coercive.

Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

We remark that, in Theorem **3.1**, if $\sup\{\Psi(x) : \Phi(x) \leq r\} = 0$, then it is possible to consider the interval of parameters $\left(\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, +\infty\right)$.

P r o o f. **(Proof of Theorem 1.1)**. We consider the reflexive real Banach space E^α and the functionals $\Phi : E^\alpha \to \mathbb{R}$ and $\Psi : E^\alpha \to \mathbb{R}$ given, respectively, by

$$\Phi(u) = \frac{1}{2}\int_0^T (|\,{}_0^cD_t^\alpha u(t)|^2 + a(t)u^2(t))dt$$

and

$$\Psi(u) = \int_0^T F(t, u(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^n J_j(u(t_j)),$$

where

$$F(t,u)=\int_0^u f(t,s)ds,\ \ J_j(u)=\int_0^u I_j(t)dt.$$

Similarly to the proof of Theorem 1.3 in [5], it is clear that $\inf_{x\in E^\alpha}\Phi(x)=\Phi(0)=0$ and $\Psi(0)=\int_0^T F(t,0)dt-\frac{\mu}{\lambda}\sum_{j=1}^{n}J_j(0)=0$. Furthermore, the mapping $\Phi: E^\alpha\to\mathbb{R}$ is sequentially weakly lower semi continuous and coercive, what can be easily deduced from $\Phi(u)\geq\frac{1}{2}\|u\|_\alpha^2\to+\infty$, as $\|u\|_\alpha\to+\infty$. Both mappings Φ and Ψ are continuously Gâteaux differentiable and, besides, for every u, $v\in(E^\alpha)^*$, we have

$$<\Phi'(u),v>=\int_0^T({}_0^cD_t^\alpha u(t))({}_0^cD_t^\alpha v(t))+a(t)u(t)v(t))dt,$$

$$<\Psi'(u),v>=\int_0^T f(t,u(t))v(t)dt-\frac{\mu}{\lambda}\sum_{j=1}^{n}I_j(u(t_j))v(t_j).$$

In addition, the operator $\Psi': E^\alpha\to(E^\alpha)^*$ is continuous and compact [14, Proposition 26.2], and the mapping $\Phi: E^\alpha\to\mathbb{R}$ is sequentially weakly lower semi continuous with $\Phi': E^\alpha\to(E^\alpha)^*$ a coercive, hemicontinuous and uniformly monotone operator, what allows to deduce that Φ' admits a continuous inverse on $(E^\alpha)^*$ [14, Theorem 26.A].

If we consider x with $\Phi(x)=\frac{1}{2}\|x\|_\alpha^2\leq r$, then, by (1.1), we have $\Phi(x)\geq\frac{1}{2R}\max_{t\in[0,T]}|x(t)|^2$, so that

$$\begin{aligned}\{x\in E^\alpha:\ \Phi(x)\leq r\}&\subseteq\left\{x:\ \frac{1}{2R}\max_{t\in[0,T]}|x(t)|^2\leq r\right\}\\&=\left\{x:\max_{t\in[0,T]}|x(t)|^2\leq 2Rr\right\}\\&=\left\{x:\max_{t\in[0,T]}|x(t)|\leq\sqrt{2Rr}\right\}.\end{aligned}$$

Hence, using that $\lambda>0$, $\mu\geq0$,

$$\begin{aligned}\sup\{\Psi(x):\Phi(x)\leq r\}&=\sup\left\{\int_0^T F(t,x(t))dt-\frac{\mu}{\lambda}\sum_{j=1}^{n}J_j(x(t_j)):\Phi(x)\leq r\right\}\\&\leq\int_0^T\max_{|x|\leq\sqrt{2Rr}}F(t,x)dt+\frac{\mu}{\lambda}\max_{|x|\leq\sqrt{2Rr}}\sum_{j=1}^{n}(-J_j(x))<\frac{r}{\lambda},\end{aligned}$$

which trivially holds if $\max\limits_{|x|\le\sqrt{2Rr}}\sum\limits_{j=1}^{n}(-J_j(x)) = 0$ since $\lambda < A_r$ and it is also true for $\mu \in [0,\gamma)$ if $\max\limits_{|x|\le\sqrt{2Rr}}\sum\limits_{j=1}^{n}(-J_j(x)) > 0$.

On the other hand,

$$\Psi(w) = \int_0^T F(t,w(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^{n} J_j(w(t_j)) > \frac{\Phi(w)}{\lambda},$$

where we have used again that $\mu < \gamma$. This proves that

$$\frac{\Psi(w)}{\Phi(w)} > \frac{1}{\lambda} > \frac{\sup\{\Psi(x) : \Phi(x) \le r\}}{r}$$

and hypothesis (i) in Theorem **3.1** holds.

Now, for each λ in $\Lambda_r = (A_l, A_r)$, the functional $\Phi - \lambda\Psi$ is coercive. In fact, we prove that it is coercive for every $\lambda > 0$ following a procedure similar to the proof of Theorem 1.3 in [5]. Using (1.4), we have, for $x \in E^\alpha$,

$$\begin{aligned}\int_0^T F(t,x(t))dt &\le L\int_0^T \left(1+|x(t)|^\beta\right) \le LT + LT\|x\|_\infty^\beta \\ &\le LT + LT\left(\sqrt{R}\right)^\beta \|x\|_\alpha^\beta.\end{aligned}$$

Also, from condition (1.5), we obtain, for $j = 1,\ldots,n$,

$$-J_j(x(t_j)) \le L_j(1+|x(t_j)|^{d_j}) \le L_j(1+\|x\|_\infty^{d_j}) \le L_j\left(1+\left(\sqrt{R}\right)^{d_j}\|x\|_\alpha^{d_j}\right)$$

which clearly implies that $\sum\limits_{j=1}^{n}(-J_j(x(t_j))) \le \sum\limits_{j=1}^{n} L_j\left(1+\left(\sqrt{R}\right)^{d_j}\|x\|_\alpha^{d_j}\right)$.

The previous inequalities, due to the sign of the term $\frac{\mu}{\lambda} \ge 0$, produce, for $x \in E^\alpha$, that

$$\begin{aligned}\Psi(x) &\le LT + LT\left(\sqrt{R}\right)^\beta\|x\|_\alpha^\beta + \frac{\mu}{\lambda}\sum_{j=1}^{n} L_j\left(1+\left(\sqrt{R}\right)^{d_j}\|x\|_\alpha^{d_j}\right) \\ &= LT + \frac{\mu}{\lambda}\sum_{j=1}^{n} L_j + LT\left(\sqrt{R}\right)^\beta\|x\|_\alpha^\beta + \frac{\mu}{\lambda}\sum_{j=1}^{n} L_j\left(\sqrt{R}\right)^{d_j}\|x\|_\alpha^{d_j}.\end{aligned}$$

This proves, for $\lambda > 0$ and $x \in E^\alpha$, that the following inequality holds:

$$\Phi(x) - \lambda\Psi(x) \geq \frac{1}{2}\|x\|_\alpha^2 - \lambda\left(LT + \frac{\mu}{\lambda}\sum_{j=1}^{n} L_j + LT\left(\sqrt{R}\right)^\beta \|x\|_\alpha^\beta + \frac{\mu}{\lambda}\sum_{j=1}^{n} L_j\left(\sqrt{R}\right)^{d_j}\|x\|_\alpha^{d_j}\right)$$

which implies, similarly to [5], that $\Phi - \lambda\Psi$ is coercive since β, $d_j < 2$.

Therefore, applying Theorem **3.1** ([4], Theorem 3.6), we obtain that, for each $\lambda \in \Lambda_r$, the functional $\varphi : E^\alpha \to \mathbb{R}$ defined as $\varphi = \Phi - \lambda\Psi$ has at least three different critical points in the space E^α. □

Note that if, in Theorem **1.1**, we consider negative values for the parameter μ, then we have the following result.

THEOREM **3.2**. *Suppose that $1/2 < \alpha < 1$ and assumption $(H1)$ holds. Assume also that:*

$(H2^*)$ There exist positive constants $L,\ L_1, \dots, L_n,\ \beta,\ d_1, \dots, d_n$, with $\beta < 2$ and $d_j < 2$, $j = 1, \dots, n$, such that

$$F(t,x) \leq L(1 + |x|^\beta), \quad \forall (t,x) \in [0,T] \times \mathbb{R} \tag{3.1}$$

and

$$J_j(x) \leq L_j(1 + |x|^{d_j}), \quad \forall x \in \mathbb{R}. \tag{3.2}$$

Suppose that there exist $r > 0$ and $w \in E^\alpha$ such that $\frac{1}{2}||w||_\alpha^2 > r$,

$$\int_0^T F(t, w(t))dt > 0, \quad \sum_{j=1}^{n} J_j(w(t_j)) < 0$$

and (1.6) holds, where $R > 0$ satisfies (1.1).

Then, for every λ in $\Lambda_r = (A_l, A_r)$, there exists

$$\gamma^* := \max\left\{\frac{\lambda\int_0^T \max_{|x|\leq\sqrt{2Rr}} F(t,x)\,dt - r}{\max_{|x|\leq\sqrt{2Rr}} \sum_{j=1}^{n} J_j(x)}, \frac{\lambda\int_0^T F(t,w(t))dt - \frac{1}{2}||w||_\alpha^2}{\sum_{j=1}^{n} J_j(w(t_j))}\right\}$$

such that, for each $\mu \in (\gamma^*, 0]$, the problem (P) has at least three classical solutions, which are three distinct critical points of functional φ in E^α.

P r o o f. Similarly to the proof of Theorem **1.1**, since $\lambda > 0$ and $\mu \in (\gamma^*, 0]$, then we have

$$\sup\{\Psi(x):\Phi(x)\le r\}=\sup\left\{\int_0^T F(t,x(t))dt-\frac{\mu}{\lambda}\sum_{j=1}^n J_j(x(t_j)):\Phi(x)\le r\right\}$$
$$\le\int_0^T \max_{|x|\le\sqrt{2Rr}}F(t,x)dt-\frac{\mu}{\lambda}\max_{|x|\le\sqrt{2Rr}}\sum_{j=1}^n J_j(x)<\frac{r}{\lambda},$$

which is obviously valid if $\max_{|x|\le\sqrt{2Rr}}\sum_{j=1}^n J_j(x)=0$ since $\lambda<A_r$ and it is also true for $\mu\in(\gamma^*,0]$ if $\max_{|x|\le\sqrt{2Rr}}\sum_{j=1}^n J_j(x)>0$.

Moreover, for $\mu\in(\gamma^*,0]$,

$$\Psi(w)=\int_0^T F(t,w(t))dt-\frac{\mu}{\lambda}\sum_{j=1}^n J_j(w(t_j))>\frac{\Phi(w)}{\lambda},$$

which proves that

$$\frac{\Psi(w)}{\Phi(w)}>\frac{1}{\lambda}>\frac{\sup\{\Psi(x):\Phi(x)\le r\}}{r}.$$

Now, for each λ in $\Lambda_r=(A_l,A_r)$, the functional $\Phi-\lambda\Psi$ is coercive. Indeed, from (3.1), we get, for $x\in E^\alpha$,

$$\int_0^T F(t,x(t))dt\le LT+LT\left(\sqrt{R}\right)^\beta\|x\|_\alpha^\beta$$

and, using (3.2), we have, for $j=1,\dots,n$,

$$J_j(x(t_j))\le L_j(1+|x(t_j)|^{d_j})\le L_j\left(1+\left(\sqrt{R}\right)^{d_j}\|x\|_\alpha^{d_j}\right),$$

which implies

$$\sum_{j=1}^n J_j(x(t_j))\le\sum_{j=1}^n L_j\left(1+\left(\sqrt{R}\right)^{d_j}\|x\|_\alpha^{d_j}\right).$$

Now, since $-\frac{\mu}{\lambda}\ge 0$, we get, for $x\in E^\alpha$, that

$$\begin{aligned}\Psi(x) &= \int_0^T F(t,x(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^n J_j(x(t_j)) \\ &\le LT + LT\left(\sqrt{R}\right)^\beta \|x\|_\alpha^\beta - \frac{\mu}{\lambda}\sum_{j=1}^n L_j\left(1+\left(\sqrt{R}\right)^{d_j}\|x\|_\alpha^{d_j}\right) \\ &= LT - \frac{\mu}{\lambda}\sum_{j=1}^n L_j + LT\left(\sqrt{R}\right)^\beta \|x\|_\alpha^\beta - \frac{\mu}{\lambda}\sum_{j=1}^n L_j\left(\sqrt{R}\right)^{d_j}\|x\|_\alpha^{d_j}.\end{aligned}$$

Hence, for $\lambda > 0$ and $x \in E^\alpha$, we have

$$\begin{aligned}&\Phi(x) - \lambda\Psi(x) \ge \frac{1}{2}\|x\|_\alpha^2 \\ &- \lambda\left(LT - \frac{\mu}{\lambda}\sum_{j=1}^n L_j + LT\left(\sqrt{R}\right)^\beta \|x\|_\alpha^\beta - \frac{\mu}{\lambda}\sum_{j=1}^n L_j\left(\sqrt{R}\right)^{d_j}\|x\|_\alpha^{d_j}\right),\end{aligned}$$

and $\Phi - \lambda\Psi$ is clearly coercive again since β, $d_j < 2$. □

Theorem **3.3**. *Suppose that $1/2 < \alpha < 1$ and assumptions $(H1)$ and $(H2)$ hold. Suppose also that there exist two constants $r > 0$ and $\delta > 0$ such that*

$$\begin{aligned} r < \widetilde{W} := &\frac{1}{(\Gamma(1-\alpha))^2}\frac{\delta^2}{2}\left(\frac{T}{4}\right)^{1-2\alpha}\frac{6\alpha^2-19\alpha+16}{(1-\alpha)^2(3-2\alpha)(2-\alpha)} \\ &+ \frac{1}{2}\frac{4^2\delta^2}{T^2}\int_0^{\frac{T}{4}}[a(t)+a(T-t)]t^2\,dt + \frac{1}{2}\delta^2\int_{\frac{T}{4}}^{\frac{3T}{4}} a(t)\,dt \end{aligned} \tag{3.3}$$

and

$$\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\delta)dt > 0, \quad \int_0^{\frac{T}{4}}\left[F\left(t,\frac{4\delta}{T}t\right) + F\left(T-t,\frac{4\delta}{T}t\right)\right]dt \ge 0. \tag{3.4}$$

Let $R > 0$ be such that (1.1) holds and assume also that

$$A_l := \frac{\widetilde{W}}{\int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\delta)dt} < A_r := \frac{r}{\int_0^T \max_{|x|\le\sqrt{2Rr}} F(t,x)\,dt}. \tag{3.5}$$

Then, for every λ in $\Lambda_r = (A_l, A_r)$, there exists

$$\gamma := \min \left\{ \frac{r - \lambda \int_0^T \max_{|x| \le \sqrt{2Rr}} F(t,x)\,dt}{\max_{|x| \le \sqrt{2Rr}} \sum_{j=1}^n (-J_j(x))}, \frac{\lambda \int_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\delta)dt - \widetilde{W}}{\max_{x \in [0,\delta]} \sum_{j=1}^n J_j(x)} \right\}$$

such that, for each $\mu \in [0, \gamma)$, the problem (P) has at least three classical solutions, which are three distinct critical points of functional φ in E^α.

P r o o f. Similarly to the choice in [2, 3, 5], we consider the function $w : [0,T] \longrightarrow \mathbb{R}$ given by

$$w(t) = \begin{cases} \frac{4\delta}{T} t, & t \in [0, \frac{T}{4}], \\ \delta, & t \in [\frac{T}{4}, \frac{3T}{4}], \\ \frac{4\delta}{T}(T-t), & t \in [\frac{3T}{4}, T], \end{cases}$$

with

$$w'(t) = \begin{cases} \frac{4\delta}{T}, & t \in (0, \frac{T}{4}), \\ 0, & t \in (\frac{T}{4}, \frac{3T}{4}), \\ -\frac{4\delta}{T}, & t \in (\frac{3T}{4}, T). \end{cases}$$

In consequence, from the calculations in [5], we get

$$|\,{}_0^cD_t^\alpha w(t)| = \frac{1}{\Gamma(1-\alpha)} \left(\int_0^t (t-s)^{-\alpha} w'(s)\,ds \right)$$

$$= \frac{1}{\Gamma(1-\alpha)} \begin{cases} \frac{4\delta}{T} \int_0^t (t-s)^{-\alpha}\,ds = \frac{4\delta}{T} \frac{t^{1-\alpha}}{1-\alpha}, & t \in [0, \frac{T}{4}], \\ \frac{4\delta}{T} \int_0^{\frac{T}{4}} (t-s)^{-\alpha}\,ds = \frac{4\delta}{T(1-\alpha)} (\frac{T}{4})^{1-\alpha} \\ = \frac{\delta}{(1-\alpha)} (\frac{T}{4})^{-\alpha}, & t \in (\frac{T}{4}, \frac{3T}{4}], \\ \frac{4\delta}{T} \int_0^{\frac{T}{4}} (t-s)^{-\alpha}\,ds - \frac{4\delta}{T} \int_{\frac{3T}{4}}^t (t-s)^{-\alpha} \\ = \frac{\delta}{(1-\alpha)} (\frac{T}{4})^{-\alpha} - \frac{4\delta}{T} \frac{(t-\frac{3T}{4})^{1-\alpha}}{1-\alpha}, & t \in (\frac{3T}{4}, T], \end{cases}$$

and hence

$$\Phi(w) = \frac{1}{2}\int_0^T (|\,{}_0^cD_t^\alpha w(t)|^2 + a(t)w^2(t))dt$$

$$= \frac{1}{(\Gamma(1-\alpha))^2}\frac{1}{2}\left(\begin{array}{c} \int_0^{\frac{T}{4}} \frac{4^2\delta^2}{T^2}\frac{t^{2(1-\alpha)}}{(1-\alpha)^2}\,dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \frac{\delta^2}{(1-\alpha)^2}\left(\frac{T}{4}\right)^{-2\alpha}\,dt \\ + \int_{\frac{3T}{4}}^{T}\left(\frac{\delta}{(1-\alpha)}\left(\frac{T}{4}\right)^{-\alpha} - \frac{4\delta}{T}\frac{(t-\frac{3T}{4})^{1-\alpha}}{1-\alpha}\right)^2\,dt \end{array}\right)$$

$$+ \frac{1}{2}\left(\frac{4^2\delta^2}{T^2}\int_0^{\frac{T}{4}} a(t)t^2\,dt + \delta^2\int_{\frac{T}{4}}^{\frac{3T}{4}} a(t)\,dt + \frac{4^2\delta^2}{T^2}\int_{\frac{3T}{4}}^{T} a(t)(T-t)^2\,dt\right).$$

Since

$$\int_{\frac{3T}{4}}^{T}\left(\frac{\delta}{(1-\alpha)}\left(\frac{T}{4}\right)^{-\alpha} - \frac{4\delta}{T}\frac{(t-\frac{3T}{4})^{1-\alpha}}{1-\alpha}\right)^2 dt = \frac{2\delta^2}{(3-2\alpha)(2-\alpha)}\left(\frac{T}{4}\right)^{1-2\alpha},$$

we have

$$\Phi(w) = \frac{1}{(\Gamma(1-\alpha))^2}\frac{1}{2}$$
$$\times\left(\frac{\delta^2}{(1-\alpha)^2(3-2\alpha)}\left(\frac{T}{4}\right)^{1-2\alpha} + \frac{2\delta^2}{(1-\alpha)^2}\left(\frac{T}{4}\right)^{1-2\alpha} + \frac{2\delta^2}{(3-2\alpha)(2-\alpha)}\left(\frac{T}{4}\right)^{1-\alpha}\right)$$
$$+ \frac{1}{2}\left(\frac{4^2\delta^2}{T^2}\int_0^{\frac{T}{4}} a(t)t^2\,dt + \delta^2\int_{\frac{T}{4}}^{\frac{3T}{4}} a(t)\,dt + \frac{4^2\delta^2}{T^2}\int_{\frac{3T}{4}}^{T} a(t)(T-t)^2\,dt\right) = \widetilde{W}.$$

Since, by hypotheses, $\widetilde{W} > r$, we prove that $\Phi(w) > r$.

Now, using that $\|x\|_\infty \le \sqrt{R}\|x\|_\alpha = \sqrt{R}\sqrt{2\Phi(x)}$, we get that $\Phi(x) \le r$ implies that $\|x\|_\infty \le \sqrt{2Rr}$. Hence, for $\lambda > 0$, $\mu \ge 0$, the following inequality holds:

$$\frac{1}{r}\sup_{\Phi(x)\le r}\Psi(x) \le \frac{1}{r}\int_0^T \max_{|x|\le\sqrt{2Rr}} F(t,x)\,dt + \frac{1}{r}\frac{\mu}{\lambda}\max_{|x|\le\sqrt{2Rr}}\sum_{j=1}^n(-J_j(x)) < \frac{1}{\lambda},$$

which is obvious if $\max_{|x|\le\sqrt{2Rr}}\sum_{j=1}^{n}(-J_j(x))=0$ since $\lambda<A_r$ and it is also true if $\max_{|x|\le\sqrt{2Rr}}\sum_{j=1}^{n}(-J_j(x))>0$ taking $\mu\in[0,\gamma)$.

On the other hand, we calculate a lower bound for the expression

$$\begin{aligned}\Psi(w)&=\int_0^T F(t,w(t))dt+\frac{\mu}{\lambda}\sum_{j=1}^{n}(-J_j(w(t_j)))\\&\ge\int_0^T F(t,w(t))dt-\frac{\mu}{\lambda}\max_{x\in[0,\delta]}\sum_{j=1}^{n}J_j(x)\end{aligned}$$

taking into account that

$$\begin{aligned}\int_0^T F(t,w(t))dt&=\int_0^{\frac{T}{4}}F\left(t,\frac{4\delta}{T}t\right)dt+\int_{\frac{T}{4}}^{\frac{3T}{4}}F(t,\delta)dt+\int_{\frac{3T}{4}}^{T}F\left(t,\frac{4\delta}{T}(T-t)\right)dt\\&=\int_0^{\frac{T}{4}}F\left(t,\frac{4\delta}{T}t\right)dt+\int_{\frac{T}{4}}^{\frac{3T}{4}}F(t,\delta)dt+\int_0^{\frac{T}{4}}F\left(T-u,\frac{4\delta}{T}u\right)du\\&=\int_0^{\frac{T}{4}}\left[F\left(t,\frac{4\delta}{T}t\right)+F\left(T-t,\frac{4\delta}{T}t\right)\right]dt+\int_{\frac{T}{4}}^{\frac{3T}{4}}F(t,\delta)dt\\&\ge\int_{\frac{T}{4}}^{\frac{3T}{4}}F(t,\delta)dt.\end{aligned}$$

Hence

$$\Psi(w)\ge\int_{\frac{T}{4}}^{\frac{3T}{4}}F(t,\delta)dt-\frac{\mu}{\lambda}\max_{x\in[0,\delta]}\sum_{j=1}^{n}J_j(x)>\frac{\widetilde{W}}{\lambda},$$

which is obvious from $\lambda>A_l$ if $\max_{x\in[0,\delta]}\sum_{j=1}^{n}J_j(x)=0$ and it is deduced from $\mu<\gamma$ if $\max_{x\in[0,\delta]}\sum_{j=1}^{n}J_j(x)>0$. In consequence, we have proved that

$$\frac{\Psi(w)}{\Phi(w)} = \frac{\Psi(w)}{\widetilde{W}} > \frac{1}{\lambda}$$

and hypothesis (i) in Theorem **3.1** holds. Further, the proof is completed similarly to Theorem **1.1**. Note that, in this result, the case $\max\limits_{x\in[0,\delta]} \sum\limits_{j=1}^{n} J_j(x) = 0$ is admissible. If any of the denominators in γ is null, we understand that the corresponding quotient is $+\infty$. □

THEOREM **3.4**. *Suppose that $1/2 < \alpha < 1$ and assumptions $(H1)$ and $(H2^*)$ hold. Suppose that there exist two constants $r > 0$ and $\delta > 0$ such that (3.3) (3.4) and (3.5) hold, where $R > 0$ satisfies (1.1).*

Then, for every λ in $\Lambda_r = (A_l, A_r)$, there exists

$$\gamma^* := \max\left\{\frac{\lambda\int\limits_0^T \max\limits_{|x|\le\sqrt{2Rr}} F(t,x)\,dt - r}{\max\limits_{|x|\le\sqrt{2Rr}} \sum\limits_{j=1}^{n} J_j(x)}, \frac{\lambda\int\limits_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\delta)dt - \widetilde{W}}{\min\limits_{x\in[0,\delta]} \sum\limits_{j=1}^{n} J_j(x)}\right\}$$

such that, for each $\mu \in (\gamma^*, 0]$, the problem (P) has at least three classical solutions, which are three distinct critical points of functional φ in E^α.

P r o o f. For $\lambda > 0$, $\mu \le 0$, we have

$$\frac{1}{r}\sup_{\Phi(x)\le r} \Psi(x) \le \frac{1}{r}\int\limits_0^T \max_{|x|\le\sqrt{2Rr}} F(t,x)\,dt - \frac{1}{r}\frac{\mu}{\lambda}\max_{|x|\le\sqrt{2Rr}}\sum_{j=1}^{n} J_j(x) < \frac{1}{\lambda}$$

which is trivially true if $\max\limits_{|x|\le\sqrt{2Rr}} \sum\limits_{j=1}^{n} J_j(x) = 0$ using that $\lambda < A_r$ and it is obvious if $\max\limits_{|x|\le\sqrt{2Rr}} \sum\limits_{j=1}^{n} J_j(x) > 0$ using that $\mu \in (\gamma^*, 0]$. Moreover,

$$\begin{aligned}\Psi(w) &= \int\limits_0^T F(t,w(t))dt - \frac{\mu}{\lambda}\sum_{j=1}^{n} J_j(w(t_j)) \\ &\ge \int\limits_{\frac{T}{4}}^{\frac{3T}{4}} F(t,\delta)dt - \frac{\mu}{\lambda}\min_{x\in[0,\delta]}\sum_{j=1}^{n} J_j(x) > \frac{\widetilde{W}}{\lambda},\end{aligned}$$

which comes from $\lambda > A_l$ if $\min_{x\in[0,\delta]} \sum_{j=1}^{n} J_j(x) = 0$ and it is deduced from $\mu > \gamma^*$ if $\min_{x\in[0,\delta]} \sum_{j=1}^{n} J_j(x) < 0$.

Hence,

$$\frac{\Psi(w)}{\Phi(w)} = \frac{\Psi(w)}{\widetilde{W}} > \frac{1}{\lambda}$$

and the proof is completed similarly to Theorem **3.2**. If some of the denominators in the terms in γ^* is zero, then the corresponding term is assumed to represent $-\infty$. □

The following results are particular cases of Theorems **1.1**, **3.2**, **3.3** and **3.4** for the case $f(t,x) = f(t)g(x)$.

Corollary **3.1**. *Suppose that $\frac{1}{2} < \alpha < 1$ and $f : [0,T] \longrightarrow \mathbb{R}$, $g : \mathbb{R} \longrightarrow \mathbb{R}$, $I_j : \mathbb{R} \longrightarrow \mathbb{R}$, $j = 1,2,\dots,n$, are continuous functions, f nonnegative, $a \in C([0,T])$, $0 < a_1 \le a(t) \le a_2$ such that*

$(H3)$ *There exist positive constants $l,\ L_1,\dots,L_n,\ \beta,\ d_1,\dots,d_n$, with $\beta < 2$ and $d_j < 2$, $j = 1,\dots,n$, such that*

$$G(x) \le l(1 + |x|^{\beta}), \quad \forall x \in \mathbb{R}, \tag{3.6}$$

where $G(x) = \int_0^x g(s)\,ds$, and also that (1.5) holds, that is,

$$-J_j(x) \le L_j(1 + |x|^{d_j}), \quad \forall x \in \mathbb{R}.$$

Suppose that there exists $r > 0$ and $w \in E^{\alpha}$ such that $\frac{1}{2}\|w\|_{\alpha}^2 > r$. Let $R > 0$ be such that (1.1) holds and suppose also that

$$\int_0^T f(t)G(w(t))\,dt > 0, \quad \sum_{j=1}^{n} J_j(w(t_j)) > 0$$

and that

$$A_l := \frac{\frac{1}{2}\|w\|_{\alpha}^2}{\int_0^T f(t)G(w(t))\,dt} < A_r := \frac{r}{\max_{|x|\le\sqrt{2Rr}} G(x) \int_0^T f(t)\,dt} . \tag{3.7}$$

Then, for every λ in $\Lambda_r = (A_l, A_r)$, there exists

$$\gamma := \min \left\{ \frac{r - \lambda \max\limits_{|x| \le \sqrt{2Rr}} G(x) \int\limits_0^T f(t)\, dt}{\max\limits_{|x| \le \sqrt{2Rr}} \sum\limits_{j=1}^{n} (-J_j(x))}, \frac{\lambda \int_0^T f(t) G(w(t))\, dt - \frac{1}{2}\|w\|_\alpha^2}{\sum_{j=1}^{n} J_j(w(t_j))} \right\}$$

such that, for each $\mu \in [0, \gamma)$, *the problem*

$$(P2) \begin{cases} {}_tD_T^\alpha \left({}_0^cD_t^\alpha u(t)\right) + a(t)u(t) = \lambda f(t) g(u(t)), & t \neq t_j, \text{a.e. } t \in [0,T], \\ \Delta({}_tD_T^{\alpha-1}({}_0^cD_t^\alpha u))(t_j) = \mu I_j(u(t_j)), & j = 1, 2, \ldots, n, \\ u(0) = u(T) = 0 & \end{cases}$$

has at least three classical solutions, which are three distinct critical points of the functional $\widetilde{\varphi}$

$$\begin{aligned} \widetilde{\varphi}(u) =& \frac{1}{2} \int\limits_0^T (|\, {}_0^cD_t^\alpha u(t)|^2 + a(t) u^2(t)) dt \\ & - \lambda \left(\int\limits_0^T f(t) G(u(t)) dt - \frac{\mu}{\lambda} \sum_{j=1}^{n} J_j(u(t_j)) \right) \end{aligned}$$

in E^α.

Corollary **3.2**. *Suppose that* $\frac{1}{2} < \alpha < 1$ *and* $f : [0,T] \longrightarrow \mathbb{R}$, $g : \mathbb{R} \longrightarrow \mathbb{R}$, $I_j : \mathbb{R} \longrightarrow \mathbb{R}$, $j = 1, 2, \ldots, n$, *are continuous functions,* f *nonnegative,* $a \in C([0,T])$, $0 < a_1 \le a(t) \le a_2$ *such that*

($H3^*$) *There exist positive constants* l, $L_1, \ldots, L_n$, β, $d_1, \ldots, d_n$, *with* $\beta < 2$ *and* $d_j < 2$, $j = 1, \ldots, n$, *such that (3.6) and (3.2) holds.*

Suppose that there exist $r > 0$ *and* $w \in E^\alpha$ *such that* $\frac{1}{2}||w||_\alpha^2 > r$. *Let* $R > 0$ *be such that (1.1) holds and suppose also that*

$$\int\limits_0^T f(t) G(w(t)) dt > 0, \quad \sum_{j=1}^{n} J_j(w(t_j)) < 0$$

and that (3.7) holds.

Then, for every λ in $\Lambda_r = (A_l, A_r)$, there exists

$$\gamma^* := \max\left\{ \frac{\lambda \max\limits_{|x|\le\sqrt{2Rr}} G(x) \int\limits_0^T f(t)\,dt - r}{\max\limits_{|x|\le\sqrt{2Rr}} \sum\limits_{j=1}^n J_j(x)}, \frac{\lambda \int\limits_0^T f(t)G(w(t))dt - \frac{1}{2}\|w\|_\alpha^2}{\sum\limits_{j=1}^n J_j(w(t_j))} \right\}$$

such that, for each $\mu \in (\gamma^*, 0]$, the problem (P2) has at least three classical solutions, which are three distinct critical points of functional $\widetilde{\varphi}$ in E^α.

Taking the function w in the proof of Theorem **3.3**, we get the following particular cases.

Corollary **3.3**. *Suppose that $\frac{1}{2} < \alpha < 1$ and $f : [0,T] \longrightarrow \mathbb{R}$, $g : \mathbb{R} \longrightarrow \mathbb{R}$, $I_j : \mathbb{R} \longrightarrow \mathbb{R}$, $j = 1, 2, \ldots, n$, are continuous functions, f nonnegative, $a \in C([0,T])$, $0 < a_1 \le a(t) \le a_2$ such that $(H3)$ holds.*

Suppose that there exist two constants $r > 0$ and $\delta > 0$ such that (3.3) holds and

$$G(\delta) \int\limits_{\frac{T}{4}}^{\frac{3T}{4}} f(t)dt > 0, \quad \int\limits_0^{\frac{T}{4}} [f(t) + f(T-t)]\, G\left(\frac{4\delta}{T}t\right) dt \ge 0. \tag{3.8}$$

Let $R > 0$ be such that (1.1) holds and assume also that

$$A_l := \frac{\widetilde{W}}{G(\delta) \int\limits_{\frac{T}{4}}^{\frac{3T}{4}} f(t)dt} < A_r := \frac{r}{\max\limits_{|x|\le\sqrt{2Rr}} G(x) \int\limits_0^T f(t)\,dt}. \tag{3.9}$$

Then, for every λ in $\Lambda_r = (A_l, A_r)$, there exists

$$\gamma := \min\left\{ \frac{r - \lambda \max\limits_{|x|\le\sqrt{2Rr}} G(x) \int\limits_0^T f(t)\,dt}{\max\limits_{|x|\le\sqrt{2Rr}} \sum\limits_{j=1}^n (-J_j(x))}, \frac{\lambda G(\delta) \int\limits_{\frac{T}{4}}^{\frac{3T}{4}} f(t)dt - \widetilde{W}}{\max\limits_{x\in[0,\delta]} \sum\limits_{j=1}^n J_j(x)} \right\}$$

such that, for each $\mu \in [0, \gamma)$, the problem $(P2)$ has at least three classical solutions, which are three distinct critical points of functional $\widetilde{\varphi}$ in E^α.

COROLLARY **3.4**. *Suppose that $\frac{1}{2} < \alpha < 1$ and $f : [0,T] \longrightarrow \mathbb{R}$, $g : \mathbb{R} \longrightarrow \mathbb{R}$, $I_j : \mathbb{R} \longrightarrow \mathbb{R}$, $j = 1, 2, \ldots, n$, are continuous functions, f nonnegative, $a \in C([0,T])$, $0 < a_1 \le a(t) \le a_2$ such that $(H3^*)$ holds.*

Suppose that there exist two constants $r > 0$ and $\delta > 0$ such that (3.3), (3.8) and (3.9) hold, where $R > 0$ satisfies (1.1).

Then, for every λ in $\Lambda_r = (A_l, A_r)$, there exists

$$\gamma^* := \max\left\{ \frac{\lambda \max\limits_{|x|\le\sqrt{2Rr}} G(x) \int\limits_0^T f(t)\,dt - r}{\max\limits_{|x|\le\sqrt{2Rr}} \sum\limits_{j=1}^n J_j(x)}, \frac{\lambda G(\delta) \int\limits_{\frac{T}{4}}^{\frac{3T}{4}} f(t)dt - \widetilde{W}}{\min\limits_{x\in[0,\delta]} \sum\limits_{j=1}^n J_j(x)} \right\}$$

such that, for each $\mu \in (\gamma^*, 0]$, the problem $(P2)$ has at least three classical solutions, which are three distinct critical points of functional $\widetilde{\varphi}$ in E^α.

EXAMPLE **3.1**. Let $a = T = n = 1$. Let us take the function $w = w_d(t)$, as

$$w(t) = \begin{cases} 3dt, & t \in [0, \frac{1}{3}], \\ d, & t \in [\frac{1}{3}, \frac{2}{3}], \\ 3d(1-t), & t \in [\frac{2}{3}, 1], \end{cases}$$

where $d > 0$. A straightforward computation shows that $w \in E_\alpha$ and $||w||_\alpha^2 = C(\alpha)d^2$, where

$$C(\alpha) = \frac{5}{9} + \frac{2(2\alpha^2 - 6\alpha + 5)}{\Gamma^2(1-\alpha)(1-\alpha)^2(3-2\alpha)(2-\alpha)3^{1-2\alpha}} > 0,$$

for $\alpha \in (\frac{1}{2}, 1)$. Take $r = \frac{1}{2k}||w||_\alpha^2 = \frac{C(\alpha)}{2k}d^2$, $k > 1$ and $R = R(\alpha) = \frac{1}{\Gamma^2(\alpha)(2\alpha-1)}$ such that $2rR = 1$, which implies that $d = \sqrt{\frac{k}{C(\alpha)R(\alpha)}}$. The inequality (1.6) of Theorem **1.1** takes the form

$$k \max_{|x|\le 1} F(x) < \int_0^1 F(w(t))\,dt, \tag{3.10}$$

if $\int_0^1 F(w(t))\,dt > 0$. If $F \ge 0$ on $[0,d]$, then (3.10) is satisfied if

$$k \max_{|x|\le 1} F(x) < \int_{1/3}^{2/3} F(w(t))\,dt = \frac{1}{3} F(d). \tag{3.11}$$

Suppose that $F = F(x)$ is an increasing function, positive for $x > 0$, $F(0) = 0$ and $F(x) \le c(1 + |x|^\beta)$, $\beta < 2$, according to (H3). Then (3.11) holds if $3kF(1) < F(d)$ and we need to have $d = \sqrt{\frac{k}{C(\alpha)R(\alpha)}} > 1$. The last inequality is true for $k \geq 6$ for some value of α. For the validity of (1.6), we need to impose that

$$F\left(\sqrt{\frac{k}{C(\alpha)\,R(\alpha)}}\right) > 3kF(1). \tag{3.12}$$

Taking $k = 8$, computer experiment by Mathematica shows that the inequality $\frac{8}{C(\alpha)R(\alpha)} > 1$ is satisfied for $\alpha \in (\alpha_1, 1) \subset \left(\frac{1}{2}, 1\right)$, where $\alpha_1 \simeq 0.583596$.

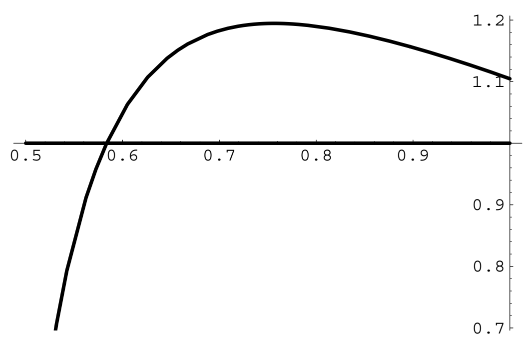

Fig. 1: The graph of the function $8/C(\alpha)\,R(\alpha)$

We have $d^* = \sqrt{\frac{8}{C(0.7)R(0.7)}} \simeq 1.18294$ and can choose

$$F(x) = \begin{cases} 0, & x \leq 0, \\ \gamma(x), & 0 \leq x \leq d^*, \\ 15\left(1 + x^{\frac{3}{2}}\right), & d^* \leq x, \end{cases}$$

where $\gamma(x)$ is a function which is differentiable and increasing on $[0, d^*]$ and such that $\gamma(0) = 0$, $\gamma'(0) = 0$, $\gamma(1) = 1$, $\gamma(d^*) = 15\left(1 + (d^*)^{\frac{3}{2}}\right)$ and $\gamma'(d^*) = \frac{45}{2}(d^*)^{\frac{1}{2}}$.

Then, in this case, $F(1) = 1$, $F(d^*) = \gamma(d^*) = 15\left(1 + (d^*)^{\frac{3}{2}}\right) > 30$ and (3.12) holds, so that the inequality (1.6) is satisfied, which offers us an example regarding Theorem **1.1**.

Denote $J(x) = \int_0^x I(t)dt$ and take I and J such that $J(w(t_1)) > 0$ and

$$-J(x) \leq L_1(1 + |x|^{d_1}), \quad \forall x \in \mathbb{R},$$

where $d_1 < 2$. Take $\alpha = 0.7$, $d = d^*$ and $k = 8$. Then, by Theorem **1.1**, for every λ in

$$\Lambda_r = \left(\frac{\frac{1}{2}C(\alpha)d^2}{2\int\limits_0^{\frac{1}{3}} F(3dt)\,dt + \frac{1}{3}F(d)}, \frac{\frac{1}{2k}C(\alpha)d^2}{F(1)} \right),$$

there exists

$$\gamma := \min \left\{ \frac{\frac{1}{2k}C(\alpha)d^2 - \lambda F(1)}{\max\limits_{|x|\le 1}(-J(x))}, \frac{\lambda\left(2\int\limits_0^{\frac{1}{3}} F(3dt)dt + \frac{1}{3}F(d)\right) - \frac{1}{2}C(\alpha)d^2}{J(w(t_1))} \right\}$$

such that, for each $\mu \in [0, \gamma)$, the problem (P) has at least three classical solutions.

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