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RESEARCH PAPER

EXTENDING THE STIELTJES TRANSFORM II

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Dedicated to Professor Ivan Dimovski
on the occasion of his 80th anniversary

Abstract

The Stieltjes transform has recently been extended to a subspace of Boehmians. In this note, additional results are obtained which include an inversion formula plus Abelian type theorems.

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1. Introduction

Recently [9], the Stieltjes transform was extended onto a space $B_r(\mathbb{R})$ of generalized functions by iterating the Laplace transform. The Stieltjes transform of an element of $B_r(\mathbb{R})$ is an analytic function in the half-plane $\operatorname{Re} s > 0$, which can be analytically extended to a region containing the half-plane. Several authors [1, 3, 6, 10, 11] have investigated the Stieltjes transform on the space of distributions $J'(r)$, which consists of distributions of the form $T = D^k f$ (for some $k \in \mathbb{N}$), where f is a locally integrable function supported on the interval $[0, \infty)$ and satisfies a growth condition at infinity. The space $J'(r)$ can be identified with a proper subspace of $B_r(\mathbb{R})$.

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Some new results for the Stieltjes transform on the space $B_r(\mathbb{R})$ are obtained in the present paper. Among these results is a real inversion formula for the Laplace transform, which is used to obtain an inversion formula for the Stieltjes transform. Additionally, some Abelian type theorems for the Stieltjes transform are established.

This article is organized as follows. Section **2** contains notation and the construction of a space of Boehmians $\beta(\mathbb{R})$ [4]. Section **3** is concerned with the Laplace transform on a subspace of $\beta(\mathbb{R})$. A real inversion formula is established for the Laplace transform. The Stieltjes transform on $B_r(\mathbb{R}) \subset \beta(\mathbb{R})$ is studied in Section **4**.

2. The Space of Boehmians

Let $C(\mathbb{R})$ denote the space of all continuous functions on the real line $\mathbb{R}$, and let $\mathcal{D}(\mathbb{R})$ denote the subspace of $C(\mathbb{R})$ of all infinitely differentiable functions with compact support.

Definition **2.1**. A sequence of real-valued functions $\varphi_n \in \mathcal{D}(\mathbb{R})$ is called a *delta sequence* provided:

(i) $\int \varphi_n = 1$ for all $n \in \mathbb{N}$,
(ii) $\varphi_n(x) \geq 0$, for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$,
(iii) For every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\varphi_n(x) = 0$ for $|x| > \varepsilon$ and $n > n_\varepsilon$.

A pair of sequences (f_n, φ_n) is called a *quotient of sequences* (q.s.) if $f_n \in C(\mathbb{R})$ for $n \in \mathbb{N}$, $\{\varphi_n\}$ is a delta sequence, and $f_n * \varphi_k = f_k * \varphi_n$ for all $k, n \in \mathbb{N}$, where $*$ denotes convolution:

$$(f * \varphi)(x) = \int_{-\infty}^{\infty} f(x-t)\varphi(t)dt. \tag{2.1}$$

Two quotients of sequences (f_n, φ_n) and (g_n, ψ_n) are said to be equivalent if $f_n * \psi_k = g_k * \varphi_n$ for all $k, n \in \mathbb{N}$. A straightforward calculation shows that this is an equivalence relation. The equivalence classes are called Boehmians. The space of all Boehmians will be denoted by $\beta(\mathbb{R})$ and a typical element of $\beta(\mathbb{R})$ will be written as $W = \left[\frac{f_n}{\varphi_n}\right]$.

The operations of addition, scalar multiplication, and differentiation are defined as follows: $\left[\frac{f_n}{\varphi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n*\psi_n+g_n*\varphi_n}{\varphi_n*\psi_n}\right]$, $\gamma\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\gamma f_n}{\varphi_n}\right]$ where $\gamma \in \mathbb{C}$, $D^m\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{f_n*\varphi_n^{(m)}}{\varphi_n*\varphi_n}\right]$, $m \in \mathbb{N}$.

Define the map $\iota : C(\mathbb{R}) \to \beta(\mathbb{R})$ by

$$\iota(f) = \left[\frac{f * \varphi_n}{\varphi_n}\right], \tag{2.2}$$

where $\{\varphi_n\}$ is any fixed delta sequence.

It is not difficult to show that the mapping ι is an injection which preserves the algebraic properties of $C(\mathbb{R})$. Thus, $C(\mathbb{R})$ can be identified with a proper subspace of $\beta(\mathbb{R})$. Likewise, the space of Schwartz distributions $\mathcal{D}'(\mathbb{R})$ [13] can be identified with a proper subspace of $\beta(\mathbb{R})$. Using this identification, the Dirac measure δ corresponds to the Boehmian $\left[\frac{\varphi_n}{\varphi_n}\right]$, where $\{\varphi_n\}$ is any delta sequence.

For $\psi \in \mathcal{D}(\mathbb{R})$ and $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$, $\psi * W$ is defined as

$$\psi * \left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\psi * f_n}{\varphi_n}\right]. \tag{2.3}$$

Definition **2.2**. A Boehmian W is said to *vanish on an open set* $\Omega \subset \mathbb{R}$, denoted $W(x) = 0$ on Ω, provided that there exists a delta sequence $\{\varphi_n\}$ such that $W * \varphi_n \in C(\mathbb{R})$, $n \in \mathbb{N}$, and $W * \varphi_n \to 0$ uniformly on compact subsets of Ω as $n \to \infty$.

Boehmians W, V are equal on an open set Ω, denoted $W(x) = V(x)$ on Ω, provided $W - V$ vanishes on Ω.

The support of a Boehmian W is the complement of the largest open set on which W vanishes. If $W = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$ such that $\operatorname{supp} W \subseteq [-b, \infty)$ for some $b \geq 0$, then $\operatorname{supp} f_n \subseteq [-(b+\varepsilon_n), \infty)$, where $\operatorname{supp} \varphi_n \subset (-\varepsilon_n, \varepsilon_n)$, $n \in \mathbb{N}$.

The space of Boehmians with compact support is denoted by $\beta_c(\mathbb{R})$.

Definition **2.3**. Let $W_n, W \in \beta(\mathbb{R})$ for $n = 1, 2, \ldots$. We say that the sequence $\{W_n\}$ is *δ-convergent to W* if there exists a delta sequence $\{\varphi_n\}$ such that for each k and n, $W_n * \varphi_k \in C(\mathbb{R})$, and for each $k, W_n * \varphi_k \to W * \varphi_k$ uniformly on compact sets as $n \to \infty$. This will be denoted by $\delta\text{-}\lim_{n\to\infty} W_n = W$.

3. The Laplace Transform

In the next section, the Stieltjes transform is defined on a subspace of Boehmians by iteration of the Laplace transform. Thus, in this section, we give a brief review for the Laplace transform of a Boehmian, and also develop some needed material for the Laplace transform. The main result of this section establishes a real inversion formula for the Laplace transform, which leads to an inversion formula for the Stieltjes transform in the next section.

Inversion formulae for integral transforms play an important role in theory as well as in applications. There are complex as well as real inversion formulae.

Real inversion formulae are of interest since they use only the data of the transform on the real line and there is no need to continue the transform analytically as in complex inversion formulae.

With some mild condition on a locally integrable function f, the following classical real inversion formula for the Laplace transform may be found in [2]:

$$\int_{-\infty}^{t} f(x)\,dx = \lim_{\sigma\to\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k!}\,F(k\sigma)\,e^{k\sigma t}, \tag{3.1}$$

where $F(s) = \int_{-\infty}^{\infty} e^{-st}\,f(t)\,dt$.

In 1966, Zemanian [12] extended formula (3.1) to include Schwartz distributions. That is, for a transformable distribution f,

$$f(t) = \lim_{\sigma\to\infty}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k-1)!}\,\sigma F(k\sigma)e^{k\sigma t}, \tag{3.2}$$

where the convergence is in the sense of $\mathcal{D}'(\mathbb{R})$.

In this section, formula (3.2) is established for the space of Laplace transformable Boehmians.

Let f be a locally integrable function whose support is bounded on the left, and $f(t) = O(e^{\alpha t})$ as $t \to \infty$, for some $\alpha \in \mathbb{R}$. Then, the *Laplace transform of* f is given by

$$\widehat{f}(s) = \int_{-\infty}^{\infty} e^{-st} f(t)dt, \;\; \mathrm{Re}\, s > \alpha.$$

A Boehmian W having support bounded on the left is called *transformable* provided there exist a quotient of sequences (f_n, φ_n) and $\alpha \in \mathbb{R}$ such that $W = \left[\frac{f_n}{\varphi_n}\right]$ with $f_n(t) = O(e^{\alpha t})$ as $t \to \infty$, for all $n \in \mathbb{N}$.

The *Laplace transform of* W, denoted $\widehat{W}$, is given by

$$\widehat{W}(s) = \lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-st} f_n(t)\,dt, \;\mathrm{Re}\; s > \alpha. \tag{3.3}$$

The sequence of analytic functions on the right-hand side of (3.3) converges uniformly on compact subsets of the half-plane $\mathrm{Re}\, s > \alpha$.

REMARKS **3.1**.

1. The Laplace transform operator on the space of transformable Boehmians ([5], [7]) has many of the same properties as the classical Laplace transform.

2. The Laplace transform for a transformable Boehmian is independent of the representative. That is, suppose that there exist $\alpha, \gamma \in \mathbb{R}$ and q.s. (f_n, φ_n) and (g_n, ψ_n) such that $\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{g_n}{\psi_n}\right]$, $f_n(t) = O(e^{\alpha t})$ as $t \to \infty$, $g_n(t) = O(e^{\gamma t})$ as $t \to \infty$, $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-st} f_n(t)\, dt = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-st} g_n(t)\, dt$ for $\operatorname{Re} s > \max\{\alpha, \gamma\}$.

3. $\widehat{W}$ is an analytic function in the half-plane $\operatorname{Re} s > \alpha$. If $W \in \beta_c(\mathbb{R})$, then $\widehat{W}$ is an entire function.

4. Let K be a compact subset of $\mathbb{R}$ and $\{\varphi_n\}$ a delta sequence. Then, $\widehat{\varphi}_n \to 1$ uniformly on K as $n \to \infty$.

EXAMPLES **3.1.**

1. Let f be a locally integrable function which vanishes on $(-\infty, -b)$, for some $b \geq 0$. Suppose that $f(t)e^{-\alpha t}$ is bounded as $t \to \infty$ for some $\alpha \geq 0$. Then, $W_f = \left[\frac{f * \varphi_n}{\varphi_n}\right]$ is a transformable Boehmian, and

$$\begin{aligned} \widehat{W}_f(s) &= \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-st} (f * \varphi_n)(t)\, dt \\ &= \lim_{n\to\infty} \widehat{f * \varphi_n}(s) \\ &= \widehat{f}(s) \lim_{n\to\infty} \widehat{\varphi}_n(s) \\ &= \widehat{f}(s), \qquad \operatorname{Re} s > \alpha. \end{aligned}$$

2. Let $W = \sum_{n=0}^{\infty} \frac{D^n \delta}{(2n)!} = \left[\frac{\sum_{n=0}^{\infty} \frac{\varphi_k^{(n)}}{(2n)!}}{\varphi_k}\right]$, where $\{\varphi_k\}$ is a delta sequence such that for all $k, n \in \mathbb{N}$, $\left|\varphi_k^{(n)}(t)\right| \leq M_k \left(\frac{1}{2}\right)^n (2n)!$, $t \in \mathbb{R}$ $(M_k > 0)$. Then,

$$\begin{aligned} \widehat{W}(s) &= \lim_{k\to\infty} \int_{-\infty}^{\infty} e^{-st} \left(\sum_{n=0}^{\infty} \frac{\varphi_k^{(n)}(t)}{(2n)!} \right) dt \\ &= \lim_{k\to\infty} \sum_{n=0}^{\infty} \frac{\widehat{\varphi_k^{(n)}}(s)}{(2n)!} \end{aligned}$$

$$= \left(\sum_{n=0}^{\infty} \frac{s^n}{(2n)!}\right) \lim_{k\to\infty} \widehat{\varphi}_k(s)$$

$$= \sum_{n=0}^{\infty} \frac{s^n}{(2n)!} = \cosh\sqrt{s}, \quad s \in \mathbb{C}.$$

LEMMA **3.1**. *Let W be a transformable Boehmian such that*

$$\operatorname{supp} W \subseteq [-b, \infty) \text{ for some } b \geq 0. \tag{3.4}$$

Then, for each $a > b$, $\widehat{W}(\sigma) = O(e^{a\sigma})$ as $\sigma \to \infty$.

P r o o f. Let $W = \left[\frac{f_n}{\varphi_n}\right]$, where

$$\operatorname{supp} \varphi_n \subset (-\varepsilon_n, \varepsilon_n), \quad \text{for all } n \in \mathbb{N} \tag{3.5}$$

($\varepsilon_n \to 0$ as $n \to \infty$), and for each $n \in \mathbb{N}$,

$$f_n(t) = O(e^{\alpha t}) \text{ as } t \to \infty. \tag{3.6}$$

By the Mean Value Theorem and the fact that $\{\varphi_n\}$ is nonnegative, it follows that for each $n \in \mathbb{N}$,

$$\widehat{\varphi}_n(\sigma) \geq e^{-\varepsilon_n|\sigma|}, \text{ for all } \sigma \in \mathbb{R}.$$

Now, by (3.6), for each $n \in \mathbb{N}$,

$$\widehat{f}_n(\sigma) \text{ exists for all } \sigma > \alpha \text{ (we may assume } \alpha \geq 0.)$$

By using

$$f_n * \varphi_k = f_k * \varphi_n \text{ (for all } k, n \in \mathbb{N}),$$

we obtain for $\sigma > \alpha$,

$$\widehat{f}_n(\sigma) = \frac{\widehat{f}_k(\sigma)}{\widehat{\varphi}_k(\sigma)} \widehat{\varphi}_n(\sigma), \quad \text{for all } k, n \in \mathbb{N}.$$

Thus, for all $\sigma > \alpha$,

$$\widehat{W}(\sigma) = \lim_{n\to\infty} \widehat{f}_n(\sigma) = \frac{\widehat{f}_k(\sigma)}{\widehat{\varphi}_k(\sigma)}, \text{ for all } k \in \mathbb{N}.$$

Thus, for all $k \in \mathbb{N}$ and $\sigma > \alpha$,

$$\left|\widehat{W}(\sigma)\right| = \frac{\left|\widehat{f}_k(\sigma)\right|}{|\widehat{\varphi}_k(\sigma)|} \leq \left|\widehat{f}_k(\sigma)\right| e^{\varepsilon_k \sigma}. \tag{3.7}$$

By (3.4) and (3.5),

$$\operatorname{supp} f_k \subseteq [-(b + \varepsilon_k), \infty), \; k \in \mathbb{N}.$$

Let $\sigma_0 > \alpha$. Then, for all $\sigma \geq \sigma_0$,

$$\left|\widehat{f}_k(\sigma)\right| \leq M_k e^{(b+\varepsilon_k)\sigma} \text{ (where } M_k > 0).$$

By (3.7) and above, for all $\sigma \geq \sigma_0$,

$$\left|\widehat{W}(\sigma)\right| \leq M_k\, e^{(b+2\varepsilon_k)\sigma}\,, \text{ for all } k \in \mathbb{N}.$$

The conclusion follows. □

In [9], a version of the classical complex inversion formula was proven for transformable Boehmians.

We now establish a real inversion formula for transformable Boehmians.

THEOREM **3.1**. *Let W be a transformable Boehmian. Then,*

$$W = \delta\text{-}\lim_{\sigma\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!}\, \sigma \widehat{W}(k\sigma) e^{k\sigma(\cdot)}. \tag{3.8}$$

REMARKS **3.2**.

1. The above means, for every $\sigma_n \to \infty$ as $n \to \infty$, $W = \delta\text{-}\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!}\, \sigma_n \widehat{W}(k\sigma_n) e^{k\sigma_n t}$.
2. The series converges in $\mathcal{D}'(\mathbb{R})$ as well as in $\beta(\mathbb{R})$.

P r o o f. (Proof of Theorem) By Lemma **3.1** there exist positive constants a, α, and M such that for all $\sigma > \alpha$, $\left|\widehat{W}(\sigma)\right| \leq M e^{a\sigma}$. First we show that for each $\sigma > \alpha$,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!}\, \sigma \widehat{W}(k\sigma) e^{k\sigma t} \quad \text{converges in } \mathcal{D}'(\mathbb{R}).$$

Let $\varphi \in \mathcal{D}(\mathbb{R})$. Then, for $m = 1, 2, \ldots$

$$\langle \sum_{k=1}^{m} \frac{(-1)^{k+1}}{(k-1)!}\, \sigma \widehat{W}(k\sigma) e^{k\sigma t}, \varphi(t)\rangle = \sum_{k=1}^{m} \frac{(-1)^{k+1}}{(k-1)!}\, \sigma \widehat{W}(k\sigma) \int_{-\infty}^{\infty} e^{k\sigma t}\varphi(t)dt. \tag{3.9}$$

It suffices to show that the above converges as $m \to \infty$.

Now, for $\sigma > \alpha$ and $k = 1, 2, \ldots$

$$\left|\frac{(-1)^{k+1}}{(k-1)!}\, \sigma \widehat{W}(k\sigma) \int_{-\infty}^{\infty} e^{k\sigma t}\varphi(t)dt\right| \leq \frac{Ae^{Bk\sigma}}{(k-1)!} \text{ (for some } A > 0 \text{ and } B > 0).$$

It follows that the sequence in (3.9) converges. Thus, for $\sigma > \alpha$, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!} \sigma \widehat{W}(k\sigma) e^{k\sigma t}$ converges in $\mathcal{D}'(\mathbb{R})$.

Now suppose $W = \left[\frac{f_n}{\varphi_n}\right]$ and $\sigma_n \to \infty$ as $n \to \infty$. Then, for all $p, n \in \mathbb{N}$

$$\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!}\sigma_n \widehat{W}(k\sigma_n)e^{k\sigma_n t}\right) * \varphi_p$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!}\sigma_n \widehat{W * \varphi_p}(k\sigma_n)e^{k\sigma_n t}$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!}\sigma_n \widehat{f_p}(k\sigma_n)e^{k\sigma_n t}.$$

The first equality follows from the fact that the mapping $\Lambda_{\varphi_p} : \mathcal{D}'(\mathbb{R}) \to \mathcal{D}'(\mathbb{R})$ given by $\Lambda_{\varphi_p}(T) = T * \varphi_p$ is continuous.

Now, for all $p \in \mathbb{N}$,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!}\sigma_n \widehat{f_p}(k\sigma_n)e^{k\sigma_n t} \to f_p$$

as $n \to \infty$, where the convergence is in $\mathcal{D}'(\mathbb{R})$, [12].

Thus, for all $p \in \mathbb{N}$,

$$\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!}\sigma_n \widehat{W}(k\sigma_n)e^{k\sigma_n t}\right) * \varphi_p \to W * \varphi_p \quad \text{as} \quad n \to \infty,$$

where the convergence is in $\mathcal{D}'(\mathbb{R})$.

Let K be a compact subset of $\mathbb{R}$. Then, for each $p \in \mathbb{N}$, $\varphi_p \in \mathcal{D}(\mathbb{R})$ and the set $\{\varphi_p(x - \cdot) : x \in K\}$ is bounded in $\mathcal{D}(\mathbb{R})$. Since a sequence in $\mathcal{D}'(\mathbb{R})$ converges in the weak topology if and only if it converges in the strong topology, it follows that for each $p \in \mathbb{N}$,

$$\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!}\,\sigma_n \widehat{W}(k\sigma_n)e^{k\sigma_n t}\right) * \varphi_p * \varphi_p \to W * \varphi_p * \varphi_p \quad \text{as} \quad n \to \infty,$$

where the convergence is uniform on compact sets.

Since $\{\varphi_p * \varphi_p\}$ is a delta sequence,

$$W = \delta\text{-}\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!}\sigma_n \widehat{W}(k\sigma_n)e^{k\sigma_n t}.$$

□

Various conditions have been placed on the function spaces used to construct Boehmian spaces in order to study the Laplace transform. Three of the spaces are denoted by β_τ, $\beta_{\mathcal{L}}$, and β_L, where β_τ denotes the space of transformable Boehmians constructed in this paper, $\beta_{\mathcal{L}}$ denotes the space constructed in [5], and β_L the space in [7]. Then,

$$\beta_L \subset \beta_\tau \subset \beta_{\mathcal{L}}.$$

Since elements of β_L are supported on $[0,\infty)$, the first inclusion is proper. The delta sequences used in [5] are more general than the ones used in this paper. Also, the elements of $\beta_{\mathcal{L}}$ have no support restrictions. Thus, the second inclusion is also proper.

4. The Stieltjes Transform

In this section, after reviewing the Stieltjes transform for Boehmians, which was developed in [9], some new results are obtained. Included in these results are Abelian type theorems of the initial and final type as well as an inversion theorem for the Stieltjes transform.

The Stieltjes transform of index r of a suitably restricted function f is given by

$$S_s^r f = \int_0^\infty \frac{f(t)\,dt}{(s+t)^{r+1}}\,, \quad s \in \mathbb{C}\backslash(-\infty,0].$$

Let $T \in J'(r)$, $r > -1$. That is, $T \in \mathcal{D}'(\mathbb{R})$ such that $T = D^k f$, where $k \in \mathbb{N}$, D is the differentiation operator on $\mathcal{D}'(\mathbb{R})$, $f \in L^1_{loc}(\mathbb{R})$, $\operatorname{supp} f \subseteq [a,\infty)$ for some $a \geq 0$, and $f(t)t^{-r-k+\alpha}$ is bounded as $t \to \infty$ (for some $\alpha > 0$). Then, the Stieltjes transform of T, which is an analytic function in the region $\mathbb{C}\backslash(-\infty,0]$, is given by $S_s^r T = \langle T_t, (s+t)^{-r-1}\rangle$, for $s \in \mathbb{C}\backslash(-\infty,0]$. Several authors ([1, 3, 6, 10, 11]) have used the space $J'(r)$ to investigate the Stieltjes transform for distributions.

In this section, we consider a subspace of $\beta^+(\mathbb{R})$, the space of Boehmians supported on $[0,\infty)$. This subspace, $B_r(\mathbb{R})$, contains $J'(\mathbb{R})$ as a proper subspace.

Let $r > -1$. A Boehmian W is in $B_r(\mathbb{R})$ provided $W \in \beta^+(\mathbb{R})$ and $W(t) = D^k f(t)$ as $t \to \infty$ for some $k \in \mathbb{N}$, where $f \in L^1_{loc}(\mathbb{R})$ such that $\operatorname{supp} f \subseteq [a,\infty)$ (some $a \geq 0$) and $f(t)t^{-r-k+\alpha}$ is bounded as $t \to \infty$ for some $a > 0$.

That is, $W = V + D^k f$, where $\operatorname{supp} V \subseteq [0,a]$ and $\operatorname{supp} f \subseteq [a,\infty)$ such that $f(t)t^{-r-k+\alpha}$ is bounded as $t \to \infty$.

This implies that if $W \in B_r(\mathbb{R})$, then there exist a quotient of sequences (f_n,φ_n) and $\alpha > 0$ such that $W = \left[\frac{f_n}{\varphi_n}\right]$ and $f_n(t) = O(t^{r+k-\alpha})$ as $t \to \infty$ for all $n \in \mathbb{N}$.

The *Stieltjes transform* for $W = \left[\frac{f_n}{\varphi_n}\right] \in B_r(\mathbb{R})$ is given by

$$\Lambda_s^r W = \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-st} t^r \widehat{W}(t)\,dt, \quad \operatorname{Re} s > 0, \tag{4.1}$$

where Γ is the gamma function.

REMARKS **4.1**.

1. The Stieltjes transform is a linear mapping from $B_r(\mathbb{R})$ into the space of analytic functions in the half-plane $\operatorname{Re} s > 0$.

2. The Stieltjes transform $\Lambda^r_{(\cdot)}$ is consistent on $J'(r)$.

3. For $\sigma \geq 0$, $\mathcal{A}_\sigma = \{s \in \mathbb{C} : |s| \leq \sigma, \operatorname{Re} s \leq 0\} \cup (-\infty, 0)$. Let $W \in B_r(\mathbb{R})$. Then there exists $\sigma \geq 0$ such that $\Lambda^r_s W$ can be analytically extended to the region $\mathbb{C} \backslash \mathcal{A}_\sigma$.

PROPERTIES **4.1**. *Let $W \in B_r(\mathbb{R})$. Then for $\operatorname{Re} s > 0$,*

1. $\Lambda^r_s \tau_c W = \Lambda^r_{s+c} W$, $c > 0$, *where* $\tau_c \left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\tau_c f_n}{\varphi_n}\right]$ *and* $(\tau_c f)(t) = f(t - c)$.

2. $\Lambda^r_s D^m W = \frac{\Gamma(r+m+1)}{\Gamma(r+1)} \Lambda^{r+m}_s W$, $m = 1, 2, \dots$.

3. $\frac{d^m}{ds^m} \Lambda^r_s W = \frac{(-1)^m \Gamma(r+m+1)}{\Gamma(r+1)} \Lambda^{r+m}_s W = (-1)^m \Lambda^r_s D^m W$, $m = 1, 2, \dots$.

4. $\Lambda^{r+m}_s (t^m W) = \sum_{k=0}^m (-1)^k \binom{m}{k} s^k \Lambda^{r+k}_s W$, $m = 1, 2, \dots$, *where* $t \left[\frac{f_n}{\varphi_n}\right] = \left[\frac{(tf_n) * \varphi_n - f_n * (t\varphi_n)}{\varphi_n * \varphi_n}\right]$ *and* $t^m \left[\frac{f_n}{\varphi_n}\right]$ *is defined by induction.*

EXAMPLE **4.1**. Let $r > -1$ and $W = \sum_{n=0}^\infty \frac{D^n \delta}{(n!)^2}$ (see [8] for the convergence of the series). Then $\widehat{W}(t) = \sum_{n=0}^\infty \frac{t^n}{(n!)^2}$, and

$$\begin{aligned}
\Lambda^r_s W &= \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-st} t^r \left(\sum_{n=0}^\infty \frac{t^n}{(n!)^2} \right) dt \\
&= \frac{1}{\Gamma(r+1)} \sum_{n=0}^\infty \int_0^\infty e^{-st} \frac{t^{r+n}}{(n!)^2} dt \\
&= \frac{1}{\Gamma(r+1)} \sum_{n=0}^\infty \frac{\Gamma(r+n+1)}{(n!)^2 s^{r+n+1}} .
\end{aligned}$$

Thus,

$$\Lambda^r_s W = \sum_{n=0}^\infty \frac{(r+1)_n}{(n!)^2 s^{n+r+1}}, \quad s \in \mathbb{C} \backslash (-\infty, 0],$$

where $(r+1)_n = (r+1)(r+2) \dots (r+n)$.

In particular, if $r = k \in \mathbb{N} \cup \{0\}$, then

$$\Lambda^k_s W = \frac{(-1)^k}{k!} \frac{d^k}{ds^k} \left(\frac{1}{s} \exp\left(\frac{1}{s}\right) \right), \quad s \in \mathbb{C} \backslash \{0\}.$$

THEOREM **4.1**. *Let $r > -1$ and $V \in \beta^+(\mathbb{R})$ with supp $V \subseteq [0,\sigma]$, for some $\sigma \geq 0$. Then, $\Lambda_s^r V = \frac{1}{\Gamma(r+1)} \sum_{n=0}^{\infty} \frac{c_n \Gamma(n+r+1)}{s^{n+r+1}}$ in the region $\{s \in \mathbb{C}\backslash(-\infty,0) : |s| > \sigma\}$, where $c_n = \frac{\widehat{V}^{(n)}(0)}{n!}$, $n = 0,1,2,\ldots$.*

P r o o f. See the proof of Theorem 3.5 in [9]. □

The proof of the following theorem follows from Theorem **4.1**.

THEOREM **4.2**. *Let $r > -1$ and $V \in \beta^+(\mathbb{R}) \cap \beta_c(\mathbb{R})$. Then $s^{r+1}\Lambda_s^r V \to \widehat{V}(0)$ as $s \to \infty$, $|\arg s| < \pi$.*

THEOREM **4.3**. *Let $r > -1$ and $W \in B_r(\mathbb{R})$. Then, there exists $\gamma > 0$ such that*

$$\Lambda_s^r W = O(s^{-\gamma}) \quad \text{as} \quad s \to \infty, \ |\arg s| < \pi.$$

P r o o f. Let $W = V + D^k f$, where V and f satisfy the necessary conditions for $W \in B_r(\mathbb{R})$. By Theorem **4.2**, $\Lambda_s^r W = O(s^{-(r+1)})$ as $s \to \infty$, $|\arg s| < \pi$.

Also, since $\Lambda_{(\cdot)}^r$ is consistent on $J'(r)$, there exists $\alpha > 0$ such that

$$\Lambda_s^r D^k f = O(s^{-\alpha}) \quad \text{as} \quad s \to \infty, \ |\arg s| < \pi \quad \text{(see [6])}.$$

By linearity of the transform, the conclusion follows. □

Previously [9], we established some Abelian type theorems which described the behavior of the Stieltjes transform of a Boehmian as $s \to 0$ ($s \to \infty$) in a wedge in the half-plane $\operatorname{Re} s > 0$ provided that the Boehmian behaved like the function $f(t) = t^\nu$ as $t \to 0^+$ ($t \to \infty$).

We now establish some Abelian type theorems which describe the behavior of the transform of a Boehmian as $s \to 0^+$ ($s \to \infty$) on the real line provided that the Boehmian behaves like the function $f(t) = t^\nu L(t)$ as $t \to 0^+$ ($t \to \infty$), where $L(t)$ is a slowly varying function.

A real-valued function $L(t)$ is *slowly varying at infinity*, if it is positive, measurable on $[a,\infty)$ for some $a > 0$, and such that for each $\lambda > 0$

$$\lim_{n\to\infty} \frac{L(\lambda t)}{L(t)} = 1.$$

A function $L_0(t)$ is *slowly varying at zero* if $L_0\left(\frac{1}{t}\right)$ is slowly varying at infinity.

For each $\alpha > 0$, $t^\alpha L(t) \to \infty$ as $t \to \infty$, and $t^{-\alpha} L(t) \to 0$ as $t \to \infty$.

We use similar notions as the ones in [3] that were used to describe the behavior of a distribution at the origin (infinity) to describe the behavior of a Boehmian at the origin (infinity).

DEFINITION **4.1**. Let $W \in \beta^+(\mathbb{R})$ and $\nu > -1$. Then $W(t) \approx t^\nu L_0(t)_+$ as $t \to 0^+$ provided there exist $a > 0$ and $T \in \mathcal{D}'(\mathbb{R})$ such that $\operatorname{supp} T \subseteq [0, a]$, $W(t) = t^\nu L_0(t)_+ + T(t)$ on $(-\varepsilon, a + \varepsilon)$ (some $\varepsilon > 0$), and

$$t^{-\nu-1} L_0^{-1}(t)_+ \langle T(x), \varphi\left(\frac{x}{t}\right)\rangle \to 0 \quad \text{as} \quad t \to 0^+, \tag{4.2}$$

for every function φ which is infinitely differentiable on some neighborhood of $[0, \infty)$. ($L_0(t)_+ = L_0(t)H(t)$, where $H(t)$ is the Heaviside function.)

LEMMA **4.1**. [9] *Let $V \in \beta^+(\mathbb{R})$ such that $\operatorname{supp} V \subseteq [a, b]$, where $a > 0$. Then, for $r > -1$, $\Lambda_s^r V$ is analytic in a neighborhood of the origin.*

THEOREM **4.4**. (Initial Value Theorem) *Let $W \in B_r(\mathbb{R})$ and $\nu > -1$. If $W(t) \approx t^\nu L_0(t)_+$ as $t \to 0^+$, then for $r > \nu$,*

$$\Lambda_t^r W \sim \frac{\Gamma(r-\nu)\Gamma(\nu+1)}{\Gamma(r+1)} t^{\nu-r} L_0(t) \quad \textit{as} \quad t \to 0^+.$$

P r o o f. Following the argument in the corresponding proof for distributions, that is, Theorem 2.1 in [3], we obtain

$$W = t^\nu L_0 + T + V + D^k f, \tag{4.3}$$

where $f \in L^1_{loc}(\mathbb{R})$ and $f(t)t^{-r-k+\alpha}$ is bounded on $[b, \infty)$ for some $\alpha > 0$. The supports of T, V, and f are in $[0, a]$, $[a, b]$, and $[b, \infty)$, respectively, $(0 < a < b)$, and T satisfies (4.2).

Apply the Stieltjes transform to both sides of (4.3) and use the linearity of the transform, then multiply by $(t^{\nu-r}L_0(t))^{-1}$. Thus,

$$\begin{aligned}(t^{\nu-r}L_0(t))^{-1}\Lambda_t^r W = (t^{\nu-r}L_0(t))^{-1}\Lambda_t^r t^\nu L_0 + (t^{\nu-r}L_0(t))^{-1}\Lambda_t^r T \\ + (t^{\nu-r}L_0(t))^{-1}\Lambda_t^r V + (t^{\nu-r}L_0(t))^{-1}\Lambda_t^r D^k f.\end{aligned}$$

Now, by taking the limit as $t \to 0^+$ of each term on the right-hand side, we see that the limit of the first term is $\frac{\Gamma(r-\nu)\Gamma(\nu+1)}{\Gamma(r+1)}$, and the limit of each of the other terms is zero. The proof of Theorem 2.1 in [3] contains a justification for these limits, except the third one, which contains the Boehmian V.

Thus, to complete the proof, we only need to show that

$$\lim_{t \to 0^+} \frac{\Lambda_t^r V}{t^{\nu-r} L_0(t)} = 0.$$

But, this follows by Lemma **4.1** and the fact that

$$\lim_{t\to 0^+} t^{\nu-r} L_0(t) = \lim_{t\to\infty} t^{r-\nu} L_0\left(\frac{1}{t}\right) = \infty.$$

Thus, the proof is complete. □

Definition **4.2**. Let $W \in \beta^+(\mathbb{R})$ and $\nu > -1$. Then, $W(t) \approx t^\nu L(t)$ as $t \to \infty$ provided there exist $a > 0$, $k \in \mathbb{N}$, $g \in L^1_{loc}(\mathbb{R})$ such that $W(t) = D^k g(t)$ on (a, ∞) and $\dfrac{g(t)}{t^{k+\nu} L(t)} \to \dfrac{1}{(\nu+1)_k}$ as $t \to \infty$.

Theorem **4.5**. (Final Value Theorem) *Let $W \in \beta^+(\mathbb{R})$ and $\nu > -1$. If $W(t) \approx t^\nu L(t)$ as $t \to \infty$, then for $r > \nu$,*

$$\Lambda_t^r W \sim \frac{\Gamma(r-\nu)\Gamma(\nu+1)}{\Gamma(r+1)} t^{\nu-r} L(t) \quad \text{as} \quad t \to \infty.$$

P r o o f. Since $W(t) \approx t^\nu L(t)$ as $t \to \infty$,

$$W = V + D^k g, \tag{4.4}$$

where for some $a > 0$, $V \in \beta^+(\mathbb{R})$ with $\operatorname{supp} V \subseteq [0, a]$, and $k \in \mathbb{N}$ such that $W(t) = D^k g(t)$ on (a, ∞), for some $g \in L^1_{loc}(\mathbb{R})$, and $\dfrac{g(t)}{t^{k+\nu} L(t)} \to \dfrac{1}{(\nu+1)_k}$ as $t \to \infty$. It follows that $W \in B_r(\mathbb{R})$. Now, by Theorem 2.2 in [3],

$$\Lambda_t^r D^k g \sim \frac{\Gamma(r-\nu)\Gamma(\nu+1)}{\Gamma(r+1)} t^{\nu-r} L(t) \quad \text{as} \quad t \to \infty. \tag{4.5}$$

By Theorem **4.1**, there exist $\alpha > 0$ and $c_n \in \mathbb{C}\,(n \in \mathbb{N})$ such that for $r > \nu > -1$ and $t > \alpha$,

$$\frac{t^{r-\nu} \Lambda_t^r V}{L(t)} = \frac{1}{\Gamma(r+1) t^{\nu+1} L(t)} \sum_{n=0}^{\infty} \frac{c_n \Gamma(n+r+1)}{t^n}.$$

Since $\nu + 1 > 0$, $t^{\nu+1} L(t) \to \infty$ as $t \to \infty$. Thus,

$$\lim_{t\to\infty} \frac{t^{r-\nu}\Gamma(r+1)\Lambda_t^r V}{\Gamma(r-\nu)\Gamma(\nu+1)L(t)} = 0. \tag{4.6}$$

Therefore, by (4.4), (4.5), and (4.6) the conclusion follows. □

Inversion formulas are essential for integral transforms for both theory and applications. Since the Stieltjes transform for Boehmians is defined as an iterated Laplace transform, Laplace inversion formulas may be useful.

To find the Boehmian whose Stieltjes transform is known, we first apply an inversion formula for the classical Laplace transform. This gives a function whose domain is a subset of the real line. Thus, to apply another inversion formula to this function, it would be convenient to have a real inversion formula and not have to use analytic continuation in order to apply a complex inversion formula.

Thus, the real inversion formula developed for the Laplace transformation in the previous section becomes useful.

The following inversion formula for the Stieltjes transform follows directly from Theorem **3.1**.

THEOREM **4.6**. *Let W be a Stieltjes transformable Boehmian. Then,*

$$W = \delta\text{-}\lim_{\sigma\to\infty} \frac{\Gamma(r+1)}{2\pi i} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k-1)!} \frac{\sigma}{(k\sigma)^r} \left(\int_{\gamma-i\infty}^{\gamma+i\infty} e^{k\sigma s} \Lambda_s^r W \, ds \right) e^{k\sigma t}, \tag{4.7}$$

for any $\gamma > 0$.

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