Research Article

Open Access

M. Purificação Coelho, M. Antónia Duffner, and Alexander E. Guterman*

Immanant Conversion on Symmetric Matrices

Abstract: Let $\Sigma_n(\mathbb{C})$ denote the space of all $n \times n$ symmetric matrices over the complex field $\mathbb{C}$.

The main objective of this paper is to prove that the maps $\Phi : \Sigma_n(\mathbb{C}) \rightarrow \Sigma_n(\mathbb{C})$ satisfying for any fixed irreducible characters $\chi, \chi' : S_n \rightarrow \mathbb{C}$ the condition $d_\chi(A + \alpha B) = d_{\chi'}(\Phi(A) + \alpha \Phi(B))$ for all matrices $A, B \in \Sigma_n(\mathbb{C})$ and all scalars $\alpha \in \mathbb{C}$ are automatically linear and bijective. As a corollary of the above result we characterize all such maps Φ acting on $\Sigma_n(\mathbb{C})$.

Keywords: Determinant, permanent, immanant, preservers, converters, symmetric matrices

MSC: 06A06, 15A03, 15A04, 15A86

M. Purificação Coelho, M. Antónia Duffner: Departamento de Matemática, Faculdade de Ciências, Universidade de Lisboa, Bloco C6, Piso 2, Campo Grande, 1700-016 Lisboa, Portugal, E-mail: mpcoelho@cii.fc.ul.pt, mamonteiro@fc.ul.pt

***Corresponding Author: Alexander E. Guterman:** Department of Higher Algebra, Faculty of Mathematics and Mechanics, Moscow State University, Moscow, 119991, Russia, E-mail: guterman@list.ru

1 Introduction

Let $\mathbb{C}$ be the field of the complex numbers, S_n be the permutation group acting on the set $\{1, \dots, n\}$, $M_{m,n}(\mathbb{C})$ denote the space of all $m \times n$ -matrices with entries in $\mathbb{C}$, $M_n(\mathbb{C}) = M_{n,n}(\mathbb{C})$ and $\Sigma_n(\mathbb{C}) \subseteq M_n(\mathbb{C})$ denote the subspace of symmetric matrices over $\mathbb{C}$. Let $|X|$ denote the cardinality of a set X .

By $J_{m,n} \in M_{m,n}(\mathbb{C})$ we denote the matrix of all ones, by I_n the $n \times n$ identity matrix. As usual, A^t denotes the transposed matrix of A . If $A, B \in M_n(\mathbb{C})$, then the Hadamard (entrywise) product of A and B is denoted by $A \circ B$.

Let (E_{ij}) be the standard basis of $M_n(\mathbb{C})$, where the matrix E_{ij} has 1 in (i, j) -th position and 0 elsewhere. By U_{ij} we denote the matrix

$$U_{ij} = \begin{cases} E_{ii}, & \text{if } i = j \\ E_{ij} + E_{ji}, & \text{if } i \neq j \end{cases} \quad (1.1)$$

It is clear that $(U_{ij})_{1 \leq i \leq j \leq n}$ forms a basis of $\Sigma_n(\mathbb{C})$.

If σ is a permutation of S_n , we will denote by $P(\sigma) = (p_{ij})$ the permutation matrix corresponding to σ , i.e.,

$$p_{ij} = \begin{cases} 1, & \text{if } j = \sigma^{-1}(i) \\ 0, & \text{otherwise} \end{cases}$$

Definition 1.1. If $\chi : S_n \rightarrow \mathbb{C}$ is an irreducible character of S_n , the immanant associated with χ is the function $d_\chi : M_n(\mathbb{C}) \rightarrow \mathbb{C}$ defined by

$$d_\chi(A) = \sum_{\sigma \in S_n} \chi(\sigma) \prod_{i=1}^n a_{i\sigma(i)}$$

for all $A = (a_{ij}) \in M_n(\mathbb{C})$.

For details on the representation theory of S_n , we refer the reader to [8] or [11].

If χ is the alternating character of S_n , then d_χ is the determinant, and if χ is the principal character of S_n , then d_χ is the permanent.

Different immanant preserving and converting maps on $M_n(\mathbb{C})$ were studied by several authors; see for example [3, 9, 12] and references therein.

In [6] Dolinar and Šemrl characterized the surjective maps Φ on $M_n(\mathbb{C})$ which satisfy $\det(\Phi(A) + \alpha\Phi(B)) = \det(A + \alpha B)$ for all $A, B \in M_n(\mathbb{C})$ and for all $\alpha \in \mathbb{C}$. They proved that such maps must be linear and then applied the result of Frobenius [7] on linear preservers of the determinant to characterize them.

In [3] Coelho and Duffner generalized this result for arbitrary immanants.

Tan and Wang in [12] removed the surjectivity assumption from [6]. Then Kuzma [9] established a general result removing the surjectivity assumption for all immanant preserving and immanant converting maps on the space $M_n(\mathbb{C})$, in particular for the maps considered in [3].

The investigation of the corresponding problem for the space of symmetric matrices was initiated by Lim in [10]. Then it was intensively investigated in the works [1, 2, 4, 5]; see also references therein.

However the following questions remained open:

Consider the map $\Phi : \Sigma_n(\mathbb{C}) \rightarrow \Sigma_n(\mathbb{C})$ such that

$$d_\chi(A + \alpha B) = d_{\chi'}(\Phi(A) + \alpha\Phi(B)) \text{ for all } A, B \in \Sigma_n(\mathbb{C}), \alpha \in \mathbb{C}, \quad (1.2)$$

where $\chi, \chi' : S_n \rightarrow \mathbb{C}$ are arbitrary irreducible characters.

Is the equation (1.2) on the space $\Sigma_n(\mathbb{C})$ strong enough to force the map Φ to be bijective? What is the characterization of all such maps Φ on $\Sigma_n(\mathbb{C})$?

In this paper we answer both these questions. Our main result states that indeed any map satisfying equation (1.2) is bijective. Moreover, it turns out that such maps are automatically linear.

Theorem 1.2. *Let $n \geq 3$ and $\chi, \chi' : S_n \rightarrow \mathbb{C}$ be irreducible characters. If $\Phi : \Sigma_n(\mathbb{C}) \rightarrow \Sigma_n(\mathbb{C})$ is a map such that for all $A, B \in \Sigma_n(\mathbb{C})$ and for all $\alpha \in \mathbb{C}$,*

$$d_\chi(A + \alpha B) = d_{\chi'}(\Phi(A) + \alpha\Phi(B))$$

then Φ is linear and bijective.

Moreover taking $\alpha = 0$ in condition (1.2) we have that

$$d_\chi(A) = d_{\chi'}(\Phi(A)) \text{ for all } A \in \Sigma_n(\mathbb{C}).$$

Thus by [5, Theorem 1.1] where linear immanant converting maps on $\Sigma_n(\mathbb{C})$ were studied, we can conclude that there are no such transformations Φ if $\chi \neq \chi'$. Thus we have the following corollary:

Corollary 1.3. *Let $n \geq 3$ and $\chi, \chi' : S_n \rightarrow \mathbb{C}$ be irreducible characters. If $\Phi : \Sigma_n(\mathbb{C}) \rightarrow \Sigma_n(\mathbb{C})$ is a map satisfying (1.2) then $\chi = \chi'$.*

Corollary 1.4. *Let $n > 3$ and $\chi : S_n \rightarrow \mathbb{C}$ be an irreducible character. Let $\Phi : \Sigma_n(\mathbb{C}) \rightarrow \Sigma_n(\mathbb{C})$ be a map such that for all $A, B \in \Sigma_n(\mathbb{C})$ and for all $\alpha \in \mathbb{C}$,*

$$d_\chi(\Phi(A) + \alpha\Phi(B)) = d_\chi(A + \alpha B).$$

Then

1. If χ is non-linear, then there exist a permutation $\sigma \in S_n$ and a matrix $C \in \Sigma_n(\mathbb{C})$ such that $\prod_{t=1}^n c_{t\pi(t)} = 1$ whenever $\chi(\pi) \neq 0$ satisfying

$$\Phi(X) = C \circ \left(P(\sigma)XP(\sigma^{-1}) \right) \text{ for all } X \in \Sigma_n(\mathbb{C}).$$

2. If χ is the principal character, then there exist a permutation $\sigma \in S_n$ and a diagonal matrix $D \in M_n(\mathbb{C})$ such that $\text{per } D \in \{-1, 1\}$ satisfying

$$\Phi(X) = DP(\sigma)XP(\sigma^{-1})D \text{ for all } X \in \Sigma_n(\mathbb{C}).$$

3. If χ is the alternating character, then there exist $c \in \mathbb{C} \setminus \{0\}$ and a matrix $M \in M_n(\mathbb{C})$ such that it holds $\det(cM^2) = 1$ satisfying

$$\Phi(X) = cMXM^t \text{ for all } X \in \Sigma_n(\mathbb{C}).$$

Proof. To get the result we apply Theorem 1.2 with $\chi = \chi'$ and

1. [4, Theorem 2.2],
2. [2, Theorem 2.1],
3. [10, Theorem 12].

□

2 The χ - immanantal adjoint of a symmetric matrix

Let $A \in M_n(\mathbb{C})$. We denote by $A_{(i,j)}$ the $n \times n$ -matrix obtained from A by replacing the (i, j) -th entry by 1 and the other elements of i -th row and j -th column by 0.

Definition 2.1. Let $A \in M_n(\mathbb{C})$. We define the χ -adjoint matrix of A to be $C = \text{adj}_\chi(A) \in M_n(\mathbb{C})$, where $c_{ij} = d_\chi(A_{(i,j)})$.

In the following lemma we will prove that the χ -adjoint matrix of a symmetric matrix is also a symmetric matrix.

Lemma 2.2. Let $A \in \Sigma_n(\mathbb{C})$. Then $\text{adj}_\chi(A) \in \Sigma_n(\mathbb{C})$.

Proof. Let us fix (i, j) and denote $B = A_{(i,j)}$, $D = A_{(j,i)} \in M_n(\mathbb{C})$ and $C = \text{adj}_\chi(A) \in M_n(\mathbb{C})$.

Since A is symmetric, it follows that $b_{kl} = d_{lk}$ for all $k, l \in \{1, \dots, n\}$, i.e., $B = D^t$.

Hence, $d_\chi(B) = d_\chi(D^t) = d_\chi(D)$, i.e.,

$$c_{ij} = d_\chi(A_{(i,j)}) = d_\chi(A_{(j,i)}) = c_{ji}.$$

Thus $C \in \Sigma_n(\mathbb{C})$.

□

For the proof of the main theorem we need to define the following symmetric $(0, 1)$ -matrices.

Definition 2.3. For any $\sigma \in S_n$ let $B(\sigma) \in \Sigma_n(\mathbb{C})$ be defined by

$$(B(\sigma))_{i,j} = \begin{cases} 1, & \text{if } j = \sigma(i) \text{ or } j = \sigma^{-1}(i) \\ 0, & \text{otherwise.} \end{cases}$$

Remark that if σ is the identity or a product of transpositions then $B(\sigma) = P(\sigma)$.

For any $\sigma \in S_n$ the following sets related to σ were introduced in [4] and will be useful for our further discussions.

Definition 2.4. Let $\sigma = \sigma_1 \dots \sigma_k$ be the cycle decomposition of $\sigma \in S_n$. The set C_σ is

$$C_\sigma = \{\sigma_1^{p_1} \dots \sigma_k^{p_k} \mid p_i = \pm 1, i = 1, \dots, k\}.$$

Definition 2.5. For each $\sigma \in S_n$ the set D_σ is

$$D_\sigma = \{\rho \in S_n \mid \rho(t) = \sigma(t) \vee \rho(t) = \sigma^{-1}(t), \forall t \in \{1, \dots, n\}\}.$$

Since all permutations belonging to C_σ have the same cyclic structure as σ it is clear that C_σ is contained in the conjugacy class of σ . Following [4] we list below some properties of the sets C_σ and D_σ .

Proposition 2.6. [4, Remarks 3.1 and 3.2] Let $\sigma \in S_n$ be a fixed permutation. Then

1. $C_\sigma \subseteq D_\sigma$.
2. $C_\sigma = D_\sigma$ if and only if σ does not contain in its cycle decomposition any cycle of even length greater than 2.
3. If $\rho \in D_\sigma \setminus C_\sigma$ then ρ contains more transpositions in its cycle decomposition than σ does.

Lemma 2.7. Let $\sigma \in S_n$ and $A \in M_n(\mathbb{C})$ be such that $a_{ij} = 0$ whenever $(B(\sigma))_{ij} = 0$. Then

$$d_\chi(A) = \sum_{\rho \in D_\sigma} \chi(\rho) \prod_{t=1}^n a_{t\rho(t)}. \quad (2.1)$$

Moreover if A is symmetric and χ vanishes on $D_\sigma \setminus C_\sigma$, then

$$d_\chi(A) = |C_\sigma| \chi(\sigma) \prod_{t=1}^n a_{t\sigma(t)}.$$

Proof. From the definitions of the matrix $B(\sigma)$ and the set D_σ , we easily conclude that $\prod_{t=1}^n (B(\sigma))_{t\rho(t)} \neq 0$ if and only if $\rho \in D_\sigma$. Since $a_{ij} = 0$ whenever $(B(\sigma))_{ij} = 0$, we get $\prod_{t=1}^n a_{t\rho(t)} = 0$ if $\rho \notin D_\sigma$, and consequently we obtain equality (2.1).

Assume in addition that A is symmetric and $\chi(\rho) = 0$ whenever $\rho \in D_\sigma \setminus C_\sigma$. Since A is symmetric we have $\prod_{t=1}^n a_{t\rho(t)} = \prod_{t=1}^n a_{t\sigma(t)}$ for all $\rho \in C_\sigma$. On the other hand, since C_σ is contained in the conjugacy class of σ we also have $\chi(\rho) = \chi(\sigma)$ for all $\rho \in C_\sigma$. Since χ vanishes on $D_\sigma \setminus C_\sigma$, applying equality (2.1) we obtain $d_\chi(A) = \sum_{\rho \in C_\sigma} \chi(\rho) \prod_{t=1}^n a_{t\rho(t)} = |C_\sigma| \chi(\sigma) \prod_{t=1}^n a_{t\sigma(t)}$. $\square$

Lemma 2.8. Let $\sigma \in S_n$. We have that

- (a) $d_\chi(B(\sigma)) = \sum_{\rho \in D_\sigma} \chi(\rho)$.
- (b) If χ vanishes on $D_\sigma \setminus C_\sigma$, then $d_\chi(B(\sigma)) = |C_\sigma| \chi(\sigma)$.

Proof. This is a particular case of the previous lemma. $\square$

Lemma 2.9. Let $\tau = (i_1, \dots, i_s) \in S_n$ be a cycle of length $s > 2$, and $\rho \in S_n$ be a permutation such that for all $t \in \{i_1, i_2, \dots, i_{s-1}\}$, we have that $\rho(t) = \tau(t)$ or $\rho(t) = \tau^{-1}(t)$ and $\rho(i_s) \in \{i_1, \dots, i_s\} \setminus \{i_1, i_{s-1}\}$. Then one of the transpositions (i_1, i_2) or (i_{s-1}, i_{s-2}) belongs to the cycle decomposition of ρ .

Proof. If $s = 3$, the result follows. Suppose that $s > 3$. With our assumptions we conclude that $\rho(\{i_1, \dots, i_s\}) = \{i_1, \dots, i_s\}$, $\rho^{-1}(i_1) = i_2$ and $\rho^{-1}(i_{s-1}) = i_{s-2}$. We also have $\rho(i_1) = i_2$ or $\rho(i_1) = i_s$.

Note that if $\rho(i_1) = i_2$ then the transposition (i_1, i_2) belongs to the cycle decomposition of ρ .

If $\rho(i_1) = i_s$, then $\rho(i_{s-1}) \neq i_s$ and thus $\rho(i_{s-1}) = i_{s-2}$. In this case the transposition (i_{s-1}, i_{s-2}) belongs to the cycle decomposition of ρ . $\square$

Lemma 2.10. Let $\sigma, \rho \in S_n$ and $i \in \{1, \dots, n\}$ be such that $\rho(i) \neq \sigma(i)$, $\rho(i) \neq \sigma^{-1}(i)$ and for all $t \in \{1, \dots, n\} \setminus \{i\}$, $\rho(t) = \sigma(t)$ or $\rho(t) = \sigma^{-1}(t)$. Then ρ contains in its cycle decomposition more transpositions than σ does.

Proof. Let $\sigma = \tau_1 \dots \tau_k$ be the cycle decomposition of σ and assume that i lies in the support of τ_k . For each $j \in \{1, \dots, k\}$, denote by $\text{supp}(\tau_j)$ the support of τ_j .

Since, for all $t \in \{1, \dots, n\} \setminus \{i\}$, $\rho(t) \in \{\sigma(t), \sigma^{-1}(t)\}$, we easily conclude that if, for some $j < k$, τ_j is a transposition then τ_j must also belong to the cycle decomposition of ρ .

It is also clear that

$$\rho(\text{supp}(\tau_j)) \subseteq \text{supp}(\tau_j), \quad j = 1, \dots, k-1.$$

As ρ is bijective it follows that

$$\rho(\text{supp}(\tau_j)) = \text{supp}(\tau_j), \quad j = 1, \dots, k-1$$

and also

$$\rho(\text{supp}(\tau_k)) = \{1, \dots, n\} \setminus \bigcup_{j=1}^{k-1} \rho(\text{supp}(\tau_j)) = \{1, \dots, n\} \setminus \bigcup_{j=1}^{k-1} \text{supp}(\tau_j) = \text{supp}(\tau_k).$$

Therefore $\rho(i) \in \text{supp}(\tau_k)$. Since $\rho(i) \neq \sigma(i) = \tau_k(i)$, $\rho(i) \neq \sigma^{-1}(i) = \tau_k^{-1}(i)$ and for all $t \in \text{supp}(\tau_k) \setminus \{i\}$, $\rho(t) = \sigma(t) = \tau_k(t)$ or $\rho(t) = \sigma^{-1}(t) = \tau_k^{-1}(t)$, we conclude that τ_k is a cycle of length greater than two. Thus,

by Lemma 2.9, there exist distinct integers $p, q \in \text{supp}(\tau_k)$ such that the transposition $(p\ q)$ belongs to the cycle decomposition of ρ . The result follows. $\square$

Lemma 2.11. *Let $\sigma \in S_n$. Assume that $\chi(\rho) = 0$ whenever ρ contains in its cycle decomposition more transpositions than σ . If B is a matrix with the same pattern of zeros as $B(\sigma)$, then $(\text{adj}_\chi(B))_{ij} = 0$ whenever $(B(\sigma))_{ij} = 0$.*

Proof. Assume that B is a matrix with the same pattern of zeros as $B(\sigma)$, and consider $i, j \in \{1, \dots, n\}$ such that $(B(\sigma))_{ij} = 0$. Then we have $j \neq \sigma(i)$ and $j \neq \sigma^{-1}(i)$.

Case 1: $\sigma(i) = i$.

In this case the i^{th} column of B has exactly one nonzero entry which is the (i, i) -entry. Since $j \neq i$ it follows that the i^{th} column of $B_{(i,j)}$ is null and consequently $(\text{adj}_\chi(B))_{ij} = 0$.

Case 2: For some $k \neq i, j$, the transposition (i, k) belongs to the cycle decomposition of σ .

Then the k^{th} column of B has exactly one nonzero entry which is the (i, k) -entry. Since $j \neq k$ it follows that the k^{th} column of $B_{(i,j)}$ is null and $(\text{adj}_\chi(B))_{ij} = 0$.

Case 3: i belongs to the support of a cycle of σ with length greater than two.

Let $A = B_{(i,j)}$. By definition we have

(i) $a_{iq} \neq 0$ if and only if $q = j$, and

(ii) for $p \neq i$, if $a_{pq} \neq 0$ then $(B(\sigma))_{pq} \neq 0$.

We will prove that $d_\chi(A) = 0$ by showing that $\chi(\rho) \prod_{t=1}^n a_{t\rho(t)} = 0$, for all $\rho \in S_n$.

Assume that $\prod_{t=1}^n a_{t\rho(t)} \neq 0$ for some $\rho \in S_n$. Then we must have $\rho(i) = j$, and thus $\rho(i) \neq \sigma(i)$, and $\rho(i) \neq \sigma^{-1}(i)$. On the other hand, for $t \neq i$, either $\rho(t) = \sigma(t)$, or $\rho(t) = \sigma^{-1}(t)$. Using Lemma 2.10 we conclude that ρ contains in its cycle decomposition more transpositions than σ does, and consequently $\chi(\rho) = 0$. It follows that $d_\chi(A) = 0$, that is $(\text{adj}_\chi(B))_{ij} = 0$. $\square$

3 Proof of the main results

In what follows m denotes the dimension of $\Sigma_n(\mathbb{C})$, i.e., $m = \frac{n(n+1)}{2}$. Let $\Phi : \Sigma_n(\mathbb{C}) \rightarrow \Sigma_n(\mathbb{C})$ be a map satisfying (1.2). We are going to prove that Φ is linear by considering an appropriate basis $\mathcal{B}$ of $\Sigma_n(\mathbb{C})$ and stating the existence of an $m \times m$ matrix K such that for all $A \in \Sigma_n(\mathbb{C})$,

$$u_{\Phi(A)} = Ku_A,$$

where u_X denotes the column vector of X in the basis $\mathcal{B}$, i.e., if $\mathcal{B} = \{V_1, \dots, V_m\}$ and $X = \sum_{i=1}^m \beta_i V_i$, then $u_X = (\beta_1, \dots, \beta_m)^t$.

Then we conclude that Φ is bijective by proving that the matrix K is nonsingular.

We start with some lemmas which allow us to prove the existence of a basis of $\Sigma_n(\mathbb{C})$ consisting of χ -adjoints of symmetric matrices.

Lemma 3.1. *Let $\lambda \in \mathbb{C} \setminus \{0, 1, \frac{1}{1-n}\}$, and*

$$A^{(i)} = \lambda U_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n U_{jj}, \quad i = 1, \dots, n,$$

where U_{jj} are defined by (1.1). Then

$$\langle \text{adj}_\chi(A^{(1)}), \dots, \text{adj}_\chi(A^{(n)}) \rangle = \langle U_{11}, \dots, U_{nn} \rangle.$$

Proof. Let $i \in \{1, \dots, n\}$. Since $A^{(i)}$ is a diagonal matrix, for all $k \neq l$ all the entries of the l -th row of $A_{(k,l)}^{(i)}$ are zero. Thus for all $k \neq l$ we have that $d_\chi(A_{(k,l)}^{(i)}) = 0$ and the matrix $\text{adj}_\chi(A^{(i)})$ is diagonal. So we have $\langle \text{adj}_\chi(A^{(1)}), \dots, \text{adj}_\chi(A^{(n)}) \rangle \subseteq \langle E_{11}, \dots, E_{nn} \rangle = \langle U_{11}, \dots, U_{nn} \rangle$.

On the other hand $A_{(k,k)}^{(i)} = A^{(i)}$ when $k \neq i$ and $A_{(i,i)}^{(i)} = I_n$. Since $d_\chi(I_n) = \chi(id)$ and $d_\chi(A^{(i)}) = \lambda\chi(id)$, it follows that

$$adj_\chi(A^{(i)}) = \chi(id) \left(U_{ii} + \lambda \sum_{\substack{j=1 \\ j \neq i}}^n U_{jj} \right).$$

As $\chi(id) \neq 0$ and $\lambda \neq 0, 1, \frac{1}{1-n}$, using [9, Lemma 2] we easily conclude that the matrices $adj_\chi(A^{(1)}), \dots, adj_\chi(A^{(n)})$ are linearly independent, and the result follows. $\square$

Definition 3.2. Define $\mathfrak{S}_\chi$ to be a conjugacy class of S_n not containing the identity permutation and satisfying the following conditions:

- (i) χ does not vanish on $\mathfrak{S}_\chi$
- (ii) $\chi(\rho) = 0$ whenever ρ contains in its cycle decomposition more transpositions than the permutations of the class $\mathfrak{S}_\chi$.

Note that for each irreducible character χ such class $\mathfrak{S}_\chi$ does exist. Indeed there exists $\sigma \in S_n$ such that $\sigma \neq id$, and $\chi(\sigma) \neq 0$. Among these permutations we choose one with the maximal number of transpositions in its disjoint cycle decomposition, and take its conjugacy class to be $\mathfrak{S}_\chi$.

Since $\mathfrak{S}_\chi$ is a conjugacy class all $\sigma \in \mathfrak{S}_\chi$ have the same cyclic structure. Thus the class $\mathfrak{S}_\chi$ determines nonnegative integers r, s such that r is the number of elements fixed by each permutation $\sigma \in \mathfrak{S}_\chi$, and s is the number of transpositions in the cycle decomposition of σ .

For the proof of the main result we need to introduce some more notations. In the following definitions σ is an arbitrarily fixed permutation in $\mathfrak{S}_\chi$.

Definition 3.3. We denote by M_σ the subset of $\{1, \dots, n\} \times \{1, \dots, n\}$, such that

$$M_\sigma = \{(i, j) | (B(\sigma))_{ij} \neq 0 \text{ and } i \leq j\}.$$

Also we consider the following decomposition of the set M_σ into the disjoint union of the subsets:

$$M_\sigma = M_1 \dot{\cup} M_2 \dot{\cup} M_3,$$

where

$$\begin{aligned} M_1 &= \{(i, i) | \sigma(i) = i\}, \\ M_2 &= \{(i, j) | i < j, \sigma(i) = j \text{ and } \sigma(j) = i\}, \\ M_3 &= M_\sigma \setminus (M_1 \cup M_2). \end{aligned}$$

Remark 3.4. It is straightforward to see that

$$B(\sigma) = \sum_{(i,j) \in M_\sigma} U_{ij},$$

where U_{ij} are defined by equation (1.1). Also $|M_\sigma| = t$ where $t = n - s$, $|M_1| = r$ and $|M_2| = s$.

Definition 3.5. Let us order the set

$$\{U_1, \dots, U_t\} = \{U_{ij} | (i, j) \in M_\sigma\}$$

in accordance with the above decomposition of M_σ . Here $U_1, \dots, U_r$ are the matrices U_{ii} for $(i, i) \in M_1$, $U_{r+1}, \dots, U_{r+s}$ are the matrices U_{ij} for $(i, j) \in M_2$, and $U_{r+s+1}, \dots, U_t$ are the matrices U_{ij} for $(i, j) \in M_3$.

From now on consider $\lambda \in \mathbb{C} \setminus \{0, 1, \frac{1}{1-t}\}$.

Definition 3.6. We define the matrices

$$B^{(i)}(\sigma) = \lambda U_i + \sum_{\substack{j=1 \\ j \neq i}}^t U_j$$

and $C^{(i)}(\sigma) = \text{adj}_\chi(B^{(i)}(\sigma))$, $i = 1, \dots, t$.

Lemma 3.7. Let $\sigma \in \mathfrak{S}_\chi$. The matrices $B^{(1)}(\sigma), \dots, B^{(t)}(\sigma)$ are symmetric and have the same pattern of zeros as $B(\sigma)$.

Proof. Follows directly from the previous definitions. $\square$

Lemma 3.8. Let $\sigma \in \mathfrak{S}_\chi$. We have that

(a) The matrices $C^{(1)}(\sigma), \dots, C^{(t)}(\sigma)$ are symmetric.

(b) $C^{(i)}(\sigma) \in \langle U_1, \dots, U_t \rangle$, $i = 1, \dots, t$.

(c) If $(j, k) \in M_\sigma$ and $A = (B^{(i)}(\sigma))_{(j,k)}$, then

$$d_\chi(A) = \chi(\sigma) \sum_{\rho \in C_\sigma} \prod_{l=1}^n a_{lp(l)}.$$

Proof. (a) By Lemma 3.7 and Lemma 2.2.

(b) Since $\sigma \in \mathfrak{S}_\chi$ and $B^{(i)}(\sigma)$ has the same pattern of zeros as $B(\sigma)$, we can apply Lemma 2.11 to conclude that $(C^{(i)}(\sigma))_{jk} = 0$ whenever $(B(\sigma))_{jk} = 0$. It follows that $C^{(i)}(\sigma) \in \langle \{U_{jk} | (j, k) \in M_\sigma\} \rangle = \langle U_1, \dots, U_t \rangle$.

(c) Notice that if $(j, k) \in M_\sigma$ and $A = (B^{(i)}(\sigma))_{(j,k)}$, then $a_{pq} = 0$ whenever $(B(\sigma))_{pq} = 0$. Applying Lemma 2.7 it follows that $d_\chi(A) = \sum_{\rho \in D_\sigma} \chi(\rho) \prod_{l=1}^n a_{lp(l)}$. By definition of $\mathfrak{S}_\chi$ we have that $\chi(\rho) = 0$ if $\rho \in D_\sigma \setminus C_\sigma$. The result follows. $\square$

We will now describe the coefficients of each $C^{(i)}(\sigma)$ as a linear combination of $U_1, \dots, U_t$.

Lemma 3.9. Let $\sigma \in \mathfrak{S}_\chi$. We have that

(a) If $1 \leq i \leq r$, then

$$C^{(i)}(\sigma) = d_\chi(B(\sigma))(U_i + \lambda \sum_{\substack{j=1 \\ j \neq i}}^{r+s} U_j + \frac{\lambda}{2} \sum_{j=r+s+1}^t U_j).$$

(b) If $r < i \leq r + s$, then

$$C^{(i)}(\sigma) = d_\chi(B(\sigma))(\lambda U_i + \lambda^2 \sum_{\substack{j=1 \\ j \neq i}}^{r+s} U_j + \frac{\lambda^2}{2} \sum_{j=r+s+1}^t U_j).$$

(c) If $r + s < i \leq t$, then

$$C^{(i)}(\sigma) = d_\chi(B(\sigma))(\frac{1}{2} U_i + \lambda \sum_{j=1}^{r+s} U_j + \frac{\lambda}{2} \sum_{\substack{j=r+s+1 \\ j \neq i}}^t U_j).$$

Proof. Denote simply by $B^{(i)}$ the matrix $B^{(i)}(\sigma)$ and by $C^{(i)}$ the matrix $C^{(i)}(\sigma)$. In each case we will prove the desired equality by computing the entries $(C^{(i)})_{jk}$, with $(j, k) \in M_\sigma$. Recall that $\{U_1, \dots, U_t\} = \{U_{jk} | (j, k) \in M_\sigma\}$ and by the previous lemma $C^{(i)} \in \langle U_1, \dots, U_t \rangle$.

(a) In this case $U_i = U_{pp}$ for some $(p, p) \in M_1$, and $B^{(i)}$ differs from $B(\sigma)$ only at the entry (p, p) which is equal to λ and so we easily conclude that $d_\chi(B^{(i)}) = \lambda d_\chi(B(\sigma))$. We have $(B^{(i)})_{(p,p)} = B(\sigma)$, $(C^{(i)})_{pp} = d_\chi(B(\sigma))$ and if $(j, j) \in M_1$ and $j \neq p$ we have $(B^{(i)})_{(j,j)} = B^{(i)}$ and $(C^{(i)})_{jj} = \lambda d_\chi(B(\sigma))$.

For $(j, k) \in M_2$ we also have $(B^{(i)})_{(j,k)} = B^{(i)}$ and consequently $(C^{(i)})_{jk} = \lambda d_\chi(B(\sigma))$.

Suppose now that $(j, k) \in M_3$ and let $A = (B^{(i)})_{(j,k)}$. Then j belongs to the support of a cycle of length greater than 2 in the cycle decomposition of σ and either $k = k_1$ or $k = k_2$, where $k_1 = \sigma(j)$ and $k_2 = \sigma^{-1}(j)$. The nonzero entries of the j^{th} -row of $B^{(i)}$ are the entries (j, k_1) and (j, k_2) .

Assume first that $k = k_1$. Then A is obtained from $B^{(i)}$ replacing by zero the 1's at entries (j, k_2) and $(\sigma(k_1), k_1)$ and we easily conclude that for any $\rho \in C_\sigma$, $\prod_{l=1}^n a_{lp(l)} \neq 0$ if and only if $\rho(j) = k_1 = \sigma(j)$. Moreover all these products are equal to λ and the cardinality of the set $\{\rho \in C_\sigma : \rho(j) = \sigma(j)\}$ is $\frac{1}{2} |C_\sigma|$. So applying the previous lemma and Lemma 2.8 we get $(C^{(i)})_{jk} = d_\chi(A) = \frac{1}{2} \lambda |C_\sigma| \chi(\sigma) = \frac{\lambda}{2} d_\chi(B(\sigma))$.

Suppose now that $k = k_2$. Then A is obtained from $B^{(i)}$ replacing by zero the 1's at entries (j, k_1) and $(\sigma^{-1}(k_2), k_2)$ and for any $\rho \in C_\sigma$, $\prod_{l=1}^n a_{l\rho(l)} \neq 0$ if and only if $\rho(j) = k_2 = \sigma^{-1}(j)$ and we also get $(C^{(i)})_{jk} = \frac{\lambda}{2} d_\chi(B(\sigma))$.

So the equality follows.

(b) In this case $U_i = U_{pq}$, where (pq) is a transposition of the cycle decomposition of σ . The matrix $B^{(i)}$ differs from $B(\sigma)$ only at the entries (p, q) and (q, p) which are equal to λ , and we have $d_\chi(B^{(i)}) = \lambda^2 |C_\sigma| \chi(\sigma) = \lambda^2 d_\chi(B(\sigma))$. Using similar arguments to that used in the proof of (a), we conclude that:

(i) $(C^{(i)})_{pq} = \lambda d_\chi(B(\sigma))$,

(ii) if $(j, k) \in M_1 \cup M_2$ and $(j, k) \neq (p, q)$, then $(C^{(i)})_{jk} = \lambda^2 d_\chi(B(\sigma))$

(iii) if $(j, k) \in M_3$, then $(C^{(i)})_{jk} = \frac{\lambda^2}{2} d_\chi(B(\sigma))$.

The equality follows.

(c) In this case we have $U_i = U_{pq}$ for some $(p, q) \in M_3$. Thus p lies in the support of a cycle of σ with length greater than 2, and $q = \sigma(p)$ or $q = \sigma^{-1}(p)$. We will assume that $q = \sigma(p)$, since the proof for the other case is similar. We have once more that $B^{(i)}$ differs from $B(\sigma)$ only at the entries (p, q) and (q, p) which are equal to λ . Moreover for each $\rho \in C_\sigma$ we have either $\rho(p) = q$ and $\rho(q) \neq p$, or $\rho(p) \neq q$ and $\rho(q) = p$. Thus for all $\rho \in C_\sigma$, $\prod_{j=1}^n (B^{(i)})_{j\rho(j)} = \lambda$.

Arguing now as in the previous cases we get the equality. $\square$

Lemma 3.10. Let $\sigma \in \mathfrak{S}_\chi$. We have that

$$\langle \text{adj}_\chi(B^{(1)}(\sigma)), \dots, \text{adj}_\chi(B^{(t)}(\sigma)) \rangle = \langle \{U_{ij} | (i, j) \in M_\sigma\},$$

where U_{ij} are defined by (1.1).

Proof. We have already proved for each $i \in \{1, \dots, t\}$ that $C^{(i)}(\sigma) = \text{adj}_\chi(B^{(i)}(\sigma)) \in \langle \{U_{jk} | (j, k) \in M_\sigma\}$. To get the result it is enough to prove that the system $(C^{(1)}(\sigma), \dots, C^{(t)}(\sigma))$ is linearly independent.

Let v_i be the vector column of $C^{(i)}(\sigma)$ in the basis $(U_1, \dots, U_t)$, and let Z be the $t \times t$ matrix with columns $v_1, \dots, v_t$.

From the previous lemma

$$Z = d_\chi(B(\sigma)) \cdot \begin{bmatrix} \lambda J_r - (\lambda - 1)I_r & \lambda^2 J_{r,s} & \lambda J_{r,t-r-s} \\ \lambda J_{s,r} & \lambda^2 J_s - (\lambda^2 - \lambda)I_s & \lambda J_{s,t-r-s} \\ \frac{\lambda}{2} J_{t-r-s,r} & \frac{\lambda^2}{2} J_{t-r-s,s} & \frac{\lambda}{2} J_{t-r-s} - (\frac{\lambda}{2} - \frac{1}{2})I_{t-r-s} \end{bmatrix}.$$

As $\lambda \neq 0$ then

$$Z = \lambda \cdot d_\chi(B(\sigma)) \cdot \begin{bmatrix} J_r - (1 - \frac{1}{\lambda})I_r & \lambda J_{r,s} & J_{r,t-r-s} \\ J_{s,r} & \lambda J_s - (\lambda - 1)I_s & J_{s,t-r-s} \\ \frac{1}{2} J_{t-r-s,r} & \frac{\lambda}{2} J_{t-r-s,s} & \frac{1}{2} (J_{t-r-s} - (1 - \frac{1}{\lambda})I_{t-r-s}) \end{bmatrix}.$$

We are going to show that $\det Z \neq 0$.

Multiplying the columns of Z from $(r+1)$ -st until $(r+s)$ -th by λ^{-1} , and the rows of Z from $(r+s+1)$ until t by 2, we obtain

$$\begin{aligned} \det Z &= \left(d_\chi(B(\sigma))\right)^t \lambda^{t+s} 2^{-(t-r-s)} \det \begin{bmatrix} \frac{1}{\lambda} & 1 & \dots & 1 \\ 1 & \frac{1}{\lambda} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \dots & 1 & \frac{1}{\lambda} \end{bmatrix} \\ &= \left(d_\chi(B(\sigma))\right)^t \lambda^{t+s} 2^{r+s-t} \det \left(J_t - \left(1 - \frac{1}{\lambda}\right)I_t\right). \end{aligned}$$

We recall that J_t denotes the $t \times t$ matrix of all ones.

Since $\sigma \in \mathfrak{S}_\chi$ by Lemma 2.8 it follows that $d_\chi(B(\sigma)) \neq 0$. Since $\lambda \notin \{0, 1, \frac{1}{1-t}\}$ the matrix $J_t - (1 - \frac{1}{\lambda})I_t$ is nonsingular. Hence $\det Z \neq 0$, and thus $C^{(1)}(\sigma), \dots, C^{(t)}(\sigma)$ are linearly independent which concludes the proof. $\square$

Lemma 3.11. *There exist $n \times n$ symmetric matrices $B_1, \dots, B_m$ such that $(\text{adj}_X(B_1), \dots, \text{adj}_X(B_m))$ is a basis of $\Sigma_n(\mathbb{C})$.*

Proof. For each fixed pair (i, j) such that $1 \leq i < j \leq n$ we can choose $\sigma \in \mathfrak{S}_X$ such that $\sigma(i) = j$. This means that $(i, j) \in M_\sigma$, and by Lemma 3.10 there exist symmetric matrices $B^{(1)}(\sigma), \dots, B^{(t)}(\sigma)$ such that $U_{ij} \in \langle \text{adj}_X(B^{(1)}(\sigma)), \dots, \text{adj}_X(B^{(t)}(\sigma)) \rangle$.

On the other hand by Lemma 3.1 there exist symmetric matrices $A^{(1)}, \dots, A^{(n)}$ such that $U_{11}, \dots, U_{nn} \in \langle \text{adj}_X(A^{(1)}), \dots, \text{adj}_X(A^{(n)}) \rangle$.

Taking the union of all the families $(\text{adj}_X(B^{(1)}(\sigma)), \dots, \text{adj}_X(B^{(t)}(\sigma)))$ with $\sigma \in \mathfrak{S}_X$, together with $(\text{adj}_X(A^{(1)}), \dots, \text{adj}_X(A^{(n)}))$, we obtain the result choosing a maximal linearly independent set. $\square$

Consider the basis $(\tilde{U}_{ij})$ of $\Sigma_n(\mathbb{C})$, defined by

$$\begin{cases} \tilde{U}_{ii} = \sqrt{2}E_{ii}, & i = 1, \dots, n, \\ \tilde{U}_{ij} = E_{ij} + E_{ji}, & 1 \leq i < j \leq n. \end{cases}$$

For each $A \in \Sigma_n(\mathbb{C})$ we denote by u_A the column vector of A in the basis $(\tilde{U}_{ij})$ lexicographically ordered. Note that if $A = (a_{ij}) \in \Sigma_n(\mathbb{C})$, we have that

$$A = \sum_{i=1}^n a_{ii}E_{ii} + \sum_{i < j} a_{ij}(E_{ij} + E_{ji}) = \sum_{i=1}^n \frac{a_{ii}}{\sqrt{2}} \tilde{U}_{ii} + \sum_{i < j} a_{ij} \tilde{U}_{ij}.$$

Now we are ready to prove our first main result.

Proof of Theorem 1.2. Assume that $\Phi : \Sigma_n(\mathbb{C}) \rightarrow \Sigma_n(\mathbb{C})$ satisfies condition (1.2).

Since $\mathbb{C}$ is infinite it follows that for any $A, B \in \Sigma_n(\mathbb{C})$, the polynomials $d_X(A + xB)$ and $d_{X'}(\Phi(A) + x\Phi(B))$ coincide, and so their coefficient at x^{n-1} must be equal.

Claim: For any $A, B \in \Sigma_n(\mathbb{C})$, the coefficient of the polynomial $d_X(A + xB)$ at x^{n-1} equals to $2(u_{\text{adj}_X(B)})^t u_A$.

In fact given $A, B \in \Sigma_n(\mathbb{C})$ and denoting $C = \text{adj}_X(B)$, with a direct computation we conclude that this coefficient is equal to

$$\begin{aligned} \sum_{i=1}^n \sum_{j=1}^n a_{ij} d_X(B_{(i,j)}) &= \sum_{i=1}^n \sum_{j=1}^n a_{ij} (\text{adj}_X(B))_{ij} \\ &= \sum_{i,j=1}^n c_{ij} a_{ij} \\ &= \sum_{i=1}^n c_{ii} a_{ii} + 2 \sum_{i < j} c_{ij} a_{ij} \\ &= 2 \left(\sum_{i=1}^n \frac{c_{ii}}{\sqrt{2}} \frac{a_{ii}}{\sqrt{2}} + \sum_{i < j} c_{ij} a_{ij} \right) \\ &= 2(u_{\text{adj}_X(B)})^t u_A. \end{aligned}$$

So we have that

$$(u_{\text{adj}_X(B)})^t u_A = (u_{\text{adj}_{X'}(\Phi(B))})^t u_{\Phi(A)} \quad (3.1)$$

for all $A, B \in \Sigma_n(\mathbb{C})$.

Let us consider matrices $B_1, \dots, B_m \in \Sigma_n(\mathbb{C})$ in conditions of Lemma 3.11 and let $u_i = u_{\text{adj}_X(B_i)}$, and $w_i = u_{\text{adj}_{X'}(\Phi(B_i))}$, $i = 1, \dots, m$.

Consider the matrices $X = \begin{bmatrix} u_1^t \\ \vdots \\ u_m^t \end{bmatrix} \in M_m(\mathbb{C})$ and $Y = \begin{bmatrix} w_1^t \\ \vdots \\ w_m^t \end{bmatrix} \in M_m(\mathbb{C})$.

Taking in (3.1) $B = B_i$, $i = 1, \dots, m$ we conclude that

$$Xu_A = Yu_{\Phi(A)} \quad (3.2)$$

for all $A \in \Sigma_n(\mathbb{C})$. Taking $A = \tilde{U}_{ij}$ in the above equality, we obtain

$$Xu_{\tilde{U}_{ij}} = Yu_{\Phi(\tilde{U}_{ij})} \text{ for all } 1 \leq i \leq j \leq n.$$

Hence we have

$$XI_m = YK,$$

where K is the $m \times m$ matrix with columns $u_{\Phi(\tilde{U}_{ij})}$. Note that the matrix X is nonsingular by the choice of $B_1, \dots, B_m$, and therefore the same must happen to Y and K and equation (3.2) can be rewritten as

$$u_{\Phi(A)} = Ku_A,$$

for all $A \in \Sigma_n(\mathbb{C})$.

Hence Φ is linear. Since $K \in M_m(\mathbb{C})$ is nonsingular, Φ is also bijective. $\square$

Acknowledgement: The authors are grateful to the referees for the fruitful comments and suggestions.

The research was started while the third author visited the University of Lisbon, Portugal. He would like to thank the colleagues for their warm hospitality.

This research was made within the activities of the Centro de Estruturas Lineares e Combinatórias (University of Lisbon, Portugal) and was partially supported by the Fundação para a Ciência e Tecnologia (Lisbon, Portugal). The research of the third author is also partially financially supported by the grants RFFI 12-01-00140a and MD-962.2014.1.

References

- [1] C. Cao, X. Tang: Determinant preserving transformations on symmetric matrix spaces, *Electronic Journal of Linear Algebra* **11** (2004) 205-211.
- [2] M. P. Coelho: Linear preservers of the permanent on symmetric matrices, *Linear and Multilinear Algebra* **41** (1996) 1-8.
- [3] M. P. Coelho, M. A. Duffner: Immanant preserving and immanant converting maps, *Linear Algebra Appl.* **418**(2006) 177-187.
- [4] M. P. Coelho, M. A. Duffner: Linear preservers of immanants on symmetric matrices, *Linear Algebra Appl.* **255** (1997) 314-334.
- [5] M. P. Coelho, M. A. Duffner: On the conversion of an immanant into another on symmetric matrices, *Linear and Multilinear Algebra*, **51**:2 (2003), 137-145.
- [6] G. Dolinar and P. Semrl: Determinant preserving maps on matrix algebras, *Linear Algebra and its Application* **348** (2002) 189-192.
- [7] G. Frobenius, Über die Darstellung der endlichen Gruppen durch lineare Substitutionen. Sitzungsber. Preuss. Akad. Wiss. Berlin (1897), 994–1015.
- [8] G.D. James, A. Kerber: The representation theory of the symmetric group, *Encyclopedia of Mathematics and its Applications* **16**, Cambridge University Press, 1981.
- [9] B. Kuzma: A note on immanant preservers, *Fundamental and Applied Mathematics*, **13**, 4, (2007) 113-120, translated in *Journal of Mathematical Sciences (New York)* **155**:6 (2008), 872-876.
- [10] M. H. Lim: Linear transformations on symmetric matrices, *Linear and Multilinear Algebra* **7** (1979) 47-57.
- [11] D.E. Littlewood: The theory of group characters, Oxford University Press, 1958
- [12] V. Tan and F. Wang: On determinant preserver problems, *Linear Algebra and its Application* **369** (2003) 311-317.

Received August 29, 2013; accepted January 3, 2014.